abordarea unor aspecte aplicative ale matematicilor, punîndu-le în valoare prin lucrări proprii sau prin lucrări inspirate de d-sa membrilor colectivului catedrei de algebră sau doctoranzilor.

Activitatea exemplară a profesorului Ion Creangă în toate domeniile, didactic, științific, politic, însuflețită de un profund patriotism, angajată în efortul constructiv al întregului popor, s-a bucurat de aprecierea cea mai înaltă a partidului și statului. Profesorul Ion Creangă a fost decorat cu numeroase ordine și medalii ale R.S.R., a fost, din 1959 pînă în 1972, deputat în M. A. N., a fost membru al Consiliului de Stat al țării. Este profesor emerit și om de știință emerit.

Universitatea din Iași, pentru propășirea căreia profesorul doctor docent Ion Creangă și-a dedicat întreaga viață, l-a înscris pentru totdeauna în galeria marilor ei profesori. Întreaga Facultate de matematică, pe care profesorul Ion Creangă a onorat-o ca distins membru al corpului ei profesoral, îi aduce prinosul de stimă și recunoștință, urîndu-i, la trecerea în al optulea deceniu de viață, sănătate și putere de muncă, pentru a contribui în continuare la înflorirea matematicii ieșene.

Mirela Ștefănescu

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LINEAR TOPOLOGICAL SPACES WITH FAMILIES OF PSEUDONORMS

BY
OVIDIU CÂRJĂ

1. **Introduction.** Among the possibilities of developing the functional analysis on locally convex spaces there is also the possibility realised with the help of seminorm families. [5] is a full illustration of this way. As a result of [1], any linear topology can be defined with the help of a family of pseudonorms. It follows that some results of the functional analysis on linear topological spaces can be obtained by using a family of pseudonorms. This is the real aim of this paper. Among others, we point out the possibility to extend the notion of pseudotopological union at $L(X, Y)$, when X and Y are some linear topological spaces. Another general approach to the pseudotopological union at $L(X, Y)$ has been developed in [9]. This notion was introduced by G. M a r i n e s c u ([2], [3]), when X and Y are locally convex spaces, and then, extended and studied in [7], [8] for general topological spaces.

2. **Families of pseudonormes.** Let X be a linear space over Γ(Γ may be C or R). We define a family of pseudonorms on X, as a family $\mathcal{P} = \{p_i : X \to R\ ;\ i \in I\}$, I being a set of indices, that satisfies the following properties :

(i) $p_i(x) \geqslant 0$, for all $i \in I$, $x \in X$,

(ii) $p_i(\lambda x) = |\lambda| p_i(x)$, for all $i \in I$, $\lambda \in \Gamma$, $x \in X$,

(iii) for every $i \in I$, there exists $j_i \in I$ such that
$p_i(x+y) \leqslant p_{j_i}(x) + p_{j_i}(y)$, for all $x, y \in X$.

As it is known (see [1]), any topology of linear space can be defined by means of a directed family of pseudonorms, a fundamental neighbourhood system of the origin being given by

$$S_i(O, \varepsilon) = \{x\ ;\ p_i(x) < \varepsilon\},\ \varepsilon > 0,\ i \in I. \tag{2.1}$$

Such a topology is Hausdorff if and only if the family of pseudonorms is saturated. The pseudonorms which define the topology described above, are not continuous at any rate (excepting the origin).

Lemma 1. *Let V be an absorbant and balanced set of the linear topological space X, with* int $V \neq \varnothing$, *and let p_v be the Minkovski functional atached to V. Then the following statements hold :*

(a) $\operatorname{int} V \subset \{x\ ;\ p_v(x) < 1\} \subset V \subset \{x\ ;\ p_v(x) \leqslant 1\} \subset \operatorname{Cl} V$,

(b) $\{x ; p_v(x) < 1\} = \bigcup_{0<\theta<1} \theta V.$

(c) $\{x ; p_v(x) \leq 1\} = \bigcap_{\theta>1} \theta V.$

Proof. If $x \in \text{int } V$, there exists a neighbourhood W, of the origin, which is absorbant and balanced, so that $x + W \subset V$; there exists $\rho > 0$ with $\rho x \in W$ and so $x + \rho x \in V$, and $x \in 1/(1+\rho)V$. Then $p_v(x) \leq 1/(1+\rho) < 1$ and $x \in \{x ; p_v(x) < 1\}$. If $x \in \{x ; p_v(x) \leq 1\}$, we have $\theta x \in V$ with $\theta \in [0, 1)$ and so $x \in C V$. The other inclusions or equalities result from the definition of Minkowski's functional.

Lemma 2. *If V is a convex and balanced set and q is a nonnegative functional and absolutely homogeneous on X, satisfying the condition*

$$\{x ; q(x) < 1\} \subset V \subset \{x ; q(x) \leq 1\}, \tag{2.2}$$

then $q(x) = p(x)$ *on* X.

Proof. We have $\{x ; q(x) < 1\} \subset \{x ; p_v(x) \leq 1\}$ and

$$\{x ; p_v(x) < 1\} \subset \{x ; q(x) \leq 1\}.$$

From here it results that, if $q(x) \leq 1$, then for any $\varepsilon \in (0, 1)$, we have $p_v(x) \leq 1/(1-\varepsilon)$ and so $p_v(x) \leq 1$. Thus $p_v(x) \leq q(x)$ and analogously, $q(x) \leq p_v(x)$.

Theorem 1. *A topology of linear space is given by a family of pseudonorms continuous everywhere iff there exists a fundamental balanced neighbourhood system $\mathcal{V}_0$ of the origin, which satisfies the following relations:*

(iv) *for every* $V \in \mathcal{V}_0$, *there exists* $V_1 \in \mathcal{V}_0$ *such that*

$$V_1 + V_1 \subset V$$

(v) $\text{int } V = \bigcup_{0<\theta<1} \theta V$; $\text{Cl} V = \bigcap_{\theta>1} \theta V$; *for all* $V \in \mathcal{V}_0$.

Proof. We point out that the Minkowski functionals attached to a balanced neighbourhood fundamental system of the origin satisfying the relation (iv), represent a family of pseudonormes which defines the same topology.

Let $\mathcal{P} = \{p_i\}_{i \in I}$, be the family of continuous pseudonorms which determine the topology of the linear space X. For i and ε we denote $V = \{x ; p_i(x) < \varepsilon\}$, and using Lemma 1, we obtain $A = \{x ; p_v(x) < 1\} = \bigcup_{0<\theta<1} \theta V = \text{int } V$, where p_v is the Minkowski functional for an open, absorbant and balanced set V. We have also:

$$\{x ; p_i(x) < 1\} = \frac{1}{\varepsilon} V \subset \{x ; p_i(x) \leq 1\}.$$

Denoting $W = 1/\varepsilon V$ and using Lemma 2 we deduce $p_i(x) = p_w(x)$ on X. From Lemma 1, it results $\text{Cl} W = \{x ; p_i(x) \leq 1\}$, and therefore $\text{Cl } V = \{x ; p_i(x) \leq \varepsilon\} = \varepsilon\{x ; p_w(x) \leq 1\} = \varepsilon \bigcap_{\theta>1} \theta W = \bigcap_{\theta>1} \theta V$. We demonstrated that $\mathcal{V}_0 = \{V ; V = \{x ; p_i(x) < \varepsilon\}, i \in I, \varepsilon > 0\}$ is a fundamental balanced neighbourhood system of the origin satisfying condition (iv) and (v). Let us prove the „only if" part. We have $\{x ; p_v(x) < 1\} = \bigcup_{0<\theta<1} \theta V = \text{int } V$ and $\{x ; p_v(x) \leq 1\} = \text{Cl} V = \bigcap_{\theta>1} \theta V$, therefore $\{x ; p_v(x) < \varepsilon\}$ is an open set and $\{x ; p_v(x) \leq \varepsilon\}$ is closed for every $\varepsilon > 0$. The proof is complete.

Remark. Because in any topological linear space there exists a fundamental system of balanced and closed neighbourhoods of the origin satisfying condition (iv), it results that the topology of any linear space could be defined with the help of a family of lower semicontinuous pseudonorms.

Let X and Y be linear topological spaces with directed families of pseudonorms $\mathcal{P} = \{p_i\}_{i \in I}$ and $Q = \{q_j\}_{j \in J}$ respectively. The following propositions are given in [1].

Proposition 1. *A necessary and sufficient condition for the convergence of a sequence (x_n) to x, is that for any pseudonorm $p_i \in \mathcal{P}$, the sequence $p_i(x_n - x_0)$ converges to zero.*

Proposition 2. *A necessary and sufficient condition for the set $A \subset X$ to be bounded is that for any pseudonorm $p_i \in \mathcal{P}$, there exists a real number M_i such that $p_i(x) \leq M_i$, for $x \in A$.*

Proposition 3. *A necessary and sufficient condition to the linear map $T : X \to Y$ to be continuous is that for any pseudonorm $q_j \in Q$, there exists $p_i \in \mathcal{P}$ and $M_{j,i} > 0$, such that $q_j(Tx) \leq M_{j,i} p_i(x)$ on X.*

3. The pseudotopological structure for $L(X, Y)$. From Proposition 3 it results:

Proposition 4. *A necessary and sufficient condition for the linear map $T : X \to Y$ to be continuous is that for any pseudonorm $q_j \in Q$, there exists $p_i \in \mathcal{P}$, such that*

$$p_i(x) = 0 \textit{ implies } q_j(Tx) = 0 \textit{ and} \tag{3.1}$$

$$\omega_{j,i}(T) = \sup_{p_i(x) \neq 0} \frac{q_j(Tx)}{p_i(x)} < \infty. \tag{3.2}$$

Proposition 4 allows to structure the space $L(X, Y)$, when X is a linear topological space and Y is a locally convex space.

Theorem 2. *Let X be a linear topological space with the directed family of pseudonorms $\mathcal{P} = \{p_i\}_{i \in I}$ and Y be a locally convex space with the directed family of seminorms $Q = \{q_j\}_{j \in J}$. Let $\varphi : J \to I$ and $L_\varphi(X, Y)$ be the set of linear maps from X into Y, for which $p_{\varphi(j)}(x) = 0$ implies $q_j(Tx) = 0$ and $\omega_{j,\varphi(j)}(T) < \infty$.*

Then $L_\varphi(X, Y)$ is a locally convex space with the directed family of seminorms $\{\omega_{j',\varphi(j')}\}$ where $j' = \{j_1, j_2, \dots, j_n\}$ run over the set of finit parts of J and

$$\omega_{j',\varphi(j')}(T) = \sup \{\omega_{j_1,\varphi(j_1)}(T), \dots, \omega_{j_n,\varphi(j_n)}(T)\}. \tag{3.3}$$

Proof. We consider

$$\omega_{j,\varphi(j)}(T) = \sup_{p_{\varphi(j)}(x) \neq 0} \frac{q_j(Tx)}{p_{\varphi(j)}(x)}, \quad j \in J.$$

It is easy to check that they are seminorms on $L_\varphi(X, Y)$, which is a linear space. Consequently $\{\omega_{j',\varphi(j')}(T)\}$ in (3.3) are seminorms on $L_\varphi(X, Y)$, too.

Moreover, they represent a directed family of seminorms on $L_\varphi(X, Y)$. If Q is saturated then (3.3) are saturated too.

Theorem 3. *Let the hypothesis of Theorem 2 be satisfied. Then $L(X, Y)$ is the pseudotopological union of the locally convex spaces $L_\varphi(X, Y)$, where φ runs over the set Φ of the mappings from J into I.*

Proof. From proposition 4, it results that $L(X, Y)=\bigcup_{\varphi\in\Phi} L_\varphi(X, Y)$.
For $\varphi_1, \varphi_2\in\Phi$, we consider $\varphi_3\in\Phi$ so that $\varphi_3(j)=i$, where $p_{\varphi_1}(j)(x)\leqslant p_i(x)$ and $p_{\varphi_2(j)}(x)\leqslant p_i(x)$ on X.
For any $j\in J$ and $T\in L(X, Y)$ we have

(3.4) $$\omega_{j, \varphi_1(j)}(T)\leqslant \omega_{j, \varphi_3(j)}(T),$$

(3.5) $$\omega_{j, \varphi_2(j)}(T)\leqslant \omega_{j, \varphi_3(j)}(T),$$

so that

(3.6) $$L_{\varphi_1}(X, Y)\subset L_{\varphi_3}(X, Y)\ ;\ L_{\varphi_2}(X, Y)\subset L_{\varphi_3}(X, Y).$$

From (3.4) and (3.5), it results :

(3.7) $$\omega_{j', \varphi_3(j')}(T)\leqslant \omega_{j' \varphi_1(j')}(T),$$

(3.8) $$\omega_{j', \varphi_3(j')}(T)\leqslant \omega_{j', \varphi_2(j')}(T),$$

which implies the continuity of the identical embedings

$$L_{\varphi_1}(X, Y)\to L_{\varphi_3}(X, Y) \text{ and } L_{\varphi_2}(X, Y)\to L_{\varphi_3}(X, Y),$$

and the proof is completed.

If Y is a normed space then the expressions

$$\omega_i(T)=\sup_{p_i(x)\neq 0}\frac{\|Tx\|}{p_i(x)}$$

are generalized normes and consequently $L(X, Y)$ is a pseudotopological union of normed spaces, i.e. a polinormed space. We refer to [2. p. 54], [3, p. 66], for the concepts of pseudotopological union of linear spaces and polinormed spaces. (see also [9]). Particularly, we have

Theorem 4. *If X is a linear topological space and $L(X, \Gamma)=X^*$ is non-zero, then X is a polinormed space.*

Theorem 5. *If X is a normed space and Y is a topological linear space with the family of pseudonorms $Q=\{q_j\}_{j\in J}$ then $L(X, Y)$ is a linear topological space with $\Omega=\{\omega_j\}_{j\in J}$ as the family of pseudonorms, where*

(3.9) $$\omega_j(T)=\sup_{\|x\|\neq 0}\frac{q_j(Tx)}{\|x\|}.$$

Proof. ω_j is nonnegative and absolutely homogeneous for every $j\in J$. For $j\in J$, there is $j_1\in J$ so that :

$$q_j(T_1x+T_2x)\leqslant q_{j_1}(T_1x)+q_{j_1}(T_2x),\ x\in X,$$

and therefore

$$\omega_j(T_1+T_2)\leqslant \omega_{j_1}(T_1)+\omega_{j_1}(T_2).$$

Analogously, it is very easy to verify the fact that the family Ω is directed. If Q is saturated, then Ω is also saturated. The proof is completed.

4. The uniform boundedness principle. The uniform boundedness principle known when X is a Banach space and Y is a normed space was generalized by G. Marinescu [3], for locally convex spaces. Next, we shall give another generalisation – and as a consequence—we will generalize the Banach-Steinhaus theorem.

Theorem 6. *Let X be a linear topological space in which every family of lower semicontinuous pseudonorms is a family of pseudonorms continuous in the origin, and Y be a linear topological space. Consider a family $\{T_\alpha;\ \alpha\in A\}\subset L(X, Y)$. Then $\{\{T_\alpha x\};\ \alpha\in A\}$ is bounded at each $x\in X$, if and only if $\lim_{x\to 0} T_\alpha x=0$ uniformly with respect to $\alpha\in A$.*

An equivalent formulation of Theorem 6 : *Let X be a linear topological space in which every family of lower semicontinuous pseudonorms is a family of pseudonorms continuous in the origin and let $\mathcal{D}=\{p_i\}_{i\in I}$ be a directed family of pseudonorms which defines the topology of X. Let Y be a linear topological space with the directed family of pseudonorms $Q=\{q_j\}_{j\in J}$ which are lower semicontinuous. Then the set $\{\{T_\alpha x\};\ \alpha\in A\}$ is bounded at each $x\in X$ if and only if there exists a map $\varphi: J\to I$, such that for every $j\in J$, there is $M_j>0$, such that $\omega_{j, \varphi(j)}(T_\alpha)\leqslant M_j$, for every $\alpha\in A$, and :*

$$p_{\varphi(j)}(x)=0, \text{ implies } q_j(T_\alpha x)=0 \text{ for every } \alpha\in A.$$

Proof. Let $\mathcal{F}=\{f_j: X\to R;\ j\in J\}$, be a family of functions defined by $f_j(x)=\sup_{\alpha\in A} q_j(T_\alpha x)$; $j\in J$. Taking into account that the family $\{\{T_\alpha x\};\ \alpha\in A\}$ is bounded at each $x\in X$, we have $f_j(x)<\infty$. For every $j\in J$, f_j is nonnnegative and absolutely homogenous. On the other hand, for $j\in J$, there exists $j_1\in J$ such that

$$q_j(T_\alpha(x+y))\leqslant q_{j_1}(T_\alpha x)+q_{j_1}(T_\alpha y), \text{ for all } x, y\in X.$$

Then $f_j(x+y)\leqslant f_{j_1}(x)+f_{j_1}(y)$, for all $x, y\in X$. It is easy to see that f_j are lower semicontinuous and then, they are continuous in the origin. This implies that

$$\lim_{x\to 0} f_j(x)=0.$$

Hence $\lim_{x\to 0} T_\alpha x=0$ uniformely with respect to $\alpha\in A$. This ends the proof of the necessity. The sufficiency is obvious.

Theorem 7. *Let the hypotheses of Theorem 6 be satisfied. In addition let the pseudonorms q_j, $j\in J$, be continuous on Y. Then, if the sequence $\{T_n\}\subset L(X, Y)$ is strongly convergent to T_0, then $T_0\in L(X, Y)$.*

Proof. Since the sequence $\{T_n\}$ is strongly convergent, it is strongly bounded, and from Theorem 6 we may infer that

(4.1) $$q_j(T_nx)\leqslant M_jp_{\varphi(j)}(x), \text{ for every } x\in X,\ n\in N.$$

Fixing $x\in X$ and taking the limit in (4.1) for $n\to\infty$, we obtain

$$q_j(T_0x)\leqslant M_jp_{\varphi(j)}(x), \text{ for every } x\in X.$$

We have obtained that T_0 (which is linear) is continuous from X into Y. The proof is completed.

The following theorem gives an example of a space X which satisfies the hypotheses of Theorem 6.

Theorem 8. *If X is a linear topological space of the second category, then every family of lower semicontinuous pseudonorms, $Q = \{q_j\}_{j \in J}$, is a family of pseudonorms continuous in the origin.*

Proof. Let $\mathcal{P} = \{p_i\}_{i \in I}$ be the family of pseudonorms which defines the topology of X. For every $j \in J$ and $n \in N$ we denote the set $X_n^j = \{x; q_j(x) \leqslant n\}$, which is nonvoid. For $j \in J$ we can write $X = \bigcup_{n \in N} X_n^j$. On the other hand, for every $j \in J$, there exists $j_1 \in J$, so that

$$q_j(x + y) \leqslant q_{j_1}(x) + q_{j_1}(y) \text{ for all } x,\ y \in X. \tag{4.2}$$

Since X is of the second category, it follows that there exists $k \in N$, such that $X_k^{j_1}$ contains an open ball $S_i(x_0, \delta)$. We get $p_i(x - x_0) < \delta$ implies $q_{j_1}(x) \leqslant k$, hence

$$p_i(x) < \delta \text{ implies } q_{j_1}(x + x_0) \leqslant k. \tag{4.3}$$

From (4.2) and (4.3) we have

$$p_i(x) < \delta \textit{ implies } q_j(x) \leqslant k + q_{j_1}(x_0) = M_j, \tag{4.4}$$

where M_j depends on j and is independent on x_0. If $p_i(x) \neq 0$, then $x' = (\delta/2)x/p_i(x)$ satisfies $p_i(x') < \delta$, and from (4.3) we have $q_j(\delta/2)\ x/p_i(x) \leqslant M_j$ that is $q_j(x) \leqslant (2/\delta)M_j p_i(x)$. If $p_i(x) = 0$, it follows that for every $\varepsilon > 0$, we have $p_i(x/\varepsilon) = 0$ and from (4.5) we obtain $q_j(x) \leqslant \varepsilon M_j$. This fact implies $q_j(x) = 0$. Denoting $L_j = (2/\delta)M_j$, it follows that for every $j \in J$, there exists $i \in I$ and $L_j > 0$, such that $q_j(x) \leqslant L_j p_i(x)$ for all $x \in X$. This implies that q_j is continuous in the origin, as claimed.

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A GENERALIZATION OF BRÉZIS-BROWDER'S ORDERING PRINCIPLE

BY

MIHAI TURINICI

Concerning maximal elements in an abstract ordered set, a fundamental result obtained in the last years is, undoubtedly, the so-called Brézis-Browder's ordering principle [5] that may be considered as a countable version of the classical Zorn's theorem [19, p. 33]. Taking into account the great importance of this principle to a series of questions pertaining to nonlinear functional analysis (see, e.g., I. Ekeland [16] as a quasi-complete reference on this topic) an extension of it may therefore be of interest. It is the main aim of the present note to demonstrate firstly that Brézis-Browder's ordering principle may be expressed essentially as a quasi-metric maximality result and secondly that — from this new viewpoint — it can be effectively included into a general maximality principle on quasi-metric spaces comparable with a similar author's one [29]. A number of applications, especially concerning mean value theorems in Dieudonné's sense [11, ch. VIII, §5] as well as differential inequalities in Bebernes-Meister's sense [2] will be given in a forthcoming paper.

Let X be an abstract (nonempty) set and let $\leqslant$ be a *quasi-ordering* on X. For any (nonempty) subset Y of X and any $x \in Y$, $Y(x, \leqslant)$ will denote the subset of all $z \in Y$ with $x \leqslant z$. A sequence $(x_n; n \in N)$ in X will be called *monotone* iff $x_i \leqslant x_j$ whenever $i \leqslant j$, and *bounded above* iff $x_n \leqslant y$, all $n \in N$, for some $y \in X$ termed an *upper bound* of this sequence. Furthermore, suppose d is a *quasi-metric* on X. A sequence $(x_n; n \in N)$ in X will be called *quasi-asymptotic* iff for any $\varepsilon > 0$ there exists $n = n(\varepsilon) \in N$ with $d(x_n, x_{n+1}) < \varepsilon$; also, an element $z \in X$ is said to be *d-maximal* iff $z \leqslant y$ implies $d(z, y) = 0$. A satisfactory motivation for introducing these notions will be offered later; for the moment, we are only interested to state and prove the main result of the present note, a result that may be formulated as follows:

Theorem 1. *Suppose the quasi-metric space (X, d) and the ordering $\leqslant$ on X are such that:*

(i) *any monotone sequence in X is a quasi-asymptotic one,*
(ii) *any monotone Cauchy sequence in X has an upper bound.*
Then, for every $x \in X$, there is a d-maximal element $z \in X$ with $x \leqslant z$.

Proof. Firstly, we claim any $x \in X$ has the property:

(1) *for every $\varepsilon > 0$ there exists $y \geqslant x$ with $d(y, z) < \varepsilon$, for all elements $z \geqslant y$.*