By virtue of (3.30) and the obvious inequality

$$\lim_{\lambda\to 0}\int_0^T \|\psi(t)\|_{H^1(\Omega_\lambda^t)}\,dt=0 \tag{3.31}$$

we deduce (3.25) from (3.11) and (3.29).

Letting $\lambda\to 0$ in the equation

$$v_\lambda(t)+\int_0^t (A_\lambda u_\lambda(\tau)-G_\lambda u_\lambda(\tau))d\tau=v_0+\int_0^t f(\tau)d\tau, \tag{3.32}$$

which is equivalent to (2.14)–(2.15), we obtain

$$v(t)+\int_0^t (w(\tau)-Gu(\tau))d\tau=v_0+\int_0^t f(\tau)d\tau, \tag{3.33}$$

where the limit is in the sense of weak-star convergence in $L^2(0, T; W')$.

Let $\Phi\in C^1([0, T])$, $\Phi(T)=0$, $\psi\in W$. Multiplying (3.33) by $(-\Phi_t\psi)$ and integrating over Q_T, we get (1.12) and (1.13). This completes the proof of Theorem 1.1.

Remark. The problem obtained by taking $f(t, x, u)$ instead of $f(t, x)$ in (1.1)–(1.3) has a local (i.e. for T sufficiently small) solution in sense of Theorem 1.1, if $f(t, x, u)$ is measurable as a function of $(t, x)\in Q_T$ for each $u\in R^1$ continuous as a function of $u\in R^1$ for a.e. $(t, x)\in Q_T$, and if, in addition, for each $N>0$, there exists $C_N>0$ such that $|f(t, x, u)|\leqslant C_N$, for a.e. $(t, x)\in Q_T$, and $|u|\leqslant N$. We can prove this conclusion with the method used in [10].

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ON SOME NEW INTEGRODIFFERENTIAL INEQUALITIES

BY

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1. Introduction. In [5], we have obtained several new integrodifferential inequalities involving higher order derivatives. These inequalities are directly useful in studying several properties of the solutions of differential equations, see also [6] where essentially these inequalities are used. The discrete inequalities involving higher order differences share same importance and are discussed in [1]. In this paper we shall give Opial's type inequalities and use these to derive another new class of integrodifferential inequalities. An application of our results is also given.

2. Main Results. Lemma 1. *Let $y\in C^{(n-1)}[a, b]$ be such that $y^{(i)}(a)=0$ for $i=0, 1, \ldots, n-1$ where $n\geqslant 1$. Let $y^{(n-1)}$ be absolutely continuous and $\int_a^b |y^{(n)}(s)|^2\,ds<\infty$. Then*

$$\int_a^x |y^{(i)}(s)y^{(n)}(s)|\,ds\leqslant K_i(x-a)^{n-i}\int_a^x |y^{(n)}(s)|^2\,ds, \tag{1}$$

where

$$K_i=\frac{1}{2((n-i)!)}\left(\frac{n-i}{2n-2i-1}\right)^{1/2}. \tag{2}$$

Proof. In view of the assumptions on y, for any s such that $a\leqslant s\leqslant x\leqslant b$

$$y^{(i)}(s)=\frac{1}{(n-i-1)!}\int_a^s (s-t)^{n-i-1}y^{(n)}(t)dt.$$

Therefore,

$$|y^{(i)}(s)y^{(n)}(s)|\leqslant\frac{|y^{(n)}(s)|}{(n-i-1)!}\int_a^s (s-t)^{n-i-1}\,|y^{(n)}(t)|\,dt,$$

and, by Schwarz's inequality,

$$|y^{(i)}(s)y^{(n)}(s)|\leqslant\frac{|y^{(n)}(s)|}{(n-i-1)!}\,\frac{(s-a)^{(n-i-1/2)}}{(2n-2i-1)^{1/2}}\left(\int_a^s |y^{(n)}(t)|^2dt\right)^{1/2}.$$

Thus, integrating from a to x and applying Schwarz's inequality to the right side again,

$$\int_a^x |y^{(i)}(s)y^{(n)}(s)|\,ds \leqslant \frac{1}{(n-i-1)!\,(2n-2i-1)^{1/2}}\left(\int_a^x (s-a)^{2n-2i-1}\,ds\right)^{1/2} \times$$

$$\times\left(\int_a^x |y^{(n)}(s)|^2\left(\int_a^s |y^{(n)}(t)|^2\,dt\right)ds\right)^{1/2} =$$

$$= \frac{1}{(n-i-1)!\,(2n-2i-1)^{1/2}}\frac{(x-a)^{n-i}}{(2n-2i)^{1/2}}\frac{1}{\sqrt{2}}\int_a^x |y^{(n)}(s)|^2 ds.$$

The above is, in fact, (1). This completes the proof.

Remark. For $n=1$, $i=0$, $x=b$ (1) is Opial's original result [4]. For $n\geqslant 2$, $i=0$, $x=b$ (1) is sharper than Willet's inequality [7] and reduces to the same as obtained in [3], also for the best bound in this case see [2].

Lemma 2. *Let p, q be both positive satisfying $p+q>1$. Let y be as in lemma 1 and let $\int_a^b |y^{(n)}(s)|^{p+q}\,ds<\infty$. Then*

$$(3)\qquad \int_a^x |y^{(i)}(s)|^p |y^{(n)}(s)|^q\,ds \leqslant M_i\,(x-a)^{(n-i)p}\int_a^x |y^{(n)}(s)|^{p+q} ds,$$

where

$$(4)\qquad M_i=\alpha q^{q\alpha}\left[\frac{(n-i)\,(1-\alpha)}{(n-i-\alpha)}\right]^{p(1-\alpha)}((n-i)\,!)^{-p},\quad \alpha=\frac{1}{p+q}.$$

Proof. As in lemma 1, the result follows on using Hölder's inequality instead of Schwarz's inequality two times with proper indices.

Remark. For $n=1$, $i=0$, $x=b$ (3) is a sharper result similar to Yang's [8]. For $n\geqslant 1$, $i=0$, $x=b$ (3) reduces to that obtained in [3].

Theorem 3. *Let y be as in lemma 1 and let $f(x)$, $g(x)$ be nonnegative continuous functions on $[a, b]$, for which the inequality*

$$(5)\qquad |y^{(n)}(x)| \leqslant c+\int_a^x f(s)\,|y^{(n)}(s)|\,ds+\int_a^x g(s)\left(\sum_{i=0}^{n-1}\int_a^s |y^{(i)}(\tau)\,y^{(n)}(\tau)|\,d\tau\right)ds$$

holds, where c is a positive constant. Then

$$(6)\qquad |y^{(n)}(x)| \leqslant c\exp\left(\int_a^x f(s)ds\right)\Big/\left[1-c\int_a^x Q(s)\exp\left(\int_a^s f(\tau)d\tau\right)ds\right]$$

as long as $1-c\int_a^x Q(s)\exp\left(\int_a^s f(\tau)d\tau\right)ds>0$, where $Q(x)=g(x)\sum_{i=0}^{n-1}K_i(x-a)^{n-i+1}$.

Proof. Define $\varphi(x)$ as the right side of (5), then

$$(7)\qquad \varphi'(x)=f(x)\,|y^{(n)}(x)|+g(x)\sum_{i=0}^{n-1}\int_a^x |y^{(i)}(\tau)y^{(n)}(\tau)|\,d\tau,\quad \varphi(0)=c.$$

Using lemma 1, in (7) one obtains

$$(8)\qquad \varphi'(x)\leqslant f(x)\,|y^{(n)}(x)|+g(x)\sum_{i=0}^{n-1}K_i(x-a)^{n-i}\int_a^x |y^{(n)}(\tau)|^2\,d\tau.$$

Since $|y^{(n)}(x)|\leqslant\varphi(x)$ and $\varphi(x)$ is nondecreasing, it follows from (8) that

$$(9)\qquad \varphi'(x)\leqslant f(x)\,\varphi(x)+g(x)\sum_{i=0}^{n-1}K_i(x-a)^{n-i+1}\varphi^2(x).$$

Since $\varphi(x)>0$, (9) can be rewritten as

$$(10)\qquad \frac{d}{dx}\left[-\frac{1}{\varphi(x)}\exp\left(\int_a^x f(s)ds\right)\right]\leqslant Q(x)\exp\left(\int_a^x f(s)\,ds\right).$$

Integrating (10) from a to x, we find

$$-\frac{1}{\varphi(x)}\exp\left(\int_a^x f(s)ds\right)+\frac{1}{c}\leqslant\int_a^x Q(s)\exp\left(\int_a^s f(\tau)d\tau\right)ds.$$

The above produces an estimate for $\varphi(x)$ which is same as right side of (6). Now the result follows since $|y^{(n)}(x)|\leqslant\varphi(x)$.

Theorem 4. *Let p, q, y be as in lemma 2 and $f(x)$, $g(x)$ and c be as in theorem 3, for which the inequality*

$$(11)\qquad |y^{(n)}(x)|\leqslant c+\int_a^x f(s)\,|y^{(n)}(s)|\,ds+\int_a^x g(s)\left(\sum_{i=0}^{n-1}\int_a^s |y^{(i)}(\tau)|^p|y^{(n)}(\tau)|^q\,d\tau\right)ds$$

holds. Then

$$(12)\qquad |y^{(n)}(x)|\leqslant\frac{c\exp\left(\int_a^x f(s)ds\right)}{\left[1-c^{p+q-1}(p+q-1)\int_a^x R(s)\exp\left((p+q-1)\int_a^s f(\tau)d\tau\right)ds\right]^{1/p+q-1}}$$

as long as $1-c^{p+q-1}(p+q-1)\int_a^x R(s)\exp\left((p+q-1)\int_a^s f(\tau)d\tau\right)ds>0$,

where $$R(x)=g(x)\sum_{i=0}^{n-1}M_i(x-a)^{(n-i)p+1}.$$

Proof. The proof is similar to that of theorem 3.

Remark. For $p=q=1$, lemma 2 reduces to lemma 1 and theorem 4 is the same as theorem 3.

3. An Application. Here we shall give an application of our theorem 3, to provide upper estimate of the solutions of the nonlinear integrodifferential equation

$$(13) \qquad y^{(n+1)}(x)=F\left(x, y^{(n)}(x), \int_a^x h(x, s, y(s), y'(s), \ldots, y^{(n-1)}(s))y^{(n)}(s)ds\right)$$

satisfying the initial conditions

$$(14) \qquad y^{(i)}(a)=0,\ 0\leqslant i\leqslant n-1,\ y^{(n)}(a)=\alpha(\neq 0).$$

where F is continuous in its arguments, and

$$(15) \qquad |F(x, u, v)|\leqslant p(x)\,|u|+q(x)\,|v|$$

$$(16) \qquad |h(x, s, y(s), y'(s), \ldots, y^{(n-1)}(s))|\leqslant L\sum_{i=0}^{n-1}|y^{(i)}(s)|.$$

The functions $p(x)$ and $q(x)$ are continuous and nonnegative, L is a nonnegative constant. Any solution $y(x)$ of (13), (14) satisfies

$$(17) \qquad y^{(n)}(x)=\alpha+\int_a^x F\left(s, y^{(n)}(s), \int_a^s h(s, \tau, y(\tau), y'(\tau), \ldots, y^{(n-1)}(\tau))\times y^{(n)}(\tau)d\tau\right) ds.$$

Using (15) and (16) in (17), it follows that

$$(18) \qquad |y^{(n)}(x)|\leqslant|\alpha|+\int_a^x p(s)\,|y^{(n)}(s)|\,ds+L\int_a^x q(s)\left(\int_a^s\sum_{i=0}^{n-1}|y^{(i)}(\tau)y^{(n)}(\tau)|\,d\tau\right) ds.$$

Inequality (18) is similar to (5) with $c=|\alpha|$, $f(x)=p(x)$, $g(x)=Lq(x)$, thus theorem 3 provides an upper estimate for $|y^{(n)}(x)|$, in terms of the known functions. The estimates for $|y^{(i)}(x)|$, $0\leqslant i\leqslant n-1$ can now be obtained by direct integration and making use of (14).

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ASYMPTOTIC DOSING PROBLEM FOR EVOLUTION EQUATIONS IN HILBERT SPACES

BY

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0. Introduction. We propose studying the so-called asymptotic dosing problem for first-order abstract differential equations of monotone type in Hilbert spaces.

The idea of writing this paper originates from a paper of Turinici [6], who considered the asymptotic dosing problem for the finite-dimensional case. However, both our assumptions and the methods we use here are completely different from those of [6].

The main result we state in this paper, Theorem 1 below, relies on the well-known Opial's lemma and on a technique similar to that developed by Baillon and Haraux [1].

The last part of the paper (Section 2) is devoted to the study of existence of solutions to abstract evolution equations which include a measure as an inhomogeneous term. This subject is closely related to the dosing problem.

1. Asymptotic dosing. Let H be a real Hilbert space with the inner product and the associated norm denoted by $(.,.)$ and $\|\,.\,\|$, respectively. Let A be a maximal monotone operator from H into itself, whose domain and range are denoted as usual by $D(A)$ and $R(A)$, respectively. Consider the sequence of Cauchy problems (P_n):

$$(P_0) \qquad \begin{cases} \dfrac{du_0}{dt}(t)+Au_0(t)\ni f(t), & 0<t<T,\\ u_0(0)=x, \end{cases}$$

$$(P_n) \qquad \begin{cases} \dfrac{du_n}{dt}(t)+Au_n(t)\ni f(t), & nT<t<(n+1)T,\\ u_n(nT)=u_{n-1}(nT)+d_n, \end{cases}$$

for $n=1, 2, \ldots,$

where $T>0$ is fixed; $x\in\overline{D(A)}$ (the closure of $D(A)$); (d_n) is a given sequence in H;

(A_1): $f\in L^2_{\mathrm{loc}}(0, \infty; H)$ and f is T-periodic.

By „d/dt" we mean the ordinary derivative with respect to t.

As a first remark, the sequence (P_n) is well-defined only if each of the initial data of problems (P_n) belongs to $\overline{D(A)}$. The assumption