Then the proposition follows.

The five term exact sequence given by Furtado-Coelho [2, p. 600] is a particular case of the one obtained here by taking $\mathcal{V}'=\mathcal{C}$.

REFERENCES

1. Frohlich, A. — *Baer-invariants of algebras*. Trans. Amer. Math. Soc. 109 (1963), 221—244.
2. Furtado-Coelho, J. — *Homology and generalized Baer-invariants*. J. of Algebra 40 (1976), 596—609.
3. Keune, F. — *Homotopical algebra and algebraic K-thoery*. (Tesis). Univers. of Amsterdam. (1972).
4. Leedhan-Green, C.-Mckay, S. — *Baer-invariants, isologism, varietal laws and homology*. Acta Math., Vol. 137 (1976), 99—150.
5. Lue, A. — *Baer-invariants and extensions relative to a variety*. Proc. Cambridge Philos. Soc. 63 (1967), 569—578.
6. May, J. — *Simplicial objects in algebraic topology*. Van Nostrand (1967).
7. Modi, K. — *Simplicial method and the homology of groups* (Thesis). Q.M.C. University of London (1976).
8. Rinehart, G. — *Satellites and cohomology*. J. of Algebra 12 (1969), 295—329.
9. Rodriguez, C. — *Invariantes de Baer y Cohomologia en variedades de Ω-grupos*. (Tesis). Universidad de Santiago de Compostela. Spain (1980).

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PERIODIC AND QUASI-PERIODIC SOLUTIONS OF ABSTRACT DIFFERENTIAL EQUATIONS

BY

LEOPOLD HERMANN and OTTO VEJVODA

1. Introduction. The aim of this paper is to investigate the existence of quasi-periodic solutions to the following equations

$$u'(t) + Au(t) + cu(t) = g(t), \tag{1.1}$$

$$-u''(t) + Au(t) + cu(t) = g(t), \tag{1.2}$$

$$u''(t) + au'(t) + Au(t) + cu(t) = g(t), \tag{1.3}$$

where A is a linear (in general unbounded) operator in a Hilbert space H, a, c are real constants and g is a quasi-periodic function. Let us recall that an abstract function $t \in R \to f(t) \in H$ is called quasi-periodic with basic frequencies $\nu_1, \ldots, \nu_m$ if there exists a continuous function $(\theta_1, \ldots, \theta_m) \in R^m \to f^*(\theta_1, \ldots, \theta_m) \in H$ of period 2π in $\theta_1, \ldots, \theta_m$ such that $f^*(\nu_1 t, \ldots, \nu_m t) = f(t)$, $t \in R$.

A similar problem (together with the corresponding weakly non-linear case) for

$$A = -\sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}$$

is examined by M. Yamaguchi who looks for a solution either in the space of analytic functions ([10], [11]) or in a subspace of continuously differentiable functions that are characterized by certain convergence properties of their Fourier expansions [13]. In all above papers the time-space-Fourier method (see [9]) is used. In [12] he makes use of the space-Fourier method and the spaces of „mixed" type to treat (1.3) where $a=0$ and A is an elliptic operator of the form

$$A = -\sum_{j,k=1}^{n} \frac{\partial}{\partial x_j}\left(a_{jk}(x)\frac{\partial}{\partial x_k}\right) + a_0(x). \tag{1.4}$$

(In the present paper, we work in the Sobolev spaces of abstract functions and this enables us to weaken the requirements on the smoothness of the right-hand sides). The weakly non-linear problem corresponding to (1.3) with A given by (1.4), $a \neq 0$ and with a small perturbation of the form $\varepsilon f(t, x, u)$ is treated by V. B. Moseenkov [5], [7] also in the framework of Sobolev spaces. In [6] the same author deals with the case $a \to 0$.

Let us add that the equation (1.1) with $H=R^n$ and A an $n\times n$ matrix is studied by some Kazakh mathematicians, see e.g. V. H. Harasahal [1], [2], D. U. Umbetžanov [8].

2. Preliminaries. In what follows we suppose that

(2.1) H is a (complex) Hilbert space,

(2.2) $A:\operatorname{dom} A\subset H\to H$ is a linear operator that is selfadjoint, positive definite

(i.e. there is a $\gamma>0$ such that $(Au,u)\geqq\gamma\|u\|^2$, $u\in\operatorname{dom} A$), and A^{-1} is compact.

Denote by $\{\lambda_k\}_{k=1}^{\infty}$ the nondecreasing sequence of (positive) eigenvalues of A and by $\{v_k\}_{k=1}^{\infty}$ the corresponding system of eigenvectors that forms a complete orthonormal set in H.

The domain $\operatorname{dom} A^s (s\geqq 0)$ of the s-th power of A is considered a Hilbert space with the norm

$$\|u\|_{\operatorname{dom} A^s}=\|A^s u\|_H=\Big(\sum_{k=1}^{\infty}\lambda_k^{2s}|(u,v_k)_H|^2\Big)^{1/2}.$$

Let $m\in N$ and put

$$I=(0,2\pi)^m$$

If $r\in N\cup\{0\}$ and X is a Hilbert space, then we denote by $P^r(I;X)$ the closure of the set of functions $u:R^m\to X$ that are of class C^∞ and $u(\theta_1,\ldots,\ldots,\theta_l+2\pi,\ldots,\theta_m)=u(\theta_1,\ldots,\theta_l,\ldots,\theta_m)$, $l=1,\ldots,m$ in the norm of $W_2^r(I;X)$, that is

$$\|u\|_{W_2^r(I;X)}=\Big(\sum_{|\alpha|\leqq r}\int_I\|D^\alpha u(\theta)\|_X^2\,d\theta\Big)^{1/2}.$$

If $u\in P^0(I;H)$ we can write

$$u(\theta)=\sum_{j,k}u_{jk}\,e^{i(j,\theta)}v_k, \tag{2.3}$$

where

$$u_{jk}=(2\pi)^{-m}\int_I(u(\theta),v_k)_H\,e^{-i(j,\theta)}\,d\theta,$$

$j=(j_1,\ldots,j_m)$, $(j,\theta)=j_1\theta_1+\ldots+j_m\theta_m$ and $\sum\limits_{j,k}$ means the summation over $j\in Z^m$ and $k\in N$. The inclusion $u\in P^r(I;\operatorname{dom} A^s)$ holds if and only if

$$\sum_{j,k}(1+|j|^{2r})\lambda_k^{2s}|u_{jk}|^2<\infty$$

($|j|=(j_1^2+\ldots+j_m^2)^{1/2}$). The square root of the series defines an equivalent norm in $P^r(I;\operatorname{dom} A^s)$. This offers an opportunity to define $P^r(I;\operatorname{dom} A^s)$ even for $r\in R^+$ non-integer.

Using the expansion (2.3) one can obtain the following embedding theorems (cf. [4]).

Lemma 1. *Let $r,s,\rho\in R^+$ and $\rho<r$. Then $P^0(I;\operatorname{dom} A^s)\cap P^r(I;H)$ is embedded into $P^\rho(I;\operatorname{dom} A^{s(1-\rho/r)})$.*

Lemma 2. *Let $r,s\in Z^+$, $\rho\in R^+$ and $\rho+m/2<r$. Then $P^0(I;\operatorname{dom} A^s)\cap P^r(I;H)$ is embedded into $C^\rho(I;\operatorname{dom} A^{s(1-(\rho+m/2)/r)})$.*

3. Main results. Let us assume that

(3.1) $\nu_1,\ldots,\nu_m$ are positive numbers rationally independent, that is $\nu_1 j_1+\ldots+\nu_m j_m=0$, $(j_1,\ldots,j_m)\in Z^m$ holds only if $j_1=\ldots=j_m=0$.

Let us denote

$$D_\theta=\sum_{\iota=1}^m\nu_\iota\frac{\partial}{\partial\theta_\iota},\qquad D_\theta^2=\sum_{\iota=1}^m\sum_{\varkappa=1}^m\nu_\iota\nu_\varkappa\frac{\partial^2}{\partial\theta_\iota\partial\theta_\varkappa}.$$

We consider the problem

$$\rho_\Lambda D_\theta^2u(\theta)+a_\Lambda D_\theta u(\theta)+Au(\theta)+cu(\theta)=g(\theta),\quad u\in U_\Lambda^r, \tag{3.2}$$

where

$$\Lambda=P \text{ or } \Lambda=E \text{ or } \Lambda=T \text{ or } \Lambda=W,$$

$$r\geqslant 1,\ \rho_P=0,\ -\rho_E=\rho_T=\rho_W=1, \tag{3.3}$$

$$a_P=1,\ a_E\in R,\ a_T\in R\{0\},\ a_W=0,\ c\in R$$

and

$$U_P^r=\{u;\ u\in P^0(I;\operatorname{dom} A^r)\cap P^{r-1}(I;H),\ D_\theta u\in P^{r-1}(I;H)\},$$

$$U_E^r=U_T^r=U_W^r=\{u;\ u\in P^0(I;\operatorname{dom} A^r)\cap P^{2(r-1)}(I;H),\ D_\theta^2u\in P^{2(r-1)}(I;H)\}.$$

These spaces are equipped with the norms

$$\|u\|_{U_P^r}=(\|A^ru\|^2_{P^0(I;H)}+\|u\|^2_{P^{r-1}(I;H)}+\|D_\theta u\|^2_{P^{r-1}(I;H)})^{1/2}$$

and (for $\Lambda=E,T,W$)

$$\|u\|_{U_\Lambda^r}=(\|A^ru\|^2_{P^0(I;H)}+\|u\|^2_{P^{2(r-1)}(I;H)}+\|D_\theta^2u\|^2_{P^{2(r-1)}(I;H)})^{1/2},$$

respectively.

If we write $g\in P^0(I;H)$ in the form

$$g(\theta)=\sum_{j,k}g_{jk}\,e^{i(j,\theta)}v_k,\quad g_{jk}=(2\pi)^{-m}\int_I(g(\theta),v_k)_H\,e^{-i(j,\theta)}\,d\theta$$

and if we look for a solution in the form (2.3), we obtain the following system for the Fourier coefficients:

$$(-\rho_\Lambda(\nu,j)^2+ia_\Lambda(\nu,j)+\lambda_k+c)u_{jk}=g_{jk},\ j\in Z^m,\ k\in N.$$

Denote

$S_P=\{(0,k);\ k\in N,\ \lambda_k=-c\}$, $S_E=S_P$ if $a_E\neq 0$

or $S_E=\{(j,k);\ (j,k)\in Z^m\times N,\ (\nu,j)^2+\lambda_k+c=0\}$ if $a_E=0$,

$S_T=S_P$, $S_W=\{(j,k);\ (j,k)\in Z^m\times N,\ -(\nu,j)^2+\lambda_k+c=0\}$.

The following estimates are valid (cf. [9]):

$$\frac{(\nu, j)^2 + \lambda_k^2}{(\nu, j)^2 + (\lambda_k + c)^2} \leqq \text{const.}, \ (j, k) \in (Z^m \times N) \setminus S_P,$$

$$\frac{(\nu, j)^4 + \lambda_k^2}{((\nu, j)^2 + \lambda_k + c)^2} \leqq \text{const.}, \ (j, k) \in (Z^m \times N) \setminus S_E,$$

$$\frac{(\nu, j)^4 + \lambda_k^2}{(-(\nu, j)^2 + \lambda_k + c)^2} \leqq \text{const.}, \ (j, k) \in (Z^m \times N) \setminus \Xi^\varepsilon,$$

$$\Xi^\varepsilon = \{(j, k) \ ; \ (j, k) \in Z \times N, \ \lambda_k + c \geqq 0,$$
$$|(\nu, j)| = \upsilon(\lambda_k + c)^{1/2}, \ \upsilon \in (1 - \varepsilon, \ 1 + \varepsilon)\}$$

(ε is an arbitrary positive number, $\varepsilon < 1$).

$$\frac{(\nu, j)^4 + \lambda_k^2}{(-(\nu, j)^2 + \lambda_k + c)^2 + a_T^2(\nu, j)^2} \leqq \text{const.} \ |(\nu, j)|^{2\alpha} \lambda_k^{1-\alpha}, (\alpha \in [0, 1])(j, k) \in \Xi^\varepsilon \setminus S_T,$$

$$\frac{(\nu, j)^4 + \lambda_k^2}{(-(\nu, j)^2 + \lambda_k + c)^2} \leqq \text{const.} \ |(\nu, j)|^{2\alpha} \lambda_k^{1-\alpha} |j|^{4\beta}, \ (\alpha \in [0, 1]), \ (j, k(\in \Xi^\varepsilon \setminus S_W,$$

where we have assumed that $\beta \geqq 0$ is such a number that for some $C > 0$

$$\begin{aligned} &||(\nu, j)| - (\lambda_k + c)^{1/2}| \geqq C |j|^{-2\beta}, \ j \in Z^m, \\ &k \in N, \ \lambda_k + c > 0, \ |(\nu, j)| \neq (\lambda_k + c)^{1/2}. \end{aligned} \tag{3.4}$$

As a consequence of these estimates we obtain

$$\frac{(1 + (\nu, j)^2) |j|^{2(r-1)} + \lambda_k^{2r}}{(\nu, j)^2 + (\lambda_k + c)^2} \leqq \text{const.} \ (|j|^{2(r-1)} + \lambda_k^{2(r-1)}), \ (j, k) \in (Z^m \times N) \setminus S_P,$$

$$\frac{(1 + (\nu, j)^4) |j|^{4(r-1)} + \lambda_k^{2r}}{((\nu, j)^2 + \lambda_k + c)^2} \leqq \text{const.} \ (|j|^{4(r-1)} + \lambda_k^{2(r-1)}), \ (j, k) \in (Z^m \times N) \setminus S_E.$$

$$\frac{(1 + (\nu, j)^4) |j|^{4(r-1)} + \lambda_k^{2r}}{(-(\nu, j)^2 + \lambda_k + c)^2} \leqq \text{const.} \ (|j|^{4(r-1)} + \lambda_k^{2(r-1)}), \quad (j, k) \in (Z^m \times N) \setminus \Xi^\varepsilon,$$

$$\frac{(1 + (\nu, j)^4) |j|^{4(r-1)}) + \lambda_k^{2r}}{(-(\nu, j)^2 + \lambda_k + c)^2 + a_T^2(\nu, j)^2} \leqq \text{const.} \ (\nu, j)^2 (|j|^{4(r-1)} + \lambda_k^{2(r-1)}),$$
$$(j, k) \in \Xi^\varepsilon \setminus S_T,$$

$$\frac{(1 + (\nu, j)^4) |j|^{4(r-1)} + \lambda_k^{2r}}{(-(\nu, j)^2 + \lambda_k + c)^2} \leqq \text{const.} \ (\nu, j) |j|^{4\beta} (|j|^{4(r-1)} + \lambda_k^{2(r-1)}),$$
$$(j, k) \in \Xi^\varepsilon \setminus S_W.$$

Denoting

$$G_P^r = P^0(I \ ; \text{dom} \ A^{r-1}) \cap P^{r-1}(I \ ; H),$$
$$G_E^r = P^0(I \ ; \text{dom} \ A^{r-1}) \cap P^{2(r-1)}(I \ ; H),$$
$$G_T^r = \{g \ ; g, \ D_\theta g \in G_E^r\},$$
$$G_W^r = \{g \ ; g \in G_E^r, \ D_\theta g \in P^{2\beta}(I \ ; \text{dom} \ A^{r-1}) \cap P^{2(r+\beta-1)}(I \ ; H)\}$$

(the norms are defined in the evident manner) we arrive at the following result.

Theorem 1. *Let the assumptions* (2.1), (2.2), (3.1), (3.3) *be satisfied. If* $\Lambda = W$ *then, moreover, let* (3.4) *be satisfied. Let* $g \in G_\Lambda^r$.

1) *Let* $S_\Lambda = \varnothing$. *Then the problem* (3.2) *has a unique solution* u *and*

$$\| u \|_{U_\Lambda^r} \leqq \text{const.} \ \| g \|_{G_\Lambda^r}.$$

2) *Let* $S_\Lambda \neq \varnothing$. *Then the problem* (3.2) *has a solution if and only if* $P_\Lambda g = \varnothing$, *where*

(3.5) P_Λ is the orthogonal projection of $P^0(I \ ; H)$ onto $\text{cl lin} \{e^{i(j, \theta)} v_k \ ; (j, k) \in S_\Lambda\}$.

If $P_\Lambda g = 0$, *then the solution of* (3.2) *can be written in the form* $u = v + w$, *where*

$$v = \sum_{(j, k) \in S_\Lambda} d_{jk} e^{i(j, \theta)} v_k,$$

d_{jk} *are arbitrary constants,* w *is the unique solution such that* $P_\Lambda w = 0$ *and*

$$\| w \|_{U_\Lambda^r} \leqq \text{const.} \ \| g \|_{G_\Lambda^r}.$$

Remark 1. Under the assumptions of Theorem 1 the following inclusions (that can be useful when dealing with the corresponding weakly nonlinear problem) can be obtained without difficulty. If $\Lambda = P$ then $u, Au, D_\theta u \in G_P^r$, if $\Lambda = E$ then $u, Au, D_\theta^2 \in G_E^r$ and if $\Lambda = T$ (or W and $\beta = 0$) then $u, A^{1/2}u, D_\theta u \in G_\Lambda^r$.

Now, if g is a quasi-periodic function with basic frequencies $\nu_1, \ldots, \nu_m$ then the above theorem can be applied to its corresponding function g^*. In this way we obtain a function u^* which is a solution of (3.2) ; it is continuous together with $D_\theta u^*$ and $D_\theta^2 u^*$, respectively, if we choose r sufficiently large, namely

$$r > \frac{m}{2} + 2 \ \text{if} \ \Lambda = P \ \text{and} \ r > \frac{m}{4} + 2. \tag{3.6}$$

in the other cases.

The function u^* represents the corresponding function to a quasi-periodic solution of the equation

$$\rho_\Lambda u''(t) + a_\Lambda u'(t) + Au(t) + cu(t) = g(t) \tag{3.7}$$

Theorem 2. *Let the assumptions* (2.1), (2.2), (3.1), (3.3), (3.6) *be satisfied. If* $\Lambda = W$ *then, moreover, let* (3.4) *be satisfied. Let* g *be a quasi-periodic function with basic frequencies* $\nu_1, \ldots, \nu_m$ *and the corresponding function* g^* *belong to* G_Λ^r.

1) *Let* $S_\Lambda = \varnothing$. *Then the equation* (3.7) *has a unique quasi-periodic solution with the basic frequencies* $\nu_1, \ldots, \nu_m$.

2) *Let* $S_\Lambda = \varnothing$. *Then the equation* (3.7) *has a quasi-periodic solution with basic frequencies* $\nu_1, \ldots, \nu_m$ *if and only if* $P_\Lambda g^* = 0$ *where* P_Λ *is given by*

(3.5). *If this condition is satisfied, then the quasi-periodic solution to* (3.7) *can be written in the form* $u=v+w$, *where*

$$v(t)=\sum_{(j,k)\in S_\Lambda} d_{jk}\, e^{i(\nu,j)t}\, v_k,$$

d_{jk} are arbitrary constants and w is the unique quasi-periodic solution such that $P_\Lambda w^*=0$.

Remark 2. The corresponding weakly nonlinear problems can be dealt with as well : by means of the Banach contraction principle if $S_\Lambda=\varnothing$ or by means of the alternative method and the implicit function theorem if $S_\Lambda\neq\varnothing$ (cf. [3]).

REFERENCES

1. Harasahal, V. H. — *On one system of partial differential equations.* (Russian). Trudy sekt. matem. meh. Akad. Nauk Kaz SSR 2 (1963), 3—19.
2. Harasahal, V. H. — *Quasi-periodic solutions of ordinary differential equations.* (Russian.) Nauka Kaz SSR, Alma-Ata, 1970.
3. Herrmann, L. — *Periodic solutions to abstract differential equations : the Fourier method.* Czechoslovak Math. J. 30 (1980), 177—206.
4. Lions, J.-L., Magenes, E. — *Problèmes aux limites non homogènes et applications*, Vol. 1. Dunod, Paris, 1968.
5. Moseenkov, V. B. — *Quasi-periodic solutions of a non-linear wave equation with damping.* (Russian.) Ukrain. Mat. Ž. 29 (1977), 537—541.
6 Moseenkov, V. B. — *Quasi periodic solutions of a weakly dissipative nonlinear wave equation.* (Russian.) Ukrain. Mat. Ž. 2 (1978), 254—257.
7. Moseenkov, V. B. — *Quasi-periodic solutions of a non-linear hyperbolic equation with damping.* (Russian.) Differencial'nye Uravnenija 95 (1979), 695—703.
8. Umbetžanov, D. U. — *On periodic and almost periodic solutions, holomorphic in a small parameter, for a class of partial differential equations.* (Russian.) Vestnik Moskov. Univ. 3 (1969), 39—43.
9. Vejvoda, O. et al. — *Partial differential equations : time-periodic solutions.* Sijthoff-Nonrdhoff, Alphen aan den Rijn, 1981.
10. Yamaguchi, M. — *On the existence of quasi-periodic solutions of non-linear hyperbolic partial differential equations.* Proc. Japan Acad. 48 (1972), 320—322.
11. Yamaguchi, M. — *On the existence of quasi-periodic solutions of non-linear partial differential equations.* Proc. Fac. Sci. Tokai Univ. 10 (1974), 1—12.
12. Yamaguchi, M. — *Note on quasi-periodic solutions of undamped linear wave equations.* Proc. Fac. Sci. Tokai Univ. 12 (1976), 3—10.
13. Yamaguchi, M. — *Existence of quasi-periodic solutions of perturbed nonlinear and quasi-linear partial differential equations of standard types.* J. Math. Anal. Appl. 59 (1977), 15—28.

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EXISTENCE FOR A PARABOLIC EQUATION WITH NONLINEAR BOUNDARY VALUE CONDITIONS

BY

ZHOU MEIKE

1. Introduction. In this paper we are concerned with the nonlinear boundary-value problem of the form

$$\frac{\partial}{\partial t}\beta(u(t,x))-\Delta u(t,x)\ni f(t,x),\ (t,x)\in Q_T, \tag{1.1}$$

$$\frac{\partial u(t,x)}{\partial n}+\sigma(u(t,x))\ni g(x,u(t,x)),\ (t,x)\in\Sigma_T, \tag{1.2}$$

$$\beta(u(0,x))\ni v_0(x),\ x\in\Omega. \tag{1.3}$$

Here Ω is a bounded and open subset of the Euclidean space R^m with the boundary S ; $\partial u/\partial n$ is the outward normal derivative of u ; β and σ are maximal monotone graphs in R^2 (possible multi-valued) ; f, g and v_0 are given functions on $Q_T=]0,T[\times\Omega$, $S\times R$ and Ω respectively ; $\Sigma_T=]0,T[\times S$.

Problems of this type occur in the heat radiation (see e.g.H.B. Keller [1]), the absorbtion of gas in a liquid (see e.g.C.V. Pao [2]), the termostat control problems (see e.g. G. Duvaut and J.L. Lions [3]) and in description of other physical problems. For instance, if we consider the following boundary control problem governed by the Stefan problem (see Ch. Saguez [4]).

$$\frac{\partial G(\theta)}{\partial t}-\Delta\theta=0,\qquad (t,x)\in Q_T,$$
$$\frac{\partial\theta}{\partial n}+h(\theta-\theta e)=u,\qquad (t,x)\in\Sigma_T$$
$$G(\theta(0,x))\ni v_0(x),\qquad x\in\Omega,$$

where G is a maximal monotone graph in R^2, h is a positive constant, θ_e and v_0 are given functions, the existence result which will be given below allows to implement a feedback law of the following form

$$u(t,x)=g(x,u(t,x)),$$

where g satisfies conditions $(g)_1$, $(g)_2$ given below. For other results concerning problem (1.1)—(1.3) in the case $g\equiv 0$ we refer the reader to the works of J.L. Lions [5] and Ph. Benilan [6].