(3.5). *If this condition is satisfied, then the quasi-periodic solution to* (3.7) *can be written in the form* $u = v + w$, *where*

$$v(t) = \sum_{(j,k) \in S_\Lambda} d_{jk} e^{i(\nu,j)t} v_k,$$

d_{jk} are arbitrary constants and w is the unique quasi-periodic solution such that $P_\Lambda w^* = 0$.

Remark 2. The corresponding weakly nonlinear problems can be dealt with as well : by means of the Banach contraction principle if $S_\Lambda = \varnothing$ or by means of the alternative method and the implicit function theorem if $S_\Lambda \neq \varnothing$ (cf. [3]).

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EXISTENCE FOR A PARABOLIC EQUATION WITH NONLINEAR BOUNDARY VALUE CONDITIONS

BY

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1. Introduction. In this paper we are concerned with the nonlinear boundary-value problem of the form

$$\frac{\partial}{\partial t}\beta(u(t,x)) - \Delta u(t,x) \ni f(t,x),\ (t,x) \in Q_T, \tag{1.1}$$

$$\frac{\partial u(t,x)}{\partial n} + \sigma(u(t,x)) \ni g(x, u(t,x)),\ (t,x) \in \Sigma_T, \tag{1.2}$$

$$\beta(u(0,x)) \ni v_0(x),\ x \in \Omega. \tag{1.3}$$

Here Ω is a bounded and open subset of the Euclidean space R^m with the boundary S ; $\partial u/\partial n$ is the outward normal derivative of u ; β and σ are maximal monotone graphs in R^2 (possible multi-valued) ; f, g and v_0 are given functions on $Q_T =]0, T[\times\Omega$, $S \times R$ and Ω respectively ; $\Sigma_T =]0, T[\times S$.

Problems of this type occur in the heat radiation (see e.g.H.B. Keller [1]), the absorbtion of gas in a liquid (see e.g.C.V. Pao [2]), the termostat control problems (see e.g. G. Duvaut and J.L. Lions [3]) and in description of other physical problems. For instance, if we consider the following boundary control problem governed by the Stefan problem (see Ch. Saguez [4]).

$$\frac{\partial G(\theta)}{\partial t} - \Delta\theta = 0, \qquad (t,x) \in Q_T,$$
$$\frac{\partial \theta}{\partial n} + h(\theta - \theta e) = u, \qquad (t,x) \in \Sigma_T$$
$$G(\theta(0,x)) \ni v_0(x), \qquad x \in \Omega,$$

where G is a maximal monotone graph in R^2, h is a positive constant, θ_e and v_0 are given functions, the existence result which will be given below allows to implement a feedback law of the following form

$$u(t,x) = g(x, u(t,x)),$$

where g satisfies conditions $(g)_1$, $(g)_2$ given below. For other results concerning problem (1.1)–(1.3) in the case $g \equiv 0$ we refer the reader to the works of J.L. Lions [5] and Ph. Benilan [6].

We shall suppose here that the boundary S of Ω is smooth, and it consists of k disjoint parts S_i, $i=1, \dots, k$, such that $S=\bigcup_{i=1}^{k}\bar{S}_i$ and $S_i \cap S_j = \varnothing$, $1 \leqslant i < j \leqslant k$, and there exists for each S_i a cartesian coordinate system $y = T_i x$ such that

$$S_i: \quad y_m = h_i(\bar{y}),\ \bar{y}=(y_1, \dots, y_{m-1}) \in D_i, \tag{1.4}$$

$$\Omega_\lambda^i = \{T_i^{-1}y\ ;\ \bar{y} \in D_i,\ 0 < y_m - h_i(\bar{y}) < \lambda^{1/2}\} \subset \Omega, \tag{1.5}$$

where D_i is a bounded and open subset of R^{m-1}; h_i is a once continuously differentiable function in D_i, and its gradient is bounded on D_i; λ is positive and sufficiently small.

As regards the maximal monotone graphs β and σ and functions f, g and v_0, we shall suppose that

(σ) $0 \in \sigma(0)$, $D(\sigma) = R^1$;

$(\beta)_1$ $0 \in \beta(0)$, $R(\beta) = R^1$ and int $D(\beta) \ni 0$;

$(\beta)_2$ for each $N > 0$ there exists $\alpha_N > 0$ such that $(\beta u - \beta v)(u - v) \geqslant \alpha_N (u-v)^2$, for $u, v \in D(\beta)$ with $|u|, |v| \leqslant N$;

$(g)_1 \to g(x, u)$ is measurable as a function of $x \in S$ for each $u \in R^1$, and continuous as a function of $u \in R^1$ for a.e. $x \in S$. In addition, it is assumed that for each $N > 0$, there exist $M_N > 0$, $L_N > 0$, such that

$$|g(x,u)| \leqslant M_N,$$

$$|g(x,u) - g(x,v)| \leqslant L_N |u - v|,$$

a.e. $x \in S$ and $u, v \in R^1$ with $|u|, |v| \leqslant N$;

$(g)_2$ there exist $k_1 < 0$, $k_2 > 0$ such that $[k_1, k_2] \subset \operatorname{int} D(\beta)$ and $g(x, u)u \leqslant 0$ a.e. $x \in S$ and $u \notin [k_1, k_2]$;

(f) $\quad f \in L^2(0, T; L^\infty(\Omega))$;

(v_0) $\quad v_0 \in H^1(\Omega) \cap L^\infty(\Omega)$,

where $H^1(\Omega)$ denotes the usual Sobolev space $W^{1,2}(\Omega)$.

We shall denote by $H = L^2(\Omega)$. Let $W' = (H^1(\Omega))'$ be the dual space of $W = H^1(\Omega)$, i.e. W' is the completion of H under the norm

$$\|w^*\|_{W'} = \sup_{\|u\|_W = 1} (w^*, u). \tag{1.6}$$

Here $(.,.)$ is the usual inner product in H and will be also used to denote the pairing between W and W'. The norms in H, W and W' are denoted by $|.|_H$, $\|.\|_W$ and $\|.\|_{W'}$, respectively. As usual, $H^1(Q_T)$ denotes the Sobolev space $W^{1,2}(Q_T)$. Let X be a Banach space; we shall denote by $C([0, T]; X)$ the space of X-valued continuous functions on $[0, T]$. $L^p(0, T; X)$, $p \geqslant 1$, denotes the space of functions $t \to h(t)$ measurable from $[0, T]$ to X (for the measure dt) such that

$$\left(\int_0^T \|h(t)\|_X^p\, dt\right)^{1/p} = \|h\|_{L^p(0,T;X)} < +\infty, \ (p \neq +\infty),$$

$$\operatorname*{ess\,sup}_{t \in]0,T[} \|h(t)\|_X = \|h\|_{L^\infty(0,T;X)} < +\infty, \ (p = +\infty).$$

Let $A: W \to W'$ be the operator defined by

$$Au = \Big\{ w^* \in W';\ \exists z \in L^2(S),\ Z(x) \in \sigma(u(x)) \text{ a.e. } x \in S \text{ such that}$$

$$(w^*, \psi) = \int_\Omega \operatorname{grad} u \operatorname{grad} \psi\, dx + \int_S z\psi ds, \quad \forall \psi \in W \Big\} \tag{1.7}$$

for $u \in D(A) = \{u \in W;\ Au \neq \theta\}$.

For $u \in W$ such that $g(., u(.)) \in L^2(S)$, we set $Gu \in W'$ defined by

$$(Gu, \psi) = \int_S g(x, u(x))\psi(x) ds, \quad \forall \psi \in W. \tag{1.8}$$

Let $B: H \to H$ be the operator defined by

$$Bu = \{v \in H;\ v(x) \in \beta(u(x)) \text{ a.e. } x \in \Omega\} \tag{1.9}$$

for $u \in D(B) = \{u \in H;\ Bu \neq \theta\}$. As it is well known, B is a maximal monotone operator on H.

Therefore, problem (1.1)–(1.3) can be written under the following form

$$(Bu(t))' + Au(t) - Gu(t) \ni f(t),\ t \in]0, T[, \tag{1.10}$$

$$Bu(0) \ni v_0, \tag{1.11}$$

where $' = d/dt$.

The main result of this paper is the following theorem.

Theorem 1.1. *Let conditions* (σ), $(\beta)_{1-2}$, $(g)_{1-2}$, (f) *and* (v_0) *be satisfied. Then problem* (1.1)–(1.3) *has a solution* $u \in C([0, T]; H) \cap L^\infty(0, T; W) \cap L^\infty(Q_T)$ *in the sense that there exists* $v \in L^2(0, T; H)$ *with* $v(t) \in Bu(t)$ *a.e.* $t \in]0, T[$ *such that*

$$\frac{d}{dt}(v(t), \psi) + (Au(t) - gu(t), \psi) = (f(t), \psi), \text{ a.e.} t \in]0, T[, \tag{1.12}$$

$$v(0) = v_0, \text{ for all } \psi \in W. \tag{1.13}$$

The contents of this paper is outlined below. In Section 2 we shall construct an approximating problem associated with (1.1)–(1.3). We shall prove the existence and we shall establish a priori estimates for its solutions. The proof of the main theorem is delivered in Section 3.

2. Approximating problem. Let A_H be the operator $H \to H$ defined by $A_H u = Au \cap H$ for $u \in D(A_H) = \{u \in W;\ Au \cap H \neq \theta\}$. It is easy to show that $w^* \in A_H u$, if and only if u is a solution in $H^1(\Omega)$ to the problem

$$-\Delta u = w^*,\ x \in \Omega, \tag{2.1}$$

(2.2) $$\frac{\partial u}{\partial n}+\sigma(u)\ni 0,\quad x\in S.$$

We shall denote by $j: R^1\to R^1$ a convex lower semicontinuous function such that $\sigma=\partial j$ and $j\geqslant 0$ on R^1, where ∂j denotes the subdifferential of j. It is well known that such a function always exists (see, e.g., [7, pp. 59—60]).

Therefore, $A_H=-\Delta$, where $D(A_H)=\{u\in H^2(\Omega)\,;\,-\partial u/\partial n\in\sigma(u)$ a.e. $x\in S\}$, and $A_H=\partial\varphi$, where φ is the proper convex lower-semicontinuous function from H to $]-\infty,+\infty]$ defined by

(2.3) $$\varphi(u)=\begin{cases}\dfrac{1}{2}\displaystyle\int_\Omega|\operatorname{grad}u|^2dx+\int_S j(u)ds,\\ \quad\text{if } u\in H^1(\Omega) \text{ and } j(u)\in L^1(S),\\ +\infty,\ \text{otherwise.}\end{cases}$$

(see [7, pp. 63—67]).

Let $A_\lambda=A_H(1+\lambda A_H)^{-1}$ be the Yosida approximation of A_H. Next we shall approximate the operator G defined by (1.8) by a family of Lipschitz continuous operators on H. To make all what follows meaningful, let us briefly describe the physical situation from which this approximation originates.

It is well known that if problem (1.1)—(1.3) describes a diffusion process of heat in a domain Ω, $u(t,x)$ denoting the temperature of a point $x\in\Omega$ at time t, then the function g is a part of the surface density of the heat injection through the boundary S of Ω into its interior. In other words, the heat injection, which is given by surface density g in unity time through an element ΔS of surface S, is equal to integral $\int_{\Delta S}gds$. The function f is the volume density of the heat injection directly in interior of Ω. In other words, the heat injection, which is given by volume density f in unity time through an element $\Delta\Omega$ of volume Ω, is equal to integral $\int_{\Delta\Omega}fdx$. Let $\hat g$ be a function defined on Ω and satisfying the following conditions

(2.4) $$\operatorname{Supp}\hat g\subset\sum_{i=1}^{k}\Omega_\lambda^i,$$

(2.5) $$\int_{\Delta S}\hat g dS=\sum_{i=1}^{k}\int_{\Omega_\lambda^i}\hat g dx,$$

where $\lambda>0$ is sufficiently small and

(2.6) $$\Delta\Omega_\lambda^i=\{x\in\Omega_\lambda^i\,;\,\overline{x}^i\in\Delta S\cap S_i\},$$

$\overline{x}^i$ being given by

(2.7) $$\overline{x}^i=T_i^{-1}(\overline{T_ix},\,h_i(\overline{T_ix})),$$

i.e., the projection of x on S_i in the direction of y_m-axis. Then it is physically obvious that the problem obtained by substituting f and g by 0 and $f+\hat g$ respectively, is a natural approximation to the original problem.

To this aim, we set

(2.8) $$g_{\lambda,i}(x,u)=\begin{cases}\lambda^{-1/2}g(\overline{x}^i,u)\eta_i(x), & x\in\Omega_\lambda^i,\ u\in R^1,\\ 0, & \text{otherwise}\end{cases}$$

(2.9) $$g_\lambda(x,u)=\sum_{i=1}^{k}g_{\lambda,i}(x,u),\quad x\in\Omega,\ u\in R^1,$$

where

(2.10) $$\eta_i(x)=\left[1+\sum_{j=1}^{m-1}\left(\frac{\partial h_i}{\partial y_j}(\overline{T_jx})\right)^2\right]^{1/2}.$$

We define the operator $G_\lambda: H\to H$ by

(2.11) $$(G_\lambda u)(x)=g_\lambda(x,u(x)),\ \text{a.e. } x\in\Omega$$

for $u\in H$ such that $g_\lambda(.,u(.))\in H$.

Thus, for each $\lambda>0$, we consider the approximating problem

(2.12) $$(Bu_\lambda(t))'+A_\lambda u_\lambda(t)-G_\lambda u_\lambda(t)\ni f(t),\quad t\in]0,T[,$$

(2.13) $$Bu_\lambda(0)\ni v_0.$$

We have the following lemma.

Lemma 2.1. *Let assumptions* (σ), $(\beta)_{1-2}$, $(g)_{1-2}$, (f) *and* (v_0) *be satisfied. Then, problem* (2.12)—(2.13) *for* $q\geqslant 2$ *has a unique solution* $u_\lambda\in C([0,T]\,;\,L^q(\Omega))\cap L^\infty(Q_T)$ *with* $u_\lambda'\in L^\infty(0,T\,;\,L^q(\Omega))$ *in the sense that there exists* $v_\lambda\in C([0,T]\,;\,L^q(\Omega))\cap L^\infty(Q_T)$ *with* $v_\lambda'\in L^\infty(0,T\,;\,L^q(\Omega))$ *such that* $v_\lambda(t)\in Bu_\lambda(t)$ *a.e.* $t\in]0,T[$ *and*

(2.14) $$v_\lambda'(t)+A_\lambda u_\lambda(t)-G_\lambda u_\lambda(t)=f(t),\quad t\in]0,T[.$$

(2.15) $$v_\lambda(0)=v_0$$

Proof. We define

(2.16) $$g^N(x,u)=\begin{cases}g(x,u), & x\in S,\ |u|\leqslant N,\\ g(x,N), & x\in S,\ u>N,\\ g(x,-N), & x\in S,\ u<-N,\end{cases}$$

(2.17) $$g_{\lambda,i}^N(x,u)=\begin{cases}\lambda^{-1/2}g^N(\overline{x}^i,u)\eta_i(x), & x\in\Omega_\lambda^i,\ u\in R^1,\\ 0, & \text{otherwise,}\end{cases}$$

(2.18) $$g_\lambda^N(x,u)=\sum_{i=1}^{k}g_{\lambda,i}^N(x,u),\quad x\in\Omega,\ u\in R^1$$

and the operator $G_\lambda^N : H \to H$ by

$$(G_\lambda^N u)(x) = g_\lambda^N(x, u(x)), \text{ a.e. } x \in \Omega \tag{2.19}$$

for each $u \in H$.

Let $\beta_N = \beta + \partial I_N$, where

$$I_N(u) = \begin{cases} 0, & |u| \leqslant N, \\ +\infty, & \text{otherwise.} \end{cases}$$

Then, β_N is a maximal monotone graph in R^2, because $0 \in \operatorname{int} D(\partial I_N) \cap D(\beta) \neq \neq \theta$ (see [3, p. 46]). We define $B_N : H \to H$ by

$$B_N u = \{v \in H \ ; \ v(x) \in \beta_N(u(x)), \text{ a.e. } x \in \Omega\} \tag{2.20}$$

for $u \in D(B_N) = \{u \in H \ ; \ B_N u \neq \theta\}$.

First, we consider the problem

$$(B_N u_\lambda^N(t))' + A_\lambda u_\lambda^N(t) - G_\lambda^N u_\lambda^N(t) \ni f(t), \ t \in]0, T[, \tag{2.21}$$

$$B_N u_\lambda^N(0) \ni v_0, \tag{2.22}$$

or equivalently

$$u_\lambda^N(t) = B_N^{-1}\left\{v_0 + \int_0^t (f(\tau) - A_\lambda u_\lambda^N(\tau) + G_\lambda^N u_\lambda^N(\tau))d\tau\right\}. \tag{2.23}$$

Since for $u \in L^q(\Omega)$, $q \geqslant 2$, the problem

$$y(x) - \lambda\Delta y(x) = u(x), \ x \in \Omega, \tag{2.24}$$

$$\frac{\partial y(x)}{\partial n} + \sigma(y(x)) \ni 0, \ x \in S \tag{2.25}$$

has a unique solution in $H^2(\Omega) \cap L^q(\Omega)$ (see H. B r é z i s [8, pp. 58—59]), it follows that $(1 + \lambda A_H)^{-1}$ is nonexpansive on $L^q(\Omega)$, and therefore, A_λ is Lipschitz on $L^q(\Omega)$. Noticing that B_N^{-1}, G_λ^N are also Lipschitz on $L^q(\Omega)$, it follows by standard arguments that problem (2.23) has a unique solution $u_\lambda^N \in C([0, T] \ ; \ L^q(\Omega))$ with $(u_\lambda^N) \in L^\infty(0, T \ ; L^q(\Omega))$.

To conclude the proof of Lemma 2.1, we shall show that for sufficiently large N,

$$\|u^N(t)\|_{L^\infty(\Omega)} < N \text{ for all } t \in [0, T] \text{ and } \lambda > 0. \tag{2.26}$$

Let

$$v_\lambda^N(t) = v_0 + \int_0^t (f(\tau) - A_\lambda u_\lambda^N(\tau) + G_\lambda^N u_\lambda^N(\tau))d\tau. \tag{2.27}$$

Then $v_\lambda^N \in B_N u^N$, $v_\lambda^N \in C([0, T] \ ; \ L^q(\Omega))$ and

$$(v_\lambda^N(t))' + A_\lambda u_\lambda^N(t) - G_\lambda^N u_\lambda(t) = f(t), \text{ a.e. } t \in]0, T[, \tag{2.28}$$

$$v_\lambda^N(0) = v_0. \tag{2.29}$$

It is easy to show that

$$\int_0^t\int_\Omega |v_\lambda^N(\tau)|^{q-2} v_\lambda^N(\tau)(v_\lambda^N(\tau))' dx d\tau = \frac{1}{q}\|v_\lambda^N(t)\|_{L(\Omega)}^q - \frac{1}{q}\|v_0\|_{L^q(\Omega)}^q, \tag{2.30}$$

$$\int_0^t\int_\Omega f(\tau)|v_\lambda^N(\tau)|^{q-2} v_\lambda^N(\tau) dx d\tau \leqslant \int_0^t \|f(\tau)\|_{L^q(\Omega)} \|v_\lambda^N(\tau)\|_{L^q(\Omega)}^{q-1} d\tau. \tag{2.31}$$

Using proposition 1.1 in [7, p. 183], we may infer that

$$\int_0^t\int_\Omega A_\lambda u_\lambda^N(\tau)|v_\lambda^N(t)|^{q-2} v_\lambda^N(\tau) dx d\tau \geqslant 0. \tag{2.32}$$

By virtue of $(\beta)_1$, $(g)_2$ and taking $N > \max\{-k_1, k_2\}$, we deduce

$$\int_0^t\int_\Omega g_\lambda^N(x, u_\lambda^N(\tau, x))|v_\lambda^N(\tau, x)|^{q-2} v_\lambda^N(\tau, x) dx d\tau \leqslant$$

$$\leqslant \sum_{i=1}^k \lambda^{-1/2} t \operatorname{mes} \Omega_\lambda^i \cdot C_g \sup_{x \in \Omega_\lambda^i} n_i(x) \cdot C_\beta^{q-1}, \tag{2.33}$$

where $C_g = \operatorname*{ess\,sup}_{(x,u) \in S \times [k_1, k_2]} |g(x, u)|$, and $C_\beta = \sup_{u \in [k_1, k_2]} |\beta(u)|$.

Multiplying (2.28) by $|v_\lambda^N(t)|^{q-2} v_\lambda^N(t)$, integrating over Q_t, and using (2.30)—(2.33), we get

$$\|v_\lambda^N(t)\|_{L^q(\Omega)}^q \leqslant (\|v_0\|_{L^q(\Omega)}^q + CqC_\beta^{q-1}) + q\int_0^t \|f(\tau)\|_{L^q(\Omega)}(1 + \|v_\lambda^N(\tau)\|_{L^q(\Omega)}^q)d\tau. \tag{2.34}$$

According to Gronwall's inequality, we obtain

$$\|v_\lambda^N(t)\|_{L^q(\Omega)}^q \leqslant (\|v_0\|_{L^q(\Omega)}^q + CqC_\beta^{q-1} + q\|f\|_{L^1(0,t;L^q(\Omega))}) \cdot \\ \cdot (1 + q\|f\|_{L^1(0,t;L^q(\Omega))} \cdot e^{q\|f\|_{L^1(0,t;L^q(\Omega))}}). \tag{2.35}$$

Taking q-root and letting $q \to +\infty$, it follows that

$$\|v_\lambda^N(t)\|_{L^\infty(\Omega)} \leqslant b, \tag{2.36}$$

for all $t \in [0, T]$, where $b = (1 + C_\beta + \|v_0\|_{L^\infty(\Omega)})(1 + e^{\|f\|_{L^1(0,T;L^\infty(\Omega))}})$, is independent of N, λ and t.

From $(\beta)_{1-2}$, we see that β^{-1} is locally Lipschitz on R^1. Noticing that $|\beta^{-1}v| \geqslant |\beta_N^{-1}v|$ for all $v \in R^1$, we see by (3.36) that

$$\|u_\lambda^N(t)\|_{L^\infty(\Omega)} \leqslant \max\{\beta^{-1}(b), -\beta^{-1}(-b)\} \tag{2.37}$$

for all $t \in [0, T]$.

Taking $N > \max\{-k_1, k_2, \beta^{-1}(b), -\beta^{-1}(-b)\}$, we obtain (2.26) as claimed. For this N, letting $v_\lambda = v_\lambda^N$, we see that $u_\lambda = u_\lambda^N$ is a solution of problem (2.12)—(2.13). This completes the proof of Lemma 2.1.

In order to establish a priori estimates for the solution u_λ given by Lemma 2.1, we need of the following lemma.

Lemma 2.2 *If* $u \in W \cap L^\infty(\Omega)$, *then* $u \in L^\infty(S)$ *and* $\|u\|_{L^\infty(S)} \leqslant \|u\|_{L^\infty(\Omega)}$.

Proof. For $x \in \Omega_\lambda^i$, we denote (for simplicity)

$$u(y) = u(\bar{y}, y_m) = u(T_i^{-1}\ y) = u(x),$$

where $y = T_i x$. For $q \geqslant 2$ we have

$$|u(\bar{y}, h_i(\bar{y}))|^q = |u(y)|^q + q\int_{y_m}^{h_i(\bar{y})} |u(\bar{y}, \xi)|^{q-2} u(\bar{y}, \xi)\left|\frac{\partial u(,\bar{y}\xi)}{\partial \xi}\right| d\xi \leqslant$$

$$\leqslant |u(y)|^q + q\int_{h_i(\bar{y})}^{h_i(\bar{y})+\sqrt{\lambda}} |u(\bar{y}, \xi)|^{q-1}\left|\frac{\partial u(\bar{y}, \xi)}{\partial \xi}\right| d\xi,$$

and therefore,

$$(2.38) \qquad |u(\bar{y}, h_i(\bar{y}))|^q \leqslant \int_{h_i(y)}^{h_i(\bar{y})+\sqrt{\lambda}} \left(\lambda^{-1/2}|u(y)|^q + q|u(y)|^{q-1}\left|\frac{\partial u(y)}{\partial y_m}\right|\right) dy_m.$$

Multiplying (2.38) by $n_i(T_i^{-1}y)$, integrating over D_i and using the boundness of n_i, it follows that

$$\|u\|^q_{L^q(S_i)} \leqslant C\left(\lambda^{-1/2}\|u\|^q_{L^q(\Omega_\lambda^i)} + q\|u\|^{q-1}_{L^{2(q-1)}(\Omega_\lambda^i)}\left\|\frac{\partial u}{\partial y_m}\right\|_{L^2(\Omega_\lambda^i)}\right).$$

Using the inequality

$$\|u\|^q_{L^q(\Omega_\lambda^i)} \leqslant \|u\|^q_{L^{2(q-1)}(\Omega_\lambda^i)} (\mathrm{mes}\ \Omega_\lambda^i)^{1-q/2(q-1)}$$

and taking q-root, we infer

$$\|u\|_{L^q(S_i)} \leqslant C^{1/q}\Big(\lambda^{-1/2}(\mathrm{mes}\ \Omega_\lambda^i)^{1-q/[2(q-1)]}\|u\|_{L^{2(q-1)}(\Omega_\lambda^i)} +$$

$$(2.39) \qquad + q\left\|\frac{\partial u}{\partial y_m}\right\|_{L^2(\Omega_\lambda^i)}\Big)^{1/q} \|u\|^{1-1/q}_{L^{2(q-1)}(\Omega_\lambda^i)}.$$

Letting $q \to +\infty$ in (2.39), we complete the proof of Lemma 2.2

Lemma 2.3. *The solution* u_λ, *given by Lemma 2.1, satisfies the following estimates* :

$$(2.40) \qquad \lambda|A_H y_\lambda(t)|_H^2 + \|y_\lambda(t)\|_W^2 + \int_0^t |u_\lambda'(\tau)|_H^2 d\tau \leqslant C,$$

$$(2.41) \qquad \int_0^t |y_\lambda'(\tau)|_H^2 d\tau \leqslant C,$$

$$(2.42) \qquad \|y_\lambda(t)\|_{L^\infty(S)} \leqslant C,$$

$$(2.43) \qquad \|A_H y_\lambda\ (t)\|_{W'} \leqslant C,$$

where $y_\lambda(t) = (1 + \lambda A_H)^{-1} u_\lambda(t)$ *and* C *is a constant independent of* $\lambda > 0$ *and* $t \in [0, T]$.

Proof. Multiplying (2.14) by $u_\lambda'(t)$, integrating over Q_t and using the equality (see [7, p. 189]).

$$(A_\lambda u_\lambda(t),\ u_\lambda'(t)) = \frac{d}{dt}\varphi_\lambda(t) \quad \text{a.e. } t \in]0, T[$$

where $\varphi_\lambda(t) = \lambda/2|A_H u(t)|_H^2 + \varphi((1 + \lambda A_H)^{-1}u(t))$, we denote, by assumption $(\beta)_2$:

$$\frac{\alpha_N}{2}\int_0^t |u_\lambda'(\tau)|_H^2 d\tau + \varphi_\lambda(u_\lambda(t)) \leqslant \frac{1}{2\alpha_N}\|f\|^2_{L^2(Q_t)} + \varphi_\lambda(u_0) +$$

$$(2.44) \qquad + \sum_{i=1}^k \int_0^t\int_{\Omega_\lambda^i} g^N_{\lambda,i}(x, u_\lambda(\tau, x))u_\lambda'(\tau, x)dx d\tau,$$

where $u_0 = B^{-1}v_0$ (it is easy to see that $u_0 = u_\lambda(0)$). As u_λ, u_0, $\lambda^{1/2} g^N_{\lambda,i}$ and $\lambda^{-1/2}$ mes Ω_λ^i are bounded, it follows that

$$\left|\int_0^t\int_\Omega g^N_{\lambda,i}(x, u_\lambda(\tau, x))u_\lambda'(\tau,\ x)dx d\tau\right| =$$

$$(2.45) \qquad = \lambda^{-1/2}\left|\int_{\Omega_\lambda^i}\int_0^t \frac{\partial}{\partial \tau}\int_0^{u_\lambda(\tau,x)} \lambda^{1/2} g^N_{\lambda,i}(\tau, \xi)d\xi d\tau dx\right| =$$

$$= \lambda^{-1/2}\left|\int_{\Omega_\lambda^i}\int_{u_0(x)}^{u_\lambda(\tau,x)} \lambda^{1/2} g^N_{\lambda,i}(x, \xi)d\xi dx\right| \leqslant C.$$

Since $v_0 \in W \cap L^\infty(\Omega)$ and β^{-1} is locally Lipschitz, we infer that $u_0 \in W \cap L^\infty(\Omega)$. By virtue of Lemma 2.2, it follows that $u_0 \cap L^\infty(S)$. From this, we deduce that $j(u_0) \in L^1(S)$, and therefore, $u_0 \in D(\varphi)$. Using the well-known inequality (see, e.g., [7, p. 57]) $\varphi_\lambda(u) \leqslant \varphi(u)$, we deduce from (σ), (2.44) and (2.45) that

$$(2.46)\qquad \frac{\alpha_N}{2}\int_0^t |u_\lambda'(\tau)|_H^2\,d\tau+\frac{\lambda}{2}|A_H u_\lambda(t)|_H^2+\frac{1}{2}|\operatorname{grad} u_\lambda(t)|_H^2\leqslant C.$$

Since $(1+\lambda A_H)^{-1}$ is nonexpansive on H, it follows that

$$(2.47)\qquad |y(t)|_H\leqslant C,$$

$$(2.48)\qquad |y'(t)|_H\leqslant |u'(t)|_H.$$

From (2.46), (2.47) and (2.48) we get (2.40) and (2.41).

As $(1+\lambda A_H)^{-1}$ is nonexpansive on $L^q(\Omega)$ for all $q\geqslant 2$, it follows that $\|y_\lambda(t)\|_{L^\infty(\Omega)}\leqslant N$. Noticing that $y_\lambda(t)\in W$ and using Lemma 2.2, we obtain (2.42).

Finally, we shall prove (2.43). Since $-\partial y_\lambda(t)/\partial n\in\sigma(y_\lambda(t))$, it follows from (2.42) and assumption (σ) that

$$(2.49)\qquad \left\|\frac{\partial y_\lambda(t)}{\partial n}\right\|_{L^2(S)}\leqslant C.$$

Thus, from the equality

$$\|A_H y_\lambda(t)\|_{W'}=\sup_{\|\psi\|_W=1}(A_H y_\lambda(t),\psi)=\sup_{\|\psi\|_W=1}\left(\int_\Omega(\operatorname{grad} y_\lambda(t)\cdot\operatorname{grad}\psi\,dx-\int_S\frac{\partial y_\lambda(t)}{\partial n}\psi\,dS\right),$$

we deduce immediately (2.43) as claimed.

3. Proof of theorem 1.1 By virtue of Lemma 2.3, it follows that

$$(3.1)\qquad \{y_\lambda\}\text{ is bounded in } L^\infty(0,T;W),$$

$$(3.2)\qquad \{y_\lambda'\}\text{ is bounded in } L^2(0,T;H),$$

$$(3.3)\qquad |y_\lambda(t)-u_\lambda(t)|_H^2\leqslant C\lambda,$$

$$(3.4)\qquad \{A_\lambda u_\lambda\}=\{A_H y_\lambda\}\text{ is bounded in } L^\infty(0,T;W').$$

By the Arzelà-Ascoli theorem, we obtain that on some subsequence convergent to zero of λ (for simplicity denoted again by λ) we have

$$(3.5)\qquad y_\lambda\to u\ \textit{strongly in}\ C([0,T];H).$$

It is well known that (3.1) (3.4) imply that

$$(3.6)\qquad y_\lambda\to u\ \textit{weak-star in}\ L^\infty(0,T;W),$$

$$(3.7)\qquad A_\lambda u_\lambda=Ay_\lambda\to w\ \textit{weak-star in}\ L^\infty(0,T;W')$$

respectively. From (3.3) and (3.5) it follows that

$$(3.8)\qquad u_\lambda\to u\ \textit{strongly in}\ C([0,T];H).$$

Combining (3.8) and the boundness of v_λ, we infer that (on some subsequence of λ)

$$(3.9)\qquad v_\lambda\to v\ \textit{weakly in}\ L^2(0,T;H),$$

$$(3.10)\qquad v(t)\in Bu(t)\ \text{a.e.}\ t\in]0,T[.$$

Now we shall prove that

$$(3.11)\qquad y_\lambda\to u\ \textit{strongly in}\ C([0,T];L^2(S)).$$

It is well known that the embedding operator from W into $L^2(S)$. is compact. Therefore, from (3.1) it follows that for each $t\in[0,T]$,

$$(3.12)\qquad \{y_\lambda(t)|_s\ \textit{is compact in}\ L^2(S).$$

Using the inequality (see [9, pp. 47—49])

$$(3.13)\qquad \|u\|_{L^2(S)}\leqslant\delta|\operatorname{grad} u|_H+C_\delta|u|_H,$$

where $\delta>0$ is arbitrary small, we deduce

$$(3.14)\qquad \|y_\lambda(t)-y_\lambda(s)\|_{L^2(S)}\leqslant\delta|\operatorname{grad} y_\lambda(t)-\operatorname{grad} y_\lambda(s)|_H+C_\delta|y_\lambda(t)-y_\lambda(s)|_H.$$

By virtue of (2.40), (2.41), from (3.14) it follows that

$$(3.15)\qquad \|y_\lambda(t)-y_\lambda(s)\|_{L^2(S)}\leqslant C(\delta+\sqrt{|t-s|}\,C_\delta)$$

for $\lambda>0$, $t,s\in[0,T]$. This means that $\{y_\lambda\}$ is equicontinuous in $C([0,T];L^2(S))$. Combining this and (3.12) according to Arzelà-Ascoli's theorem, we obtain (3.11).

By virtue of (3.11) we can prove that

$$(3.16)\qquad w(t)\in Au(t)\ \text{a.e.}\ t\in]0,T[.$$

Indeed, if follows from (3.7) that for each $\psi\in L^1(0,T;W)$,

$$(3.17)\qquad \int_0^T(A_\lambda u_\lambda(t),\psi(t))dt\to\int_0^T(w(t),\psi(t))dt,$$

and therefore,

$$(3.18)\qquad \int_0^T(\operatorname{grad} y_\lambda(t),\operatorname{grad}\psi(t))dt+\int_0^T\int_S-\frac{\partial y_\lambda(t)}{\partial n}\psi(t)\,sS\,dt\to\int_0^T(w(t),\psi(t))\,dt.$$

From (3.5), (3.6), we may infer that for $\psi\in L^1(0,T;W)$

$$\int_0^T(y_\lambda(t),\psi(t))dt\to\int_0^T(u(t),\psi(t))dt,$$

$$\int_0^T (y_\lambda(t), \psi(t))\,dt + \int_0^T (\operatorname{grad} y_\lambda(t), \operatorname{grad} \psi(t))\,dt \to$$

$$\to \int_0^T (u(t), \psi(t))\,dt + \int_0^T (\operatorname{grad} u(t), \operatorname{grad} \psi(t))\,dt,$$

and therefore

$$\int_0^T (\operatorname{grad} y_\lambda(t), \operatorname{grad} \psi(t))dt \to \int_0^T (\operatorname{grad} u(t), \operatorname{grad} \psi(t))\,dt. \tag{3.19}$$

Because $-\delta y_\lambda(t)/\delta n \in \sigma(y_\lambda(t))$, and $\{y_\lambda\}$ is bounded on S, from (3.11) it follows that there exists a subsequence of λ (again denoted by λ) such that

$$-\frac{\partial y_\lambda}{\partial n} \to z \textit{ weakly in } L^2(\Sigma_T) \tag{3.20}$$

and

$$z(t) \in \sigma(u(t)) \text{ a.e. } t \in]0, T[\ . \tag{3.21}$$

Combining (3.18), (3.19), (3.20), we obtain

$$\int_0^T (\operatorname{grad} u(t), \operatorname{grad} \psi(t)\,dt + \int_0^T\!\!\int_S z(t)\psi(t)\,dS dt = \int_0^T (w(t), \psi(t))\,dt, \tag{3.22}$$

for each $\psi \in L^1(0, T\,;\, W)$. Since ψ was arbitrary, we have

$$(\operatorname{grad} u(t), \operatorname{grad} \psi) + \int_S z(t)\psi dS = (w(t), \psi), \tag{3.23}$$

for each $\psi \in W$ and a.e. $t \in]0, T[$, i.e. (3—16) holds.

Now we shall prove

$$G_\lambda u_\lambda \to Gu \textit{ weak-star in } L^2(0, T\,;\, W'), \tag{3.24}$$

i.e.

$$\lim_{\lambda\to 0} \int_0^T (G_\lambda u_\lambda(t) - Gu(t), \psi(t))\,dt = 0, \tag{3.25}$$

for each $\psi \in L^2(0, T\,;\, W)$.

Forst, we remind that Gu is well defined because $u \in L^\infty(0, T\,;\, W) \cap L^\infty(Q_T)$, and so $u \in L^\infty(\Sigma_T)$.

It is easy to see that

$$(Gu(t), \psi(t)) = \sum_{i=1}^k \int_{\Omega_\lambda^i} \lambda^{-1/2} g(\bar{x}^i, u(t, \bar{x}^i))\psi(t, \bar{x}^i) n_i(x)\,dx, \tag{3.26}$$

and therefore,

$$|(G_\lambda u_\lambda(t) - Gu(t), \ \psi(t))| = \Big| \sum_{i=1}^k \int_{\Omega_\lambda^i} \lambda^{-1/2} [g(\bar{x}^i, u_\lambda(t, x))\psi(t, x) -$$

$$-\psi(t, \bar{x}^i) g(\bar{x}^i, u(t, \bar{x}^i))]\, n_i(x)\,dx \Big| \leqslant \sum_{i=1}^k C \int_{\Omega_\lambda^i} \lambda^{-1/2} \ [L_N(|u_\lambda(t, x) - y_\lambda(t, x)| + \tag{3.27}$$

$$+ |y_\lambda(t, x) - y_\lambda(t, \bar{x}^i)| + |y_\lambda(t, \bar{x}^i) - u(t, \bar{x}^i)|)|\psi(t, x)| +$$

$$+ |g(\bar{x}^i\, u(t, \bar{x}^i))| . |\psi(t, x) - \psi(t, \bar{x}^i)|] n_i(x) dx.$$

Using the following inequality

$$\|u(x) - u(x^i)\|_{L^2(\Omega_\lambda^i)} \leqslant C\lambda^{1/2} |\operatorname{grad} u|_{L^2(\Omega_\lambda^i)}, \ \forall u \in W, \tag{3.28}$$

it follows from (3.27) that

$$|(G_\lambda u_\lambda(t) - Gu(t), \ \psi(t))| \leqslant C \sum_{i=1}^k [L_N(\lambda^{1/2}|A_H y_\lambda(t)|_H +$$

$$+ \|\operatorname{grad} y_\lambda(t)\|_{L^2(\Omega_\lambda^i)}) \cdot \|\psi(t)\|_{L^2(\Omega_\lambda^i)} + M_N \|\operatorname{grad} \psi\|_{L^2(\Omega_\lambda^i)} (\operatorname{mes} \Omega_\lambda^i)^{1/2}] + \tag{3.29}$$

$$+ C \sum_{i=1}^k \lambda^{-1/4} \|y_\lambda(t) - u(t)\|_{L^2(S_i)} \|\psi(t)\|_{L^2(\Omega_\lambda^i)} .$$

For $u \in W$ and $x \in \Omega_\lambda^i$, it follows by the inequality

$$|u(y)|^2 \leqslant |u(\bar{y}, h_i(\bar{y}))|^2 + 2 \int_{h_i(\bar{y})}^{h_i(\bar{y})+\sqrt{\lambda}} |u(y)| \left| \frac{\partial u(y)}{\partial y_n} \right| dy_m,$$

that

$$\int_{h_i(\bar{y})}^{h_i(\bar{y})+\sqrt{\lambda}} |u(y)|^2 dy_m \leqslant \sqrt{\lambda} \left(|u(\bar{y}, h_i(\bar{y}))|^2 + 2 \int_{h_i(\bar{y})}^{h_i(\bar{y})+\sqrt{\lambda}} |u(y)| \left| \frac{\partial u(y)}{\partial y_m} \right| dy_m \right).$$

Multiplying this inequality by $n_i(T_i^{-1}y)$ and integrating over D_i, we get

$$\|u\|^2_{L^2(\Omega_\lambda^i)} \leqslant C\sqrt{\lambda}(\|u\|^2_{L^2(S_i)} + \|u\|^2_W).$$

Using the inequality $\|u\|_{L^2(S)} \leqslant C\|u\|_W$, it follows that

$$\|u\|_{L^2(\Omega_\lambda^i)} \leqslant C\lambda^{1/4}\|u\|_W, \ \ \forall u \in W. \tag{3.30}$$

By virtue of (3.30) and the obvious inequality

$$\lim_{\lambda\to 0}\int_0^T \|\psi(t)\|_{H^1(\Omega_\lambda^t)}\,dt=0 \tag{3.31}$$

we deduce (3.25) from (3.11) and (3.29).

Letting $\lambda\to 0$ in the equation

$$v_\lambda(t)+\int_0^t (A_\lambda u_\lambda(\tau)-G_\lambda u_\lambda(\tau))d\tau=v_0+\int_0^t f(\tau)d\tau, \tag{3.32}$$

which is equivalent to (2.14)–(2.15), we obtain

$$v(t)+\int_0^t (w(\tau)-Gu(\tau))d\tau=v_0+\int_0^t f(\tau)d\tau, \tag{3.33}$$

where the limit is in the sense of weak-star convergence in $L^2(0, T; W')$.

Let $\Phi\in C^1([0, T])$, $\Phi(T)=0$, $\psi\in W$. Multiplying (3.33) by $(-\Phi_t\psi)$ and integrating over Q_T, we get (1.12) and (1.13). This completes the proof of Theorem 1.1.

Remark. The problem obtained by taking $f(t, x, u)$ instead of $f(t, x)$ in (1.1)–(1.3) has a local (i.e. for T sufficiently small) solution in sense of Theorem 1.1, if $f(t, x, u)$ is measurable as a function of $(t, x)\in Q_T$ for each $u\in R^1$ continuous as a function of $u\in R^1$ for a.e. $(t, x)\in Q_T$, and if, in addition, for each $N>0$, there exists $C_N>0$ such that $|f(t, x, u)|\leqslant C_N$, for a.e. $(t, x)\in Q_T$, and $|u|\leqslant N$. We can prove this conclusion with the method used in [10].

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ON SOME NEW INTEGRODIFFERENTIAL INEQUALITIES

BY

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1. Introduction. In [5], we have obtained several new integrodifferential inequalities involving higher order derivatives. These inequalities are directly useful in studying several properties of the solutions of differential equations, see also [6] where essentially these inequalities are used. The discrete inequalities involving higher order differences share same importance and are discussed in [1]. In this paper we shall give Opial's type inequalities and use these to derive another new class of integrodifferential inequalities. An application of our results is also given.

2. Main Results. Lemma 1. *Let* $y\in C^{(n-1)}[a, b]$ *be such that* $y^{(i)}(a)=0$ *for* $i=0, 1, \ldots, n-1$ *where* $n\geqslant 1$. *Let* $y^{(n-1)}$ *be absolutely continuous and* $\int_a^b |y^{(n)}(s)|^2 ds<\infty$. *Then*

$$\int_a^x |y^{(i)}(s)y^{(n)}(s)|\,ds\leqslant K_i(x-a)^{n-i}\int_a^x |y^{(n)}(s)|^2 ds, \tag{1}$$

where

$$K_i=\frac{1}{2((n-i)!)}\left(\frac{n-i}{2n-2i-1}\right)^{1/2}. \tag{2}$$

Proof. In view of the assumptions on y, for any s such that $a\leqslant s\leqslant x\leqslant b$

$$y^{(i)}(s)=\frac{1}{(n-i-1)!}\int_a^s (s-t)^{n-i-1}y^{(n)}(t)dt.$$

Therefore,

$$|y^{(i)}(s)y^{(n)}(s)|\leqslant\frac{|y^{(n)}(s)|}{(n-i-1)!}\int_a^s (s-t)^{n-i-1}|y^{(n)}(t)|\,dt,$$

and, by Schwarz's inequality,

$$|y^{(i)}(s)y^{(n)}(s)|\leqslant\frac{|y^{(n)}(s)|}{(n-i-1)!}\frac{(s-a)^{(n-i-1/2)}}{(2n-2i-1)^{1/2}}\left(\int_a^s |y^{(n)}(t)|^2dt\right)^{1/2}.$$

Thus, integrating from a to x and applying Schwarz's inequality to the right side again,