By using the inequality (16′), we obtain

$$
\begin{aligned}
(17)\qquad & N(n) \leqslant P_2(2) + P_3(3) + \ldots + P_n(n) + \\
& + \left[1 - \left(\frac{1}{2!} + \frac{2}{3!} + \ldots + \frac{n-2}{(n-1)!}\right)\right](c_0 + c_1 e^{-1}).
\end{aligned}
$$

The inequality (17) and the equality $\sum_{n=2}^{\infty} \frac{n-2}{(n-1)!} = 1$ impliy the theorem.

The expression (14), the inequality (16) and the above theorem lead to the following remarks:

a) If there exists a number $T_0 > 3$ such that

$$
(18)\qquad c_T \leqslant \frac{T-2}{T^{T-1}}(c_0 + c_1 e^{-1}) e^T \quad \text{for } T \geqslant T_0,
$$

then, the function $N(T)$ has a maximum point $T \in [2, T_0]$.

b) If there exists $\lim\limits_{n} c_{n+1}/c_n$ and $\lim\limits_{n} c_{n+1}/c_n < 1$, then the function $N(T)$ is bounded.

c) If there exists a number $q \in (0, 1)$ and a natural number n_0 such that $c_{n+1} < q c_n$ for $n \geqslant n_0$, then the function $N(T)$ is bounded.

d) If $c_0 \neq 0$, $c_1 \neq 0$, the sequence $\{c_n\}$ decreases and there exists a number $n_0 \geqslant 3$ such that

$$
c_{n_0} \leqslant \frac{n_0 - 2}{n_0^{n_0 - 1}} e^{n_0}(c_0 + c_1 e^{-1}),
$$

then, the function $N(T)$ is maximum for $\bar{n} \leqslant n_0$.

e) If $P_n(n) = 0$ for $n \geqslant n_0$, then the function $N(T)$ is bounded for any time.

f) The average number of the occurences of the event in the unit time satisfies the inequality

$$
\begin{aligned}
(19)\qquad & \frac{N(T)}{T} \leqslant \frac{P_2(2) + \ldots + P_T(T)}{T} + \frac{1}{T}\left[1 - \left(\frac{1}{2!} + \frac{2}{3!} + \ldots + \right.\right. \\
& \left.\left. + \frac{T-2}{(T-1)!}\right)\right](c_0 + c_1 e^{-1}).
\end{aligned}
$$

Therefore, if there exists $\lim\limits_{n} N(n)/n$ and the sequence $P_n(n)$ is convergent, then we have

$$
(20)\qquad \lim_{n} \frac{N(n)}{n} \leqslant \lim_{n} P_n(n) \leqslant 1.
$$

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THERMAL INFLUENCES ON THE SHEAR ACCELERATION WAVES IN INCOMPRESSIBLE SIMPLE FLUIDS

BY

C. FETECĂU

1. Introduction. An exact theory of the growth and decay of surfaces of discontinuity in non-linear, incompressible, simple fluids with memory undergoing steady plane-rectilinear shearing flows, is developed in [1], but no account has been taken about the dependence of the constitutive functional upon the temperature or entropy. Thus, most of those results are to be thought appropriate to circumstances when either θ or η is kept constant, or to fluids for which the response functionals are insensitive to changes in these variables.

In this paper we shall consider nonsteady shearing flows and we shall allow our constitutive functionals to be affected not only by the history of the strain, but also by the histories of the thermodynamic variables θ^t and g^t. The main results obtained here are:

1) a homothermal shear wave propagating in a region undergoing a nonsteady shearing flow, is damped out or it blows up[1] (i.e. its amplitude can become infinite in a finite time).

2) a shear wave in a linear simple fluid undergoing a nonsteady shearing flow, cannot grow, but it is always damped out.

2. Constitutive equations and smooth assumptions. The fluids to be considered here, will be characterized by constitutive equations of the form [2—4].

$$
(2.1)\qquad
\begin{aligned}
& (\mathbf{S}, \psi, \eta) = (\underset{s=0}{\overset{\infty}{\mathbf{S}, \psi,}} \eta)(\mathbf{G}(s), \theta^t(s), \mathbf{g}^t_r(s)), \\
& \mathbf{q} = \underset{s=0}{\overset{\infty}{\mathbf{q}}}(\mathbf{G}(s), \theta^t(s), \mathbf{g}^t(s)), \quad \mathbf{G}(\cdot) = \mathbf{F}^t(\cdot)^T \mathbf{F}^t(\cdot) - \mathbf{1},
\end{aligned}
$$

in which $\mathbf{S}$, ψ, η and $\mathbf{q}$ are respectively, the extra-stress tensor, the free energy, the entropy and the heat flux, while $\mathbf{F}^t(\cdot)$, $\theta^t(\cdot)$ and $\mathbf{g}^t(\cdot)$ are the histories of the relative deformation gradient, of the temperature and of its spatial gradient. By $\mathbf{g}^t_r(\cdot)$ we have denoted the past history of $\mathbf{g}^t(\cdot)$.

For a rectilinear shearing flow, whose velocity field in a Cartesian co-ordinate system x, y, z is given by

$$
(2.2)\qquad \dot{x} = 0, \quad \dot{y} = v(x, t), \quad \dot{z} = 0,
$$

[1] Cf. Coleman and Gurtin [1], for mechanical waves.

these constitutive relations may be written under forms

$$(2.3)\qquad \begin{aligned}&(\mathbf{S}, \psi, \eta)(\mathbf{S}, \overset{\infty}{\underset{s=0}{\psi}}, \eta)(\Lambda^t(s)\,;\ \theta),\\ &\mathrm{q} = \overset{\infty}{\underset{s=0}{\mathrm{q}}}\,(\Lambda^t(s)\,;\ \theta, g),\ \Lambda^t = (\alpha^t, \theta^t_r, g^t_r),\end{aligned}$$

where

$$\alpha^t(s) = \int_t^{t-s} \partial_x v(x, \delta)\,d\delta,\ g = \partial_x\theta$$

and θ was implicitly assumed to be a function only of x and t.

In the following we shall assume that our fluid obeys to the principle of fading memory. In order to state this principle in our context, we consider a fixed influence function, i.e., a positive, monotone-decreasing, measurable function with $\mathrm{sh}(s)$ square-integrable on $(0, \infty)$ and define the norm of a real-valued function $f(\cdot)$ on $(0, \infty)$ by

$$(2.4)\qquad \|f\| = \left\{\int_0^\infty f(s)^2 h(s)^2 ds\right\}^{1/2}.$$

Denoting by $\mathcal{H}$ the collection of all measurable functions $f: (0, \infty) \to R$, whose norms (2.4) are finite, and by $\mathbf{H} = \mathcal{H} \times \mathcal{H} \times \mathcal{H}$, the principle of fading memory asserts that for each value of θ and g, our functionals considered as functions of Λ^t are smooth functions over $\mathbf{H}$ with respect to the norm

$$(2.5)\qquad \|\Lambda^t\| = \|\alpha^t\| + \|\theta^t_r\| + \|g^t_r\|,$$

i.e., they are continuously Fréchet differentiable. As functions of θ and g they are also assumed to be differentiable. For our discussion it is sufficient to have continuous differentials up to the third, second and first order, respectively, for ψ, $\mathbf{T}$ and η, q.

In view of the above assumptions, we can easily show that the motion equations (when the body force is conservative) and the energy balance reduce to ([4], § 2.3).

$$(2.6)\qquad \partial_x S_{xy} + b = \rho\partial_t v,$$

and

$$(2.7)\qquad S_{xy}\partial_x v - \partial_x q_x - \rho\partial_t\varepsilon + \rho r = 0,$$

while the entropy inequality implies

$$(2.8)\qquad \begin{aligned}&S_{xy} = -\rho\delta_\alpha \overset{\infty}{\underset{s=0}{\psi}}(\Lambda^t(s)\,;\ \theta/1^t(s)),\ \eta = -\partial_\theta \overset{\infty}{\underset{s=0}{\psi}}(\Lambda^t(s)\,;\ \theta),\\ &q_x g \leqslant \rho\theta^2\sigma,\ \sigma = -\frac{1}{\theta}\Big[\delta_\alpha \overset{\infty}{\underset{s=0}{\psi}}(\Lambda^t(s)\,;\ \theta/\partial_x v^t(s)) +\\ &+ \delta_\theta \overset{\infty}{\underset{s=0}{\psi}}(\Lambda^t(s)\,;\ \theta/\partial_t\theta^t_r(s)) + \delta_g \overset{\infty}{\underset{s=0}{\psi}}(\Lambda^t(s)\,;\ \theta/\partial_t g^t_r(s))\Big] \geqslant 0,\end{aligned}$$

where b is the driving force per unit volume in the y direction,

$$(2.9)\qquad \varepsilon = \psi + \theta\eta$$

is the internal energy per unit mass, r is the heat supply, $\Lambda^t(s) = 1$ for $0 < s < \infty$, and σ is the internal dissipation.

3. **General theory of shear-acceleration waves.** In the sequel we shall study the propagation of singular surfaces across which the derivatives of the functions v and θ have jump discontinuities. Such a singular surface, which will be at each moment t a plane perpendicular to the x-axis of the fixed co-ordinate system in which (2.2) holds (see [5] Sect. 191 or 195), may be called a shear wave.

Definition 3.1. *We say that $x = x_t$ ($t \in R$) is a shear-acceleration wave or simple a shear wave if:*

a$_1$) v and θ are continuous functions of x and t, while their partial derivatives $\dot v = \partial_t v$, $\partial_x v$, $\dot\theta = \partial_t\theta$ and $\partial_x\theta$ have jump discontinuities across $x = x_t$, but are continuous in x and t jointly, everywhere else.

a$_2$) $v^t(x, .)$, $\theta^t_r(x, .)$ and $g^t_r(x, .)$, are smooth functions ([6], § 5) of x and t with respect to the norm (2.4) for all x and t.

Denoting now by $u = u(t) = dx_t/dt$, $a = [\dot v]$[2] and $\tau = [\dot\theta]$, respectively, the velocity, mechanical and thermal amplitudes of the wave and using a well known Maxwell's theorem (see [5], Sect. 175 or [6], §3) we have

$$(3.1)\qquad a = -u[\partial_x v],\ \tau = -u[\partial_x\theta],$$

and

$$(3.2)\qquad [\dot S_{xy}] = -u[\partial_x S_{xy}],\ [\dot q_x] = -u[\partial_x q_x].$$

From (2.6) we also obtain

$$(3.3)\qquad [\partial_x S_{xy}] = \rho a,$$

while, from (2.7) in the natural assumption $[r] = 0$, it results[3]

$$(3.4)\qquad \rho\theta[\dot\eta] + [\partial_x q_x] = 0.$$

Now, evaluating the jumps of $\partial_x S_{xy}$, $\dot\eta$ and $\partial_x q_x$ and introducing them in (3.3) and (3.4), we attain to

$$(3.5)\qquad \begin{aligned}&(\rho u^2 - E_\alpha)a + u e_\theta\tau = 0,\\ &(Q_\alpha - \rho u\theta H_\alpha)a - u(q_\theta - \rho u\theta h_\theta)\tau = -\partial_g q_x[\ddot\theta],\end{aligned}$$

where $E_\alpha = -\delta_\alpha \overset{\infty}{\underset{s=0}{S_{xy}}}(\Lambda^t(s)\,;\ \theta/1^t(s))$, $e_\theta = \partial_\theta \overset{\infty}{\underset{s=0}{S_{xy}}}(\Lambda^t(s), \theta)$ and H_α, h_θ, Q_α, q_θ are similar quantities corresponding to η and q_x.

Remark 3.1. For special fluids, whose first component of the heat flux q_x, do not depend of g, i.e. $\partial_g q_x = 0$, the equations (3.5) reduce to a li-

[2] Here, $[\,.\,]$ is the usual notation for the jump of a function.

[3] Indeed, in view of the smooth assumptions from the second section and of the definition 3.1, we have $[\dot\psi] = 1/\rho\, S_{xy}[\partial_x v] - \eta[\dot\theta]$ which together with (2.7), (2.8)$_{1,2}$ and (2.9) lead to (3.4).

near and homogeneous system in the unknowns a and τ. This system will have nontrivial solutions if and only if the velocity u of the wave ia a solution of the equation

$$(\rho u^2 - E_\alpha)(q_\theta - \rho u\theta h_\theta) + e_\theta(Q_\alpha - \rho u\theta H_\alpha) = 0. \tag{3.6}$$

If furthermore $e_\theta = 0$, i.e. the functional $S_{xy}(\cdot)$ do not depend of θ, then the mecanical and thermal waves propagate separately with finite speeds given, respectively, by

$$\rho u^2 - E_\alpha = 0, \text{ and } q_\theta - \rho u\theta h_\theta = 0. \tag{3.7}$$

4. The velocity and the amplitude of homothermal waves. As it was defined in [7], a shear wave is homothermal if

$$[\dot\theta] = [\partial_x\theta] = 0. \tag{4.1}$$

From $(3.5)_{1,2}$ we have

Theorem 4.1. *The intrinsec velocity u of a homothermal shear wave is given by*[4] *(see* [1] *the relation* (6.18)).

$$u = \sqrt{E_\alpha/\rho}, \tag{4.2}$$

while the jump of $\ddot\theta$ is related to a by

$$k[\ddot\theta] = (Q_\alpha - \rho u\theta H_\alpha)a, \quad k = -\partial_g q_x. \tag{4.3}$$

In order to find a formula for the rate of change of the amplitude a we use the relation (6.29) from [1], i.e.

$$-\frac{d}{dt}(\rho u a) = [\partial_t^2 S_{xy}] + \left(\frac{da}{dt} - u[\partial_t\partial_x v]\right)\rho u, \tag{4.4}$$

and follow the same way as there. So, evaluating the jump o f$\partial_t^2 S_{xy}$ we find

$$\begin{aligned}[\partial_t^2 S_{xy}] = {} & (E'_\alpha + 2E_{\alpha\alpha}\partial_x v - 2\overline{E}_{\alpha\alpha} - 2\overline{E}_{\alpha\theta} - 2\overline{E}_{\alpha g} - \\ & - 2e_{\alpha\theta}\dot\theta)[\partial_x v] + E_{\alpha\alpha}[\partial_x v]^2 + E_\alpha[\partial_{tx}^2 v] + (e_\theta - E'_{g/u})[\ddot\theta],\end{aligned} \tag{4.5}$$

where

$$E_{\alpha\alpha} = \delta_{\alpha\alpha}\overset{\infty}{\underset{s=0}{S_{xy}}}(\Lambda^t(s);\ \theta/1^t(s)/1^t(s)), \quad \overline{E}_{\alpha\alpha} = \delta_{\alpha\alpha}\overset{\infty}{\underset{s=0}{S_{xy}}}(\Lambda^t(s);\ \theta/1^t(s)/\partial_x v^t(s)),$$

$$\overline{E}_{\alpha\theta} = \delta_{\alpha\theta}\overset{\infty}{\underset{s=0}{S_{xy}}}(\Lambda^t(s);\ \theta/1^t(s)/\dot\theta_r^t(s)), \quad e_{\alpha\theta} = \partial_\theta\delta_\alpha\overset{\infty}{\underset{s=0}{S_{xy}}}(\Lambda^t(s);\ \theta/1^t(s))$$

and E'_α, E'_g are similar quantities to $G'(k_t, 0)$ from [1], § 6. Inserting now (4.5) in (4.4) and using (4.2) and (4.3) we attain to

Theorem 4.2. *The amplitude a of a homothermal shear wave in a heat conducting incompressible simple fluid obeys*

$$2\rho\sqrt{u}\,\frac{d}{dt}(a\sqrt{u}) = \lambda\left(\frac{a}{u}\right) + \mu\left(\frac{a}{u}\right)^2, \tag{4.6}$$

[4] Here the thermal influence upon the velocity u is included in the modulus E_α.

where $\lambda = \lambda_s + \lambda_n + \lambda_\theta$ *with* $\lambda_s = E'_\alpha$, $\lambda_n = 2\partial_x v E_{\alpha\alpha} - 2\overline{E}_{\alpha\alpha}$ *and* $\lambda_\theta = -2\overline{E}_{\alpha\theta} - 2\overline{E}_{\alpha g} - 2e_{\alpha\theta}\dot\theta - (ue_\theta - E'_g)(Q_\alpha - \rho u\theta H_\alpha)/k$ *and* $\mu = -E_{\alpha\alpha}$.

Obviously, this equation may be written as a differential equation of Bernoulli type

$$\frac{dc}{dt} + \gamma c + \nu c^2 = 0, \quad c = a\sqrt{u}, \quad \gamma = -\frac{\lambda}{2E_\alpha}, \quad \nu = -\frac{\mu}{2E_\alpha u^{3/2}}, \tag{4.7}$$

whose solution (see [1] the relation (6.35)) enables us to see that the amplitude a of the wave can decay to zero or, if the conditions are right, it not only increases, but it can become close to infinity in a finite time. Furthermore, the term $\gamma_\theta = -\lambda_\theta/2E_\alpha$ shows us the influence of the thermal effects upon the growth of the mechanical amplitude while $\gamma_n = -\lambda_n/2E_\alpha) = 0$ for a steady shearing flow.

5. Waves in linear fluids. The linearized constitutive equations of the fluids above considered, as they was deduced in [4], § 1.3, are given by

$$\begin{gathered}\mathrm{S} = -2\rho\int_0^\infty \zeta'(s)\mathrm{G}(s)ds, \quad \mathrm{q} = -k(0)\mathrm{g} - \int_0^\infty k'(s)\tilde{\mathrm{g}}^t(s)ds, \\ \psi = \psi_0 + \alpha(0)\theta + \int_0^\infty \alpha'(s)\theta^t(s)ds + \int_0^\infty \xi'(s)\mathrm{tr}[\mathrm{G}(s)]ds, \quad \eta = -\alpha(0),\end{gathered} \tag{5.1}$$

in which $\tilde{\mathrm{g}}^t(\cdot) = \mathrm{F}^t(\cdot)^T\mathrm{g}^t(\cdot)$ and ζ, k and α are relaxation material functions satisfying

$$\zeta'(t) \geq 0, \quad \zeta(0) = 0, \quad k(0) \geq 0, \quad k(\infty) \geq 0, \quad \alpha(\infty) \geq \alpha(0). \tag{5.2}$$

For such fluids we can easily prove that $H_\alpha = Q_\alpha = e_\theta = h_\theta = q_\theta = 0$, whi e

$$E_\alpha = 2\rho\int_0^\infty \zeta'(s)ds = 2\rho\zeta(\infty) > 0 \tag{5.3}$$

such that from (3.5) it results

Theorem 5.1. *In a linear incompressible simple fluid for which*[5] *$k(0) \neq 0$, the mechanical waves only, can propagate with a constant velocity*

$$u = \sqrt{2\zeta(\infty)}. \tag{5.4}$$

Now, evaluating $[\partial_t^2 S_{xy}]$ i.e.,

$$[\partial_t^2 S_{xy}] = -2\rho\zeta'(0)[\partial_x v] + 2\rho\zeta(\infty)[\partial_{tx}^2 v], \tag{5.5}$$

and using (3.1), (4.4) and (5.4), we attain to

Theorem 5.2. *The amplitude a of a shear acceleration wave propagating into a linear incompressible simple fluid, obeys*

$$\frac{da}{dt} = -\frac{\zeta'(0)}{2\zeta(\infty)}\,a \tag{5.6}$$

[5] If $k(0) = 0$, the thermal waves can propagate, too, but with undetermined speeds.

and then

$$a = a(t) = a(0)\exp\left(-\frac{\zeta'(0)}{2\zeta(\infty)}t\right). \quad (5.7)$$

Remark 5.1. As $\zeta'(0)/2\zeta(\infty) \geq 0$, it results from (5.7) that $a(t) \to 0$ for $t \to \infty$, i.e. a shear acceleration wave in an incompressible linear simple fluid is always damped out.

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CHARACTERIZATIONS OF BOOLEAN FUNCTIONS

BY

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0. Introduction. The concepts of polynomial and algebraic function in universal algebra reflect the fact that in every algebraic structure it is natural to work with functions related to that structure. Thus a *polynomial* of an arbitrary algebra A is a function $F: A^n \to A$ which can be obtained from variables by superpositions of the basic operations of the algebra A, while an *algebraic function* is obtained from a polynomial by substituting fixed elements of A for certain variables; cf. G. Grätzer [3]. The concepts of *simple Boolean function* and *Boolean function* are obtained by specialization of the above general ideas to the case of a Boolean algebra B. A simple Boolean function $f: B^n \to B$ is built up from variables by superpositions of the operations of disjunction, conjunction and negation, while the more general concept of Boolean function includes all the functions which are obtained from simple Boolean functions by substituting fixed elements of B for certain (possibly all or none) variables. Suppose B is a Boolean algebra and let $\vee, \cdot, ', +$ denote disjunction, conjunction, negation and ring sum, respectively; then a Boolean function $f: B \to B$. can be written in the form $f(x) = ax \vee bx' = (a+b)x + b$, where $a = f(1)$ and $b = f(0)$ are fixed elements from B. Thus there are $|B|^2$ Boolean functions of one variable and only four simple Boolean functions: 0, x, x', 1. Note that the Boolean function $f(x) = ax \vee bx'$ is simple if and only if $a, b \in \{0,1\} \subset B$. The distinction between Boolean functions and simple Boolean functions has been emphasized in a previous work by one of us [5]; in the literature on Boolean algebra the term „Boolean function" usually means a simple Boolean function, while Boolean functions in our sense are often disregarded.

For other characterizations of Boolean functions see e.g., Grätzer [1], [2].

The aim of this paper is to obtain various necessary and sufficient conditions for a function $f: B \to B$ of one variable to be Boolean. Thus we obtain several characterizations of Boolean functions in terms of identities involving the functions (Theorem 1). Then we associate with each Boolean function f the orthonormal vector consisting of the four minterms in $f(0)$ and $f(1)$, that is

$$(f'(0)f'(1),\ f'(0)f(1),\ f(0)f'(1),\ f(0)f(1)), \quad (1)$$

which we call the *orthonormal characteristic* of the function f. Several properties of a Boolean function can be expressed in terms of its orthonormal