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COMPLEMENTARY VARIATIONAL PRINCIPLES FOR A CLIMATOLOGICAL MODEL

BY

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1. Introduction. The purpose of this paper is to use the theory of complementary variational principles (Arthurs, 1970) to derive maximum and minimum values and to further establish a suitable trial function that approximates closely to the solution of the non-linear differential equation

$$\frac{d}{dx}\left\{K(1-x^2)\frac{d\varphi}{dx}\right\}+C_1(x)-A_1-B_1\varphi=0, \tag{1 a}$$

subject to the boundary condition:

$$(1-x^2)^{\frac{1}{2}}\frac{d\varphi}{dx}=0 \text{ at } x=\pm 1, \tag{1 b}$$

where

φ is the mean surface temperature,

x is sin θ; θ being latitude,

K is the constant heat diffusion coefficient, $C_1=QS(1-\alpha)$; where $QS(x)$ is the mean annual insolation at latitude θ and $\alpha(x)$ is the surface albedo.

Equations (1), which parameterises the static state of the mean annual heat budget of the atmosphere, is called the *energy-balance model* and is based on the fundamental work of Sellers (1969) and Budyko (1970). Some analytical solutions have been attempted by Ghil (1976) and Drazin (1977) where further details of the model can be found.

2. Canonical Formalism. The basic equations in the theory of complementary variational principles are the generalised canonical equations:

$$T\varphi=\frac{\partial W}{\partial u} \tag{2}$$

$$T^*u=\frac{\partial W}{\partial \varphi} \tag{3}$$

where T is some linear operator and T^* is its adjoint (see Arthurs 1970). To cast equation (1) into this form, we introduce the operator

$$T=(1-x^2)^{\frac{1}{2}}\frac{d}{dx}, \tag{4}$$

so that

$$\int_{-1}^{1} uT\varphi dx = \int_{-1}^{1} u(1-x^2)^{\frac{1}{2}}\ \frac{d\varphi}{dx}dx = \int_{-1}^{1}\left[-\frac{d}{dx}\{(1-x^2)^{\frac{1}{2}}u\}\varphi\right]dx +$$

$$+[u(1-x^2)^{\frac{1}{2}}\varphi]_{-1}^{1} = \int_{-1}^{1}(T^*u)\varphi dx + [u\sigma\varphi]_{-1}^{1} = \langle T^*u, \varphi\rangle_v + (u, \sigma\varphi)_{\partial v}.$$

We can thus write the problem (1) in canonical form as

(5) $$T\varphi = u = \frac{\partial W}{\partial u} \text{ in } V=[-1,\ 1]$$

and

(6) $$T^*u = C(x) - A - B\varphi = \frac{\partial W}{\partial\varphi} \text{ in } V=[-1,\ 1],$$

where $u(-1)=0=u(1)$. That is

(7) $$(T^*T+Q)\varphi = f \text{ in } V \text{ and}$$

$$\sigma^*u = \sigma^*u_B \text{ on } \partial v.$$

Where $Q=B$, $f=C(x)-A$, $A=A_1/k$, $B=B_1/k$ and $C=C_1/k$.
From Arthurs (1973), this gives a type II boundary condition. A suitable Hamiltonian therefore is

$$W(u,\varphi) = \tfrac{1}{2}(u,u) + \langle C(x),\ \varphi\rangle - \langle A,\ \varphi\rangle - \tfrac{1}{2}\langle\varphi,\ B\varphi\rangle$$

and dual extremum principles exist if W is convex in u and concave in φ. Now,

$$\frac{\partial W}{\partial u} = u, \text{ i.e. } \frac{\partial^2 W}{\partial u^2} = 1.$$

Hence W is convex in u. Also

$$\frac{\partial W}{\partial\varphi} = C(x) - A - B\varphi$$

i.e. $\dfrac{\partial^2 W}{\partial\varphi^2} = -B$. Hence W is concave in φ if $B \geqslant 0$.

To obtain variational principles associated with the boundary value problem described by equations (5), (6) and (7), we introduce the action functional (see Arthurs, 1970)

$$I(U,\Phi) = (U, T\Phi)_v - W(U,\Phi)_v - (\varphi_B, \sigma\Phi)_{\partial v} =$$

$$= \langle T^*U, \Phi\rangle - W(U,\Phi)_v + \langle\sigma^*(U, u_B),\ \Phi\rangle_{\partial v}.$$

So that

$$\frac{\partial I}{\partial U} = 0 \Rightarrow T\Phi = \frac{\partial W}{\partial U} \text{ in } V \text{ and } \frac{\partial I}{\partial\Phi} = 0 \Rightarrow T^*U = \frac{\partial W}{\partial\Phi} \text{ in } V$$

and

$$\sigma^*U = \sigma^*u_B \text{ on } \partial v$$

Where,

$$I(U,\Phi) \text{ is stationary at } (u,\varphi).$$

§3. **Dual Extremum Principles.** (a) **Maximum Principle.**
We define $J(\Phi) = I(U(\Phi), \Phi)$ where
$U(\Phi)$ is a solution of equation (5) and
$J(\Phi)$ is stationary at φ. Then

$$J(\Phi) = (T\Phi,\ T\Phi) - W(U(\Phi),\Phi) - (u_B, \sigma\Phi)_{\partial v} = \tfrac{1}{2}(T\Phi, T\Phi) -$$

$$\langle C(x), \Phi\rangle_v + \langle A,\ \Phi\rangle_v + \tfrac{1}{2}\langle\Phi,\ B\Phi\rangle_v - (u_B, \sigma\Phi)_{\partial v}$$

(8) $$= \int_{-1}^{1}\left\{\tfrac{1}{2}(1-x^2)\left(\frac{d\Phi}{dx}\right)^2 - C(x)\Phi + A\Phi + \tfrac{1}{2}B\Phi^2\right\}dx$$

(8 a) $$= \tfrac{1}{2}\int_{-1}^{1}\left\{(1-x^2)\left(\frac{d\Phi}{dx}\right)^2 - 2C(x)\Phi + 2A\Phi + B\Phi^2\right\}dx,$$

where $I(u,\varphi) \leqslant J(\Phi)$.

(b) **Minimum Principle.**
We also define $G(U) = I(U, \Phi(U))$
where $\Phi(U)$ is a solution of equation (6) and $G(U)$ is stationary at u. Then

$$G(U) = \langle T^*U,\ \frac{1}{B}(C(x) - A - T^*U)\rangle_v - \tfrac{1}{2}(U,U) -$$

$$\langle C(x),\ \frac{1}{B}(C(x) - A - T^*U)\rangle_v + \langle A, \frac{1}{B}(C(x) - A - T^*U),\ \rangle_v +$$

$$+\tfrac{1}{2}\langle\frac{1}{B}(C(x) - A - T^*U),\ \frac{1}{B}(C(x) - A - T^*U)\rangle_v + \langle\sigma^*(U -$$

$$-u_B)_B,\ \Phi\rangle_{\partial v} = -\tfrac{1}{2}(U,U) - \tfrac{1}{2}\langle C(x) - A - T^*U,\ \frac{1}{B}(C(x) - A - T^*U)\rangle$$

(9) $$= \tfrac{1}{2}\int_{-1}^{1}\left\{U^2 + \frac{1}{B}(C(x) - A - T^*U)^2\right\}dx.$$

where $G(U) \leqslant I(u,\varphi)$. We further choose $U = T\Phi$ so that Ψ approximates to φ. Then

(9 a) $$G(T\Psi) = -\tfrac{1}{2}\int_{-1}^{1}\left\{(1-x^2)\left(\frac{\partial\Psi}{\partial x}\right)^2 + \frac{1}{B}\left(C(x) - A + \frac{d}{dx}\left[(1-x^2)\left(\frac{d\Psi}{dx}\right)^2\right]\right]\right\}dx.$$

§4. **A Suitable Trial Function.** From (9)

$$G(T\Psi) = -\tfrac{1}{2}\int_{-1}^{1}\left[(T\Psi)^2 + \frac{1}{B}(C(x) - A - T^*T\Psi)^2\right]dx =$$

$$= -\tfrac{1}{2}\int_{-1}^{1}[T\Psi, C(x) - A - T^*T\Psi]\begin{bmatrix}1 & 0\\ 0 & 1/B\end{bmatrix}\begin{bmatrix}T\Psi\\ C(x) - A - T^*T\Psi\end{bmatrix}dx$$

$$= -\tfrac{1}{2}\int_{-1}^{1}[T\Psi, C(x) - A - T^*T\Psi]Q\begin{bmatrix}T\Psi\\ c(x) - A - T^*T\Psi\end{bmatrix}dx.$$

On defining the matrix operator Q as

$$Q = \begin{bmatrix}1 & 0\\ 0 & 1/B\end{bmatrix},$$

the greatest lower bound of $I(U\ \Phi)$, denoted by $m_{I(U,\Phi)}$ is approximately equal to the smallest eigenvalue of the matrix operator Q, designated by m_Q (see I b i e j u g b a, 1980), where

(10) $$m_Q = 1/B$$

Similarly

$$J(\Phi) = \int_{-1}^{1}\left[\tfrac{1}{2}(1-x^2)\left(\frac{d\Phi}{dx}\right)^2 - 2(C(x) - A)\Phi + \tfrac{1}{2}B\Phi^2\right]dx$$

Let $N(x) = \Phi$, then

$$J(\Phi) = \int_{-1}^{1}\left[\tfrac{1}{2}(1-x^2)\left(\frac{d\Phi}{dx}\right)^2 - \frac{2(C(x) - A)\dot{\Phi}\,\Phi}{N(x)} + \tfrac{1}{2}B\Phi^2\right]dx =$$

$$= \int_{-1}^{1}[\Phi, \dot{\Phi}]\begin{bmatrix}\tfrac{1}{2}B & -\dfrac{C(x) - A}{N(x)}\\ -\dfrac{(C(x) - A)}{N(x)} & \tfrac{1}{2}(1-x^2)\end{bmatrix}\begin{bmatrix}\Phi\\ \dot{\Phi}\end{bmatrix}dx$$

$$= J(\Phi) = \int_{-1}^{1}[\Phi, \dot{\Phi}]\,\rho\begin{bmatrix}\Phi\\ \dot{\Phi}\end{bmatrix}dx.$$

Similarly, the least upper bound of $I(U, \Phi)$ denoted by $M_{I(U,\Phi)}$ is approximately given as the largest eigenvalue of the matrix operator ρ which we denote by M_ρ (see I b i e j u g b a, 1980).

But

(11) $$M_\rho = \tfrac{1}{2}\left[\frac{B}{2} + \frac{1-x^2}{2} + \sqrt{\left(\frac{(B+1-x^2)^2}{2}\right) - 4\left(\frac{B}{4}(1-x^2) - \frac{(C(x) - A)^2}{N^2(x)}\right)}\right].$$

We then equate (10) and (11) to make the bounds equal. That is

$$\tfrac{1}{2}\left[\frac{B}{2} + \frac{1-x^2}{2} + \sqrt{\frac{(B+1-x^2)^2}{2} - 4\left(\frac{B}{4}(1-x^2) - \frac{(C(x) - A)^2}{(N(x))^2}\right)}\right] = 1/B.$$

Where from:

$$N(x) = \Phi \simeq 2\,\frac{(C(x) - A)}{\sqrt{B}}\quad \text{i.e. } \Phi \simeq \frac{2x(C(x) - A)}{\sqrt{B}}$$

gives a suitable trial function.

§5. **Numerical Computation.** We now proceed to use this function to find an approximate form for the mean surface temperature in the Northern Hemisphere. The values of the constituents parts; C, B, and A (the radiation flux from the atmosphere into space) have all been evaluated from S m a g o r i n s k y's (1963) formulation and has been successfully employed in setting up several atmospheric general circulation models (eg. O g u n m o y e l a and E v e r s o n (1979). The values are displayed in Table 1 and the resulting pattern of Φ is displayed in Fig. 1. The form compares very well with observation except at the polar latitudes where there is, from about 70° N, a steady increase in temperature.

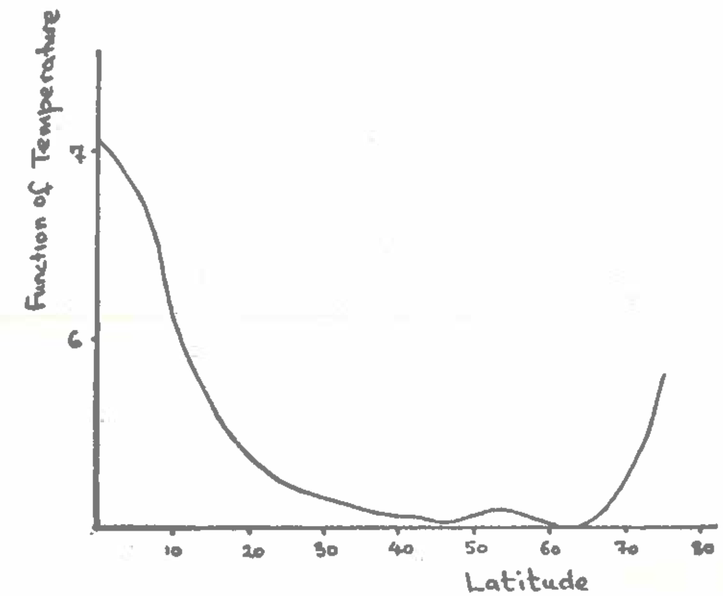

Fig. 1

TABLE 1

The constituent terms terms of the formulation ; A in 10^{-2} cal/day/cm^2 ; B in cal/day/cm^2/^{0} C^{-1}

X	C	A	B	Φ
0.0872	6 140	539	204	6.8
0.1737	6 031	526	989	6.1
0.2588	5 915	508	2 460	5.6
0.3420	5 734	485	4 438	5.4
0.4226	5 492	452	6 656	5.2
0.5000	5 287	417	8 911	5.2
0.5736	5 028	380	10 951	5.1
0.6428	4 825	341	12 622	5.1
0.7071	4 363	295	13 423	5.0
0.7660	4 055	248	13 431	5.0
0.8192	3 718	207	12 966	0.1
0.8660	3 270	164	11 577	5.1
0.9063	2 828	123	9 622	5.0
0.9397	2 375	87	7 381	5.0
0.9659	1 996	57	5 187	5.2
	1 744	35	3 358	5.8

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SEMANTICS IN COMPLETE LATTICES FOR RELATIONAL DATABASE FUNCTIONAL DEPENDENCIES

BY

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§ 1. Functional dependencies were introduced in relational database analisis and design by Codd [1]. The functional dependency calculus (*FD* calculus) as an abstract manipulation of functional dependencies was axiomatized by Armstrong [2] and Beeri et al. [3]. The surprising relationships existing between the *FD* calculus and the study of certain particular boolean functions associated to the functional dependencies first appears in Delobel and Casey [4] and is further cleared in Fagin [5].

Like the propositional calculus the *FD* calculus raises two problems : a proof or deduction problem of syntactical nature and a truth or validity problem of semantic nature.

Let U be a finite set whose elements are called *symbolic attributes* (this is the syntactic appearance of the concept of attribute). By *set of attributes* we mean a subset of U. A functional dependency statement (*FD* statement) between two sets of attributes is simply a pair (X, Y) of subsets of U, that we write $X \leftarrow Y$ and read „X depends on Y". The deductions in *FD* calculus are based on the following axioms (the set $\{A\} = A$ is denoted simply by A) :

A1. $\forall A \in X \subseteq U,\ A \leftarrow X$;

A2. $\forall X, Y, Z \subseteq U$, *from* $X \leftarrow Y$ *and* $Y \leftarrow Z$ *deduce* $X \leftarrow Z$;

A3. $\forall X, Y \subseteq U$, *from* $A \leftarrow Y$ *for all* $A \in X$ *deduce* $X \leftarrow Y$.

Note that only *AI* is an axiom scheme while A 2 and $A3$ are inference rules.

Given a set π of functional dependencies called premises, a *deduction* from π is a finite sequence of *FD* statements so that each of them is either a premise or an axiom from $A1$, or can be inferred using $A2$ or $A3$ from some of the *FD* statements preceding it in the sequence. We say that the last *FD* statement in the sequence may be deduced from premises π (or that π syntactically implies it). Denote by Ded (π) the set of *FD*'s that can be deduced from π. Note that $A3$ works as an inferrence rule only because U is finite. In order to check for the validity of a *FD* we use two kinds of models for U : table-like models and 2-value models.

A table-like model φ consists of a set of *entries* (rows), E, and an interpretation that maps every symbolic attribute A into a *table attribute* (column)