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BRIOT-BOUQUET DIFFERENTIAL EQUATIONS WITH ANALYTICAL SOLUTIONS IN BANACH SPACES

BY

ION ONCIULESCU

In this paper, the existence of analytical Michal-Martin type solutions [2] of Briot-Bouqet differential equations with Fréchet derivatives, is discussed. These equations are generalizations of equations studied by L. Reich [3] in Banach spaces. A majorant method has been used : formal solutions for the given equations have been determined, and these solutions are majored by analytical solutions of corresponding scalar differential equations.

1. Homogeneous Polynomials an Analytical Functions in Banach Spaces. Definition 1. [2] *Let X be a linear normed space and Y a Banach space, defined over the same field of scalars Γ. A function $P_n(x)$ defined on X having values in Y is called a homogeneous polynomial of n-th degree if there exists an n-linear function $h_n(x_1,\dots, x_n)$, defined on X^n with values in Y, symmetrical in $x_1, x_2,\dots, x_n$ such that $P_n(x) = h_n(x,\dots, x)$ for any $x \in X$. $h_n(x_1,\dots, x_n)$ is called a „polar form" of the homogeneous polynomial $P_n(x)$.*

Proposition 1. [2] *The polar form $h_n(x_1,\dots, x_n)$ of a homogeneous polynomial $P_n(x)$ of the n-th degree is unique. If $P_n(x)$ is continuous in the origin of X, then $h_n(x_1,\dots, x_n)$ is continuous in the origin of X^n.*

Remarks 1. A homogeneous polynomial $P_n(x)$ is continuous on X if and only if its polar form $h_n(x_1,\dots, x_n)$ is continuous on X^n.

2. A specific norm can be defined in the case of continuous and homogeneous n-th degre polynomials, namely : $\|P_n\| = \inf\{\mu\}$, such that $\|h_n(x,\dots,x)\| \leqslant \mu\|x\|^n$, $\forall x \in X$. Generally the following relationship between $\|P_n\|$ and $\|h_n\|$: $1 \leqslant \dfrac{\|h_n\|}{\|P_n\|} \leqslant \dfrac{n^n}{n!}$, holds.

If the field of scalars is C (the field of complex numbers) then $\|h_n\| = \|P_n\|$, [2].

In the following, we shall denote both the polynomial and its polar form by the same letter. The norm of a polynomial to be used is the norm of the polar form of the polynomial.

3. Let $f(x_1,\dots, x_n)$ be an n-linear function defined on X and having values in Y. Then $P_n(x) = f(x,\dots, x)$ is a homogeneous polynomial of n-th degree whose polar form is :

$$h(x_1, \ldots, x_n) = \frac{1}{n!} \sum_{(i_1, i_2, \ldots, i_n)} f(x_{i_1}, \ldots, x_{i_n}),$$

$(i_1, \ldots, i_n)$ being a permutation of the set $\{1, 2, \ldots, n\}$.

4. Let $f(x_1, \ldots, x_s, u)$ be a function defined on $X^s \times Y$, with values in Y, $s+1$-linear and symmetrical with respect to the variables $x_1, \ldots, x_s$. If $P_k(x)$ is a homogeneous polynomial of the k-th degree, defined on X with values in Y, then $h_{k+s} = f(x, \ldots, x, P_k(x)) = f(x^s, P(x^k))$ is a homogeneous polynomial of the $k+s$-th degree and

$$h_{k+s}(x_1, x_2, \ldots, x_s) =$$

$$= \frac{s!\,k!}{(k+s)!} \sum_{(i_1, \ldots, i_s, i_{s+1}, \ldots, i_{k+s})} f(x_{i_1}, \ldots, x_{i_s}, P(x_{i_{s+1}}, \ldots, x_{i_{k+s}})),$$

where $\{i_1, \ldots, i_s\}$, $\{i_{s+1}, \ldots, i_{k+s}\}$ are subsets of the set $\{1, 2, \ldots, k+s\}$.

5. Let $y_0 \in Y$ and $l(x)$ be a linear functional defined on X. Then $P_n(x) = [l(x)]^n y_0$ is a continuous polynomial of the n-th degree. We have the following equality $P_n(x_1, \ldots, x_n) = l(x_1)\, l(x_2) \ldots l(x_n) y_0$.

Proposition 2. *Let $f(x_1, \ldots, x_s, y)$ be a $s+1$-linear function, defined on $X^s \times Y$, continuous, symmetrical with respect to $x_1, x_2, \ldots, x_s$. Consider the equation*

$$f(x^s, y_k(x^{k-s}, x_0^s)) = A y_k(x) + P_k(x^k), \tag{1}$$

where :

a) $x_0 \in X$, $x_0 \neq \theta$,

b) *P_k is a continuous polynomial of k-th degree,*

c) *$A : Y \to Y$ is a continuous linear operator, having $A^{-1} : Y \to Y$,*

d) *$A_0 : Y \to Y$, $A_0 y = f(x_0^s, y)$, has an inverse $A_0^{-1} : Y \to Y$ and*

$$\frac{k!}{(k-s)!\,s!} \|A_0^{-1}\| \, \|A\| < 1.$$

Then, the equation (1) *has one and only one solution y_k in the set of continuous, homogeneous polynomials of k-th degree.*

Proof. For simplicity, we shall consider $s=1$. From the proposition 1 and the remark 4, we conclude that the equation (1) is equivalent to

$$\begin{gathered}\frac{1}{k} f(x_1, y_k(x_2, \ldots, x_k, x_0)) + f(x_2, y_k(x_1, x_3, \ldots, x_k, x_0)) + \\ + \ldots + f(x_k, y_k(x_1, \ldots, x_{k-1}, x_0)) = A y_k(x_1, x_2, \ldots, x_k) + P_k(x_1, \ldots, x_k), \\ \forall (x_1, x_2, \ldots, x_k) \in X^k.\end{gathered} \tag{1'}$$

Denote $y_k(x_0^k) = v_0$, $y_k(x_1, x_0^{k-1}) = v_1(x_1), \ldots, y_k(x_1, \ldots, x_{k-1}) = v_{k-1}(x_1, \ldots, x_{k-1})$. The equation (1′) is then equivalent to the system

$$\left\{\begin{aligned} & f(x_0, v_0) = A v_0 + P_k(x_0^k), \\ & \frac{k-1}{k} f(x_0, v_1(x_1)) + \frac{1}{k} f(x_1, v_0) = A v_1(x_1) + P_k(x_1, x_0^{k-1}), \\ & \frac{k-2}{k} f(x_0, v_2(x_1, x_2)) + \frac{1}{k} [f(x_1, v_1(x_2)) + f(x_2, v_1(x_1))] = \\ & \qquad = A v_2(x_1, x_2) + P_k(x_1, x_2, x_0^{k-2}), \\ & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ & \frac{1}{k} [f(x_1, v_{k-1}(x_2, \ldots, x_k)) + \ldots + f(x_k, v_{k-1}(x_1, \ldots, x_{k-1}))] = A y_k(x_1, \ldots, x_k) + \\ & \qquad + P_k(x_1, \ldots, x_k). \end{aligned}\right. \tag{2}$$

Indeed, taking in (1′) successively $x_1 = x_2 = \ldots = x_k = x_0$, then $x_2 = x_3 = \ldots = x_k = x_0, \ldots$, then $x_k = x_0$, we obtain the system (2).

Conversely, if $(v_0, v_1, \ldots, v_{k-1}, y_k)$ is a solution of the system (2), we have the following equalities $v_1(x_0) = v_0$, $v_2(x_1, x_0) = v_1(x_1), \ldots, v_k(x_1, x_2, \ldots, x_{k-1}, x_0) = v_{k-1}(x_1, x_2, \ldots, x_{k-1})$. Obviously, taking $x_1 = x_0$ in the second equation from (2) and substracting from it the first equation, we obtain $\frac{k-1}{k} [f(x_0, v_1(x_0)) - f(x_0, v_0)] = A(v_1(x_0) - v_0)$, whence, under the assumptions of the proposition, we get first $v_1(x_0) - v_0 = \frac{k}{k-1} A_0^{-1} A (v_1(x_0) - v_0)$ and then $v_1(x_0) - v_0 = \theta$.

In order to obtain the equality $v_2(x_1, x_0) = v_1(x_1)$, we substitute in the third equation (2), $x_2 = x_0$ and we apply a similar method. The fact that $v_1(x_0) = v_0$ should also be kept in view. Thus, step by step we conclude the whole statement.

If we substitute in the last equation (2)

$$v_{k-1}(x_2, \ldots, x_k) = y_k(x_2, \ldots, x_k, x_0), \ldots, v_{k-1}(x_1, \ldots, x_{k-1}) = y_k(x_1, \ldots, x_{k-1}, x_0)$$

we obtain (1′) and the proof of the statement (1′) $\Leftrightarrow$ (2) is now complete. By the assumptions, (c) (d) the system (2) has a unique solution. This completes the proof of the proposition 2.

Definition 2. *Let $X_1, X_2, \ldots, X_k$ be normed linear spaces and Y a Banach space over the field of scalars. Consider the spheres $B_i = \{x_i \mid x_i \in X_i, \|x_i\| < r_i\}$, $i = \overline{1, k}$, and let $f(x_1, \ldots, x_k)$ be a function defined on $B = B_1 \times B_2 \times \ldots \times B_k$, having values in Y. The function f is called analytical on B if the following equality holds*

(1) $f(x_1, x_2, \ldots, x_k^p) = \sum\limits_{i_1, \ldots, i_k = \overline{0, \infty}} p_{i_1 \ldots i_k}(x_1, x_2, \ldots, x_k)$, *in which $p_{i_1 \ldots i_k}$ is a continuous homogeneous polynomial of i_1-th degree in $x_1, \ldots$, of the i_k-th degree in x_k and such that*

$$\sum_{i_1, \ldots, i_k = \overline{0, \infty}} \|p_{i_1 \ldots i_k}\| \lambda_1^{i_1} \lambda_2^{i_2} \ldots \lambda_k^{i_k} < \infty \text{ for } \lambda_i \in \Gamma, \ |\lambda_i| < r_i, \quad i = \overline{1, k},$$

Remarks. 1. An analytical function $f(x_1, x_2, ..., x_k) = \sum_{i_1,...,i_k=0,\infty} p_{i_1 ... i_k}(x_1,..., x_k)$ is Fréchet differentiable any number of times in the convergence domain B and we have:

$$p_{i_1 ... i_k}(x_1,..., x_k) = \frac{\partial^{i_1+ ... +i_k} f(\theta; x_1, x_2, ..., x_k)}{\partial^{i_1} x_1 ... \partial^{i_k} x_k}.$$

Hence it follows that if $f^i(x_1,..., x_k)$ $(i = 1,2)$ are analytical in the neighbourhoods $B_i (i = 1,2)$ of the origin of $X_1 \times X_2 \times ... \times X_k$,

$$f^i = \sum_{i_1,...,i_k=0,\infty} p^i_{i_1 ... i_k}(x_1,..., x_k)$$

$(i = 1,2)$ and $f^1(x_1,..., x_k) = f^2(x_1,..., x_k)$ in a neighbourhood $B_3 \subset B_1 \cap B_2$, then $p^1_{i_1 ... i_k}(x_1, ..., x_k) = p^2_{i_1 ... i_k}(x_1,..., x_k)$.

2. If: a) $u_i(x_1,..., x_k) (i = \overline{1, p})$ are analytical in a neighbourhood of the origin of $X_1 \times X_2 \times ... \times X_k$, having values in $Y_i (i = \overline{1, p})$ and $u_i(0) = 0$,

b) $f(u_1,..., u_p)$ is analytical in a neighbourhood of the origin of $Y_1 \times Y_2 \times ... \times Y_p$, having values in Y, then there exists $h(x_1,..., x_k) = f(u_1(x_1,..., x_k),..., u_p(x,..., x_k))$, analytical in a neighbourhood of the origin of $X_1 \times X_2 \times ... \times X_k$.

3. Taking in (*I*) $p_{i_1 ... i_k}(x_1,..., x_k) = l_1(x_1) ... l_k(x_k) y_{i_1 ... i_k}$ where $l_i(x_i)$ $(i = \overline{1,k})$ are linear functionals continuous on X_i, and $y_{i_1 ... i_k} \in Y$, f is an analytical function in a certain neighbourhood of the origin of $X_1 \times X_2 \times ... \times X_k$.

2. A theorem of uniqueness for a Briot-Bouquet differential equation in Banach spaces. Let X be a normed linear space and Y a Banach space defined on the same field of scalars Γ.

Consider the equation

$$f\left(x^s, \frac{d^s y(x_1 x_0)}{dx^s}\right) = Ay(x) + B_n(x^n) + P(x, y), \tag{3}$$

where:

a) s, n are integers $\geqslant 1$, $s \leqslant n$, $x_0 \in X$, $x_0 \neq \theta_i$

b) $f(x_1, x_2,..., x_s, u)$ is a $s+1$-linear function, continuous, defined on $X^s \times Y$, with values in Y, symmetrical with respect to the variables x_i,

c) A is a continuous linear operator, defined on Y, with values in Y,

d) $B_n(x^n) = B_n(x, x,..., x)$ is a homogeneous continuous polynomial of n-th degree, defined on X with values in Y.

e) $P(x, y)$ is a function analytical in a neighbourhood of the origin of $X \times Y$, having the form:

$$P(x, y) = \sum_{\substack{i,j=0,\infty \\ i+j \geqslant n}} p_{ij}(x^i, y^j).$$

We shall use the following notation:

$A_0 u = f(x_0, u)$, $u \in Y$, and assume that

(i_1) A_0 and A are invertible, with $A_0^{-1}: Y \to Y$ and $A^{-1}: Y \to Y$,

$$\frac{1}{s!} \|A^{-1}\| \, \|A_0\| < 1. \tag{i_2}$$

Then we have

Theorem 1. *Under the assumptions (i_1) and (i_2), the equation (3) can have only one solution analytical of the form*

$$y(x) = y_n(x) + y_{n+1}(x) + ... + y_{n+k}(x) + ... \tag{4}$$

where $y_{n+k} (k = \overline{0, \infty})$ are continuous polynomials of $n+k$-th degree.

Remark. The equation (3) represents a generalization in Banach spaces of differential equations having the following form:

$$z^s \frac{d^s w}{dz^s} = aw + bz^n + Q(z, w), \tag{*}$$

where $z \in C$, $a, b \in C$ and $Q(z, w)$ is a power series in z and w, convergent in a neighbourhood of the origin of $C \times C$, such that $\operatorname{ord} Q(z, z^n) > n$, s and n being integers $\geqslant 1$, $s \leqslant n$.

In case $a \neq n \cdot s$, $n \cdot s + 1$,.... then the equation (*) has a unique analytical solution of the form

$$w(z) = c_n z^n + c_{n+1} z^{n+1} + ... + c_{n+k} z^{n+k} + ... [3, \text{th. } 1]$$

Proof of the theorem 2. If:

$$y(x) = y_n(x) + y_{n+1}(x) + ... + y_{n+k}(x) + ... \tag{4}$$

is an analytical solution of the equation (3), then we have

$$\frac{d^s y(x, x_0)}{dx^s} = \frac{n!}{(n-s)!} y_n(x^{n-s}, x_0^s) + \frac{(n+1)!}{(n+1-s)!} y_{n+1}(x^{n+1-s}, x_0^s) + . \tag{5}$$

Substituting $y(x)$ from (4) in (3). taking into account (5), it follows

$$\begin{cases} \frac{n!}{(n-s)!} f(x^s, y_n(x^{n-s}, x_0^s)) = Ay_n(x^n) + B_n(x^n), \\ \frac{(n+1)!}{(n+1-s)!} f(x^s, y_{n+1}(x^{n+1-s}, x_0^s)) = Ay_{n+1}(x^{n+1}) + \\ + p_{n+1,0}(x) + p_{11}(x, y_n(x)), \\ \frac{(n+2)!}{(n+2-s)!} f(x^s, y_{n+2}(x^{n+2-s}, x_0^s)) = Ay_{n+2}(x^{n+2}) + \\ + p_{n+2,0}(x) + p_{11}(x, y_{n+1}) p_{21}(x, y_n(x)), \\ \cdots\cdots\cdots\cdots\cdots\cdots \end{cases} \tag{6}$$

Applyng successively proposition (2) to the equations of the system (6), we find that the system has a uniquely determined solution.

Remark. If $P(x, y) = p_{n+1,0}(x) + ... + p_{n+k,0}(x)$. then the equation (3) has a unique solution of the form

$$y(x) = y_n(x) + y_{n+1}(x) + \ldots + y_{n+k}(x)$$

3. Briot-Bouquet Equations with Analytical Solutions in Banach Spaces. Let $l(x)$ be a continuous linear functional defined on X, such that $l(x_0) \neq 0$. We shall study equations of the form

(3′) $$[l(x)]^s \frac{d^s y(x,x_0)}{dx^s} = Ay + B_n(x^n) + P(x,y), \text{ where } B_n(x^n) = [l(x)]^n b_n, b_n \in Y,$$

$P(x, y) = \sum_{\substack{i,j=0,\infty \\ i+nj\,n}} [l(x)]^i p_{ij}(y)$, $p_{ij}(y)$ being continuous homogeneous polynomials of j-th degree, defined on Y with values in Y, such that $\sum \|p_{ij}\|\lambda^i\mu^j$ for $\lambda, \mu \in \Gamma$, $|\lambda| < r_1$, $|\mu| < r_2$, and $A : Y \to Y$ is a continuous linear operator.

Theorem 2. *If*

(i$_3$) $\frac{(n-s)!}{n!} \cdot \frac{1}{|l(x_0)|^s} \| A \| < 1$, *equation* (3′) *has only one analytical solution* $y(x)$ *of the form*

(4′) $$y(x) = [l(x)]^n c_n + [l(x)]^{n+1} c_{n+1} + \ldots, \text{ with } c_{n+k}(k = \overline{0, \infty}) \in Y.$$

Moreover, if the operator A *is invertible with* $A^{-1} : Y \to Y$, *and*

(i$_2'$) $\frac{1}{s!} \cdot \frac{1}{|l(x_0)|^s} \|A\| < 1$, *then* $y(x)$ *from* (4) *is the only solution* $y(x)$ *of the form* (4).

Proof. In this case we may determine a unique solution $y_n(x), y_{n+1}(x), \ldots$ for the system (6) with $y_{n+k}(x)$ of the form $y_{n+k}(x) = l^{n+k}(x) c_{n+k}$, $c_{n+k} \in Y$. For this purpose it is sufficient to detemine $c_{n+k} \in Y$ such that :

(6′) $$\begin{aligned} &\frac{n!}{(n-s)!} l^s(x_0) c_n = Ac_n + b_n, \\ &\frac{(n+1)!}{(n+1-s)} l^s(x_0) c_{n+1} = Ac_{n+1} + p_{n+1,0} + p_{11}(c_n), \\ &\frac{(n+2)!}{(n+2-s)} l^s(x_0) c_{n+2} = Ac_{n+2} + p_{n+2,0} + p_{11}(c_{n+1}) + p_{21}(c_n), \\ &\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}$$

By assumption (i$_3$), the system (6′) has a unique solution. Moreover, if there exists $A^{-1} : Y \to Y$ and (i$_2'$), then we can apply theorem 1. Thus it follows that (4′) determined by the system (6′) is the unique formal solution of the equation (3) having the form (4).

In order to prove the convergence of the solution (4′) obtained by solving the system (6′) we notice first that (6′) leads to :

(7) $$\begin{aligned} &\|c_n\| \leqslant \frac{(n-s)!}{n!} \frac{1}{|l(x_0)|^s} [\|A\| \|c_n\| + \|b_n\|], \\ &\|c_{n+1}\| \leqslant \frac{(n+1-s)!}{(n+1)!} \frac{1}{|l(x_0)|^s} [\|A\| \|c_{n+1}\| + \|p_{n+1,0}\| + \|p_{11}\| \|c_n\|], \\ &\|c_{n+2}\| \leqslant \frac{(n+2-s)!}{(n+2)!} \frac{1}{|l(x_0)|^s} [\|A\| \|c_{n+2}\| + \|p_{n+2,0}\| + \|p_{11}\| \|c_{n+1}\| + \\ &+ \|p_{21}\| \|c_n\|], \\ &\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}$$

Consider now the scalar equation

(3*) $$z^s \frac{d^s w}{dz^s} = \frac{1}{|l(x_0)|^s} [\|A\| w + \|b_n\| z^n + P^*(z, w)],$$

where $P^*(z, w) = \sum_{\substack{i,j=\overline{0,\infty} \\ i+nj>n}} \|p_{ij}\| z^i w^j$

The equation (3*) has a unique analytical solution $w(z)$ of the form

(4*) $$w(z) = c_n^* z^n + c_{n+1}^* z^{n+1} + \ldots$$

We obtain $c_{n+k}^* (k = \overline{0, \infty})$ from the system

(8) $$\begin{cases} c_n^* = \frac{(n-s)!}{n!} \frac{1}{|l(x_0)|^s} [\|A\| c_n^* + \|b_n\|], \\ c_{n+1}^* = \frac{(n+1-s)!}{(n+1)!} \frac{1}{|l(x_0)|^s} [\|A\| c_{n+1}^* + |p_{n+1,0}\| + \|p_{11}\| c_n^*, \\ c_{n+2}^* = \frac{(n+2-s)!}{(n+2)!} \frac{1}{|l(x_0)|^s} [\|A\| c_{n+2}^* + \|p_{n+2,0}\| + \|p_{11}\| c_{n+1}^* + \|p_{21}\| c_n^*, \\ \ldots\ldots\ldots\ldots\ldots\ldots \end{cases}$$

From (7) and (8), by assumption (i$_3$), we get $\|c_{n+k}\| \leqslant c_{n+k}^*$. This completes the proof of the theorem.

4. Briot-Bouquet Equations with a continuous linear functional which generalize the equation (3′) from 3. Theorem 3. *Let* m, n, r, s *be integers,* $m \geqslant 1$, $n \geqslant 1$, $s \geqslant 1$, $r \geqslant 0$, $s \leqslant n$.

Consider the differential equation

(9) $$\begin{aligned} &[l(x)]_{:}^{m-1)n+r+s} \frac{d^s y((x, x_0)}{dx^s} = [l(x)]^r p_m(y^m) + \\ &+ [l(x)]^{r+n} p_{m-1}(y^{m-1}) + [l(x)]^{r+2n} p_{m-2}(y^{m-2}) + \ldots + \\ &+ [l(x)]^{s+(m-1)n} p_1(y) + [l(x)]^{r+mn} p_0 + P(x, y), \end{aligned}$$

where :

(i$_1$) $x_0 \in X$, $x_0 \neq \theta$, *and* $l(x)$ *is a linear functional, continuous on* X, *and* $l(x_0) \neq \theta$,

(i_2) *$p_i(i=\overline{0,m})$ are continuous homogeneous polynomials of the i-th degree, defined over the space Y and with values in Y,*

(i_3) *$P(x, y)$ is analytical in a neighbourhood of the origin of $X \times Y$, having the form*

$$P(x, y) = \sum_{\substack{i,j=\overline{0,\infty} \\ i+nj \geq mn+r}} [l(x)]^i p_{ij}(y).$$

(i_4) *Suppose that there exists a vector $c_n \in Y$, a solution of the equation*

$$[l(x_0)]^s c_n - p_m(c_n) - p_{m-1}(c_n) - \ldots - p_1(c_n) - p_0 = 0, \tag{10}$$

such that

$$\frac{(n+1-s)!}{(n+1)!\,|l(x_0)|^s} (m\|p_m\|\,\|c_n\|^{m-1} + (m-1)\|p_{m-1}\|\,\|c_n\|^{m-2} + \ldots + \|p_1\|) < 1. \tag{11}$$

Then the equation (9) has an analytical solution of the form

$$y(x) = [l(x)]^n c_n + [l(x)]^{n+1} c_{n+1} + \ldots \tag{4'}$$

Remarks 1) This theorem is a generalization of theorem 1 from [3] over Banach spaces.

2) For $m=1$, $r=0$, the equation (9) becomes the equation (3') from 3.

Proof of the theorem 3. The proof of this theorem is similar to that of theorem 2. Substituting (4) in (9) and equalizing the coefficients which we obtain for $[l(x)]^{mn+r+k}(k=0, 1, 2,\ldots)$ in both sides we get for $c_n, c_{n+1},\ldots, c_{n+k},\ldots$, the system

$$\begin{cases} \dfrac{n!}{(n-s)!}[l(x_0)]^s c_n = p_m(c_n^m) + p_{m-1}(c_n^{m-1}) + \ldots + p_0, \\ \dfrac{(n+1)!}{(n+1-s)!}[l(x_0)]^s c_{n+1} = m p_m(c_n^{m-1}, c_{n+1}) + (m-1)p_{m-1}(c_n^{m-2}, c_{n+1}) + \\ \quad + p_1(c_{n+1}) + F_1(c_n, a), \\ \quad \ldots\ldots\ldots\ldots\ldots\ldots \\ \dfrac{(n+k)!}{(n+k-s)!}[l(x_0)]^s c_{n+k} = m p_m(c_n^{m-1}, c_{n+k}) + \ldots + p_1(c_{n+k}) + \\ \quad + F_k(c_n, \ldots, c_{n+k-1}\,;\, p_1, \ldots p_m), \\ \quad \ldots\ldots\ldots\ldots\ldots\ldots \end{cases} \tag{12}$$

$F_k(c_n, c_{n+1},\ldots, c_{n+k-1}\,;\, a\,;\, p_1,\ldots, p_{m-1})$ are finite sums of terms which are values of the polar forms of p_{ij} from the development of the function $P(x, y)$, in some vectors $c_n, c_{n+1},\ldots, c_{n+k-1}$, as well as values of the polar forms of the polynomials $p_1,\ldots, p_{m-1}$, into the same $c_n,\ldots, c_{n+k-1}$. The polynomials p_{ij} from the development of $P(x, y)$, involved in the determination of c_{n+k} are denoted by „a".

By the assumption (i_4), the system (12) has a solution $c_{n+1}, c_{n+2},\ldots$, uniquely determined, corresponding to a solution c_n of the equation (10).

To prove the convergence of the solution (4'), obtained by the system (12), we shall form again a scalar equation which has an analytical solution, majorant of the series $\sum_{k=\overline{0,\infty}} \|c_{n+k}\|\lambda^{n+k}$.

From the system (12) we obtain

$$\begin{aligned} &\|c_n\| \leqslant \frac{(n-s)!}{n!}\frac{1}{[l(x_0)]^s}(\|p_m\|\|c_n\|^m + \|p_{m-1}\|\,\|c_n\|^{m-1} + \ldots + \|p_0\|), \\ &\|c_{n+1}\| \leqslant \frac{(n+1-s)!}{(n+1)!}\frac{1}{|l(x_0)|^s}[(m\|p_m\|\,\|c_n\|^{m-1} + \ldots + \|p_1\|)\,\|c_{n+1}\| + \\ &+ F_1^*(\|c_n\|, a^*)], \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ &\|c_{n+k}\| \leqslant \frac{(n+k-s)!}{(n+k)!}\frac{1}{|l(x_0)|^s}[m\|p_m\|\,\|c_n\|^{m-1} + \ldots + \|p_1\|)\|c_{n+k}\| + \\ &+ F_k^*(\|c_n\|, \ldots, \|c_{n+k-1}\|;\, a^*, \|p_1\|, \ldots, \|p_m\|), \\ &\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned} \tag{13}$$

F_k^* in (13) is obtained from F_k, by substituting for the terms of this sum, the products of the norm of the respective polynomials and the norms of the corresponding elements $c_n, c_{n+1},\ldots$. Thus, we substitute $\|p_{11}\|\,\|c_{n+k-1}$ for $p_{11}(c_{n+k-1})$. Now consider:

$$\sigma_0 = \frac{1}{|l(x_0)|^s}(\|p_m\|\,\|c_n\|^m + \|p_{m-1}\|\,\|c_n\|^{m-1} + \ldots + \|p_0\|) - \frac{n!}{(n-s)!}\,\|c_n\|. \tag{14}$$

Then, the scalar equation

$$\begin{aligned} z^{(m-1)n+r+s}\frac{d^s w}{dz^s} &= \frac{1}{|l(x_0)|^s}(z^r\|p_m\|w^m + z^{r+n}\|p_{m-1}\|w^{m-1} + \ldots + \\ &+ z^{(m-1)n+r}\|p_1\| + \frac{1}{|l(x_0)|^s}\|p_0\| - \sigma_0)\, z^{mn+r} + P^*(z, w), \end{aligned} \tag{9'}$$

where

$$P^*(z, w) = \frac{1}{|l(x)|^s}\sum_{i,j}\|p_{ij}\|z^i w^j$$

and p_{ij} belong to the expression of $P(x, y)$, has an analytical solution $w(z) = c_n^* z^n + c_{n+1}^* z^{n+1} + \ldots + c_{n+k}^* z^{n+k} + \ldots$ with $c_n^* = \|c_n\|$ (on account of the way we chose σ_0) and $c_{n+1},\ldots, c_{n+k},\ldots$ satisfying the system

$$\begin{aligned} &c_{n+1} = \frac{(n+1-s)!}{(n+1)!}\frac{1}{|l(x_0)|^s}[(m\|p_m\|c_n^{*m-1} + \ldots + \|p_1\|)c_{n+1}^* + F_1^*(c_n^*, a^*)] \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ &c_{n+k} = \frac{(n+k-s)!}{(n+k)!}\frac{1}{|l(x_0)^s|}[(m\|p_m\|c_n^{*m-1} + \ldots + \|p_1\|)\,c_{n+k}^* + \\ &+ F_k^*(c_n^*, \ldots, c_{n+k-1}^*,\, a^*,\ \|p_1\|, \ldots \|p_m\|)], \\ &\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned} \tag{15}$$

[3, th. 1].

From (13) and (15), because of the inequality (11), we get $\|c_{n+k}\| \leqslant c^*_{n+k}$ $k = \overline{1, \infty}$. The proof of the theorem is now complete.

Remarks. 1) The solution (4') corresponding to a c_n, solution of the equation (10), is unique in the class of analytical functions of the form

$$y(x) = a_0 + l(x)a_1 + \dots + [l(x)]^k a_k + \dots$$

whereas the solution (4) from the theorem (2) has been proved to be unique in the class of analytical functions of the form $y(x) = y_0 + y_1(x) + \dots + y_k(x) + \dots$ where $y_k(x)$ are any homogeneous polynomials of the k-th degree.

2) The systems of differential equations of the form (30) from [3] can be likewise generalized over Banach spaces, and we get a result similar to theorem 3.

4. Differential Equations in Banach Spaces, Generalizations of the Equation

$$z^t \frac{d^s w}{dz^s} = \lambda_r z^r + P(z, w), \quad t < s. \tag{**}$$

Theorem 4. *Consider the equation*

$$[l(x)]^t \frac{d^s y(x, x_0)}{dx^s} = [l(x)]^r b + P(x, y), \tag{16}$$

where $x_0 \in X$, $x_0 \neq \theta$, $l(x)$ is a continuous linear functional defined over X, such that $l(x_0) \neq 0$; t, r, s are integers $\geqslant 0, 1 \leqslant t < s$; $b_r \in Y$, $b_r \neq \theta$, and $P(x,y)$ is analytical of the form

$$P(x, y) = \sum_{i=r+1,\infty} [l(x)]^i p_{i0} + \sum_{\substack{i=\overline{0,\infty} \\ j=\overline{1,\infty}}} [l(x)]^i p_{ij}(y). \tag{17}$$

Then the equation (16) has an analytical solution of the form

$$y(x) = [l(x)]^n c_n + [l(x)]^{n+1} c_{n+1} + \dots \text{where } n = r - t + s. \tag{18}$$

Proof. Substituting formally (18) in (16) we obtain

$$\frac{n!}{(n-s)!}[l(x)]^{n-s+t}[l(x_0)]^s c_n + \frac{(n+1)!}{(n+1-s)!}[l(x)]^{n+1-s+t}[l(x_0)]^s c^s_{n+1} + \dots =$$
$$= [l(x)]^r b_r + P(x, y(x)).$$

This equality leads to the system

$$\begin{aligned} &\frac{n!}{(n-s)!}[l(x_0)]^s c_n = b_r, \\ &\frac{(n+1)!}{(n+1-s)!}[l(x_0)]^s c_{n+1} = F_1(c_n, a), \\ &\dots\dots\dots\dots\dots\dots\dots\dots \\ &\frac{(n+k)!}{(n+k-s)!}[l(x_0)]^s c_{n+k} = F_k(c_1, \dots, c_{n+k-1}, a). \end{aligned} \tag{19}$$

The meaning of the expressions $F_k(c_1, \dots, c_{n+k-1}, a)$ is similar to that of the expressions introduced in the proof of theorem 3.

The system (19) determines a formal solution of the form (18). The convergence of the formal solution (18) obtained from the system (19) is proved by means of the majorant scalar equation :

$$z^t \frac{d^s w}{dz^s} = \frac{\|b_r\| z^r}{|l(x_0)|^s} + P^*(z, w), \text{ where} \tag{20}$$

$$P^*(z, w) = \left(\sum_{i=r+1,\infty} \|p_{i0}\| z^i + \sum_{\substack{i=\overline{0,\infty} \\ j=1,\infty}} \|p_{ij}\| z^i w^j \right) \frac{1}{|l(x_0)|^s}.$$

The equation (20) has an analytical solution of the form $w(z) = c^*_n z^n + c^*_{n+1} z^{n+1} + \dots$ with $n = r - s + t$ [3, th 4].

The same arguments as those used in the theorems 2 and 3 show that $\|c_{n+k}\| \leqslant c^*_{n+k}$, $k = 0, 1, 2, \dots$

5. Briot-Bouquet Equations in Banach Spaces in which the Fréchet Derivatives are not Fixed on a Direction x_0. We shall consider for simplicity an equation which is a generalization of the equation of the form

$$z^s \frac{d^s w}{dz^s} = aw + bz^n + P(z, w). \tag{*}$$

Theorem 5. *Let us consider the equation*

$$[l(x)]^s \frac{d^s y(x ; h_1, \dots, h_s)}{dx^s} = l(h_1)\dots l(h_s)(Ay + l^n(x)b_n + P(x, y)) \tag{21}$$

in which l, A, b_n, P, s, n, have the same meaning as in the theorem 2, and $h_1, h_2, \dots, h_s \in X$.

Suppose that

$$\frac{(n-s)!}{n!}\|A\| < 1. \tag{i}$$

Then the equation (21) has an analytical solution of the form

$$y(x) = [l(x)]^n c_n + \dots + [l(x)]^{n+k} c_{n+k} + \dots \tag{22}$$

Proof. Substituting (22) in (21) we obtain

$$[l(x)]^s \left(\frac{n!}{(n-s)!}[l(x)]^{n-s} l(h_1)\dots l(h_s) c_n + \frac{(n+1)!}{(n+1-s)!}[l(x)]^{n+1-s} l(h_1)\dots l(h_s) \right) +$$
$$+ \dots) = l(h_1) l(h_2) \dots l(h_s) A [l(x)]^{n-s} c_n + [l(x)]^{n+1} c_{n+1} + \dots) + [l(x)]^n b_n + P(x, y(x)).$$

We get a formal solution of the form (22) if $c_n, c_{n+1}, \dots$ verify the system

$$\left\{ \begin{aligned} &\frac{n!}{(n-s)!} c_n = A c_n + b_n, \\ &\frac{(n+1)!}{(n+1-s)!} c_{n+1} = A c_{n+1} + p_{n+1,0} + p_{11}(c_n), \dots \end{aligned} \right. \tag{23}$$

Under the assumption (i), the system (23) has a unique solution.

As a majorant scalar equation we can consider the equation

$$z^s \frac{d^s w}{dz^s} = Aw + b_n z^n + P^*(z, w), \text{ where} \tag{24}$$

$$P^*(z, w) = \sum_{\substack{i,j=\overline{0,\infty} \\ i+kj>n}} \|p_{ij}\| z^i w^j.$$

6. Briot-Bouquet Differential Equations in Banach Algebras. Let $X = Y$ be a comutative Banach algebra with unit. In this case, we can consider other equations which generalize the equations from [3] and which have analytical solutions in the sense of the definition 2 from 1.

We shall give, for simplicity, a generalization of equation (*).

Theorem 6. *Consider the equation*

$$x^s \frac{dy^s(x, x_0)}{dx^s} = Ay + x^n b_n + P(x, y), \text{ where} \tag{25}$$

a) $x_0 \in X$, x_0 *with inverse* x_0^{-1},

b) s *and* n *are integers* $\geqslant 1$, $s \leqslant n$,

c) A *is a continuous linear operator, defined over* X *with values in* X,

d) $P(x, y)$ *is an analytical function in a neighbourhood of the origin of* X^2 *with values in* X, *of the form*

$$P(x, y) = \sum_{\substack{i,j=\overline{0,\infty} \\ i+nj>n}} x^i y^j a_n, \quad \text{with } a_n \in X \text{ being constants values.}$$

If (i) $\frac{(n-s)!}{n!} \|x_0^{-1}\|^s \|A\| < 1$, *then the equation* (25) *has a unique analytical solution of the form*

$$y(x) = x^n c_n + x^{n+1} c_{n+1} + \ldots \tag{26}$$

The proof is similar to that of the theorem 2.

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Department of Mathematics
Polytechnical Institute
Iaşi, R. S. România

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ON A STOCHASTIC PROCESS WITH DISCRET TIME AND IMPOSED LENGTH OF TIME

BY

C. AMIHĂESEI and P. MINUŢ

In this paper we propose a model suggested by many situations we meet frequently in practice. As for instance in the viticulture and fruitgrowing. An event (the appearance of a fertile offshoot) occurs from time to time (here and there on the cord) and the length of time (the length of the cord) is imposed. The probability of such an event is a function of time t and the total length of the time, T (the length of the cord).

We consider that an event may occurs at $t = 0, 1, \ldots, T$, and we denote by $P_T(t)$ the probability that the event occurs at the time t. Let $Q_T(t) = 1 - P_T(t)$ be the probability that the event does not occur at the time t. We consider a random variable X_t which associates the number 1 when the event occurs at the moment t, and the number 0 when the event does not occur at the instant t

$$X_t \begin{pmatrix} 1 & 0 \\ P_T(t) & Q_T(t) \end{pmatrix}, \quad t = 0, 1, \ldots, T. \tag{1}$$

The total number of the occurences of the event until the moment T is

$$\xi_T = \sum_{t=0}^{T} X_t. \tag{2}$$

We suppose that the following property holds

$$P(X_t = k / X_m = i_m, \ m = 0, 1, \ldots, t-1) = P(X_t = k), \quad k, i_m = 0, 1, \tag{3}$$

that is, the appearance of the event at the moment t does not depend on the occurences of the event at earlier instants.

An important problem is to determine T such that the expectation of ξ_T be maximum. This expected number of ξ_T is

$$N(T) = M(\xi_T) = \sum_{t=0}^{T} M(X_T) = \sum_{t=0}^{T} P_T(t). \tag{4}$$

The practice shows that the probabilities $P_T(t)$ are increasing functions on an interval $[0, t_0]$, and decreasing on $[t_0, T]$, where $t_0 \in [0, T]$.

We will consider two expressions for the probability $P_T(t)$.

Case 1. We suppose that

$$P_T(t) = -a_T t^2 + b_T t + c_T, \tag{5}$$