The following theorem gives an example of a space X which satisfies the hypotheses of Theorem 6.

Theorem 8. *If X is a linear topological space of the second category, then every family of lower semicontinuous pseudonorms, $Q = \{q_j\}_{j\in J}$, is a family of pseudonorms continuous in the origin.*

Proof. Let $\mathcal{P} = \{p_i\}_{i\in I}$ be the family of pseudonorms which defines the topology of X. For every $j \in J$ and $n \in N$ we denote the set $X_n^j = \{x; q_j(x) \leqslant n\}$, which is nonvoid. For $j \in J$ we can write $X = \bigcup_{n\in N} X_n^j$. On the other hand, for every $j \in J$, there exists $j_1 \in J$, so that

$$q_j(x+y) \leqslant q_{j_1}(x) + q_{j_1}(y) \text{ for all } x,\ y \in X. \tag{4.2}$$

Since X is of the second category, it follows that there exists $k \in N$, such that $X_k^{j_1}$ contains an open ball $S_i(x_0, \delta)$. We get $p_i(x - x_0) < \delta$ implies $q_{j_1}(x) \leqslant k$, hence

$$p_i(x) < \delta \text{ implies } q_{j_1}(x + x_0) \leqslant k. \tag{4.3}$$

From (4.2) and (4.3) we have

$$p_i(x) < \delta \textit{ implies } q_j(x) \leqslant k + q_{j_1}(x_0) = M_j, \tag{4.4}$$

where M_j depends on j and is independent on x_0. If $p_i(x) \neq 0$, then $x' = (\delta/2)x/p_i(x)$ satisfies $p_i(x') < \delta$, and from (4.3) we have $q_j(\delta/2)\, x/p_i(x) \leqslant M_j$ that is $q_j(x) \leqslant (2/\delta) M_j p_i(x)$. If $p_i(x) = 0$, it follows that for every $\varepsilon > 0$, we have $p_i(x/\varepsilon) = 0$ and from (4.5) we obtain $q_j(x) \leqslant \varepsilon M_j$. This fact implies $q_j(x) = 0$. Denoting $L_j = (2/\delta) M_j$, it follows that for every $j \in J$, there exists $i \in I$ and $L_j > 0$, such that $q_j(x) \leqslant L_j p_i(x)$ for all $x \in X$. This implies that q_j is continuous in the origin, as claimed.

REFERENCES

1. Heyers D. H. — *Linear topological spaces*, Bull. Amer. Math. Soc., 51, (1945), 1—21.
2. Marinescu G. — *Spaţii vectoriale topologice şi pseudotopologice*, Editura Academiei, Bucureşti, 1959.
3. Marinescu G. — *Tratat de analiză funcţională*, Vol. 1, Editura Academiei, Bucureşti, 1970.
4. Gheorghiu N. — *Introducere în analiza funcţională*, Editura Academiei, Bucureşti, 1974.
5. Garnir H. G., De Wilde M., Schemets J. — *Analyse fonctionelle*, 1, Birkhäuser, 1968.
6. Köthe G. — *Topologische lineare Räume*, 1. Springer-Verlag, Berlin, 1960.
7. Jarchow H. — *Marinescu Räume*, Comment. Math. Hel., 138—163 (1969).
8. Jarchow H. — *Dualität und Marinescu-Räume*, Math. Annalen, 182, 134—144 (1969).
9. Cârjă O. — *A pseudotopological structure for $L(X, Y)$, with X, Y linear topological spaces*, Rev. Roum. Math. Pures et Appl, XXV, (1980), 1311—1315.

Received 20.XII.1980

Faculty of Mathematics
University of Iaşi
6600—Iaşi
R. S. România

Analele ştiinţifice ale Universităţii „Al. I. Cuza" din Iaşi
Tomul XXVIII, s. I a, 1982, f. 1

A GENERALIZATION OF BRÉZIS-BROWDER'S ORDERING PRINCIPLE

BY

MIHAI TURINICI

Concerning maximal elements in an abstract ordered set, a fundamental result obtained in the last years is, undoubtedly, the so-called Brézis-Browder's ordering principle [5] that may be considered as a countable version of the classical Zorn's theorem [19, p. 33]. Taking into account the great importance of this principle to a series of questions pertaining to nonlinear functional analysis (see, e.g., I. Ekeland [16] as a quasi-complete reference on this topic) an extension of it may therefore be of interest. It is the main aim of the present note to demonstrate firstly that Brézis-Browder's ordering principle may be expressed essentially as a quasi-metric maximality result and secondly that — from this new viewpoint — it can be effectively included into a general maximality principle on quasi-metric spaces comparable with a similar author's one [29]. A number of applications, especially concerning mean value theorems in Dieudonné's sense [11, ch. VIII, §5] as well as differential inequalities in Bebernes-Meister's sense [2] will be given in a forthcoming paper.

Let X be an abstract (nonempty) set and let $\leqslant$ be a *quasi-ordering* on X. For any (nonempty) subset Y of X and any $x \in Y$, $Y(x, \leq)$ will denote the subset of all $z \in Y$ with $x \leqslant z$. A sequence $(x_n\,;\, n \in N)$ in X will be called *monotone* iff $x_i \leqslant x_j$ whenever $i \leqslant j$, and *bounded above* iff $x_n \leqslant y$, all $n \in N$, for some $y \in X$ termed an *upper bound* of this sequence. Furthermore, suppose d is a *quasi-metric* on X. A sequence $(x_n\,;\, n \in N)$ in X will be called *quasi-asymptotic* iff for any $\varepsilon > 0$ there exists $n = n(\varepsilon) \in N$ with $d(x_n, x_{n+1}) < \varepsilon$; also, an element $z \in X$ is said to be *d-maximal* iff $z \leqslant y$ implies $d(z, y) = 0$. A satisfactory motivation for introducing these notions will be offered later; for the moment, we are only interested to state and prove the main result of the present note, a result that may be formulated as follows:

Theorem 1. *Suppose the quasi-metric space (X, d) and the ordering $\leqslant$ on X are such that:*

(i) *any monotone sequence in X is a quasi-asymptotic one,*
(ii) *any monotone Cauchy sequence in X has an upper bound.*

Then, for every $x \in X$, there is a d-maximal element $z \in X$ with $x \leqslant z$.

Proof. Firstly, we claim any $x \in X$ has the property:

(I) *for every $\varepsilon > 0$ there exists $y \geqslant x$ with $d(y, z) < \varepsilon$, for all elements $z \geqslant y$.*

Indeed, suppose (1) were not true; then, there must be a number $\varepsilon>0$ such that, for any $y\geqslant x$ there exists $z\geqslant y$ with $d(y, z)\geqslant\varepsilon$. It immediately follows that a monotone sequence $(y_n ; n\in N)$ in $X(x, \leqslant)$ may be found with $d(y_n, y_{n+1})\geqslant\varepsilon$, all $n\in N$, contradicting (i) and proving our claim. In this case, given $x\in X$, a monotone sequence $(x_n ; n\in N)$ in $X(x, \leqslant)$ may be constructed with

$$n\in N,\ y\in X,\ x_n\leqslant y \textit{ imply } d(x_n, y)<(1/2)^n. \tag{2}$$

By (2), $(x_n ; n\in N)$ appears as a monotone Cauchy sequence so that, by (ii), $x_n\leqslant z$, all $n\in N$, for some $z\in X$. Clearly, $x_n\to z$. Now, suppose $y\in X$ is such that $z\leqslant y$; then, $x_n\leqslant y$, all $n\in N$ so that, by (2), $x_n\to y$ and this gives $d(z, y)=0$. Consequently, z satisfies all the requirements of our theorem and the proof is complete. Q.E.D.

Our main intention is now to express the above maximality principle on quasi-metric spaces as an „abstract" one (from which the Brézis-Browder's ordering principle will follow as a simple corollary). To this end, suppose $(X, \leqslant)$ and $(Y, \leqslant)$ are quasi-ordered sets, e a metric on Y and φ a mapping from X into Y. In such a case, as a first variant of Theorem 1 we have

Theorem 2. *Suppose the above elements are such that*

(iii) *any monotone sequence in X is bounded above,*

(iv) *φ is increasing ($x\leqslant y$ implies $\varphi(x)\leqslant\varphi(y)$),*

(v) *any monotone sequence in $\varphi(X)$ is a quasi-asymptotic one.*

Then, for every $x\in X$ there is an element $z\in X$ with $x\leqslant z$ and, in addition, $z\leqslant y$ implies $\varphi(z)=\varphi(y)$.

Proof. Let us define a quasi-metric d on X by the convention

$$d(x, y)=e(\varphi(x), \varphi(y)),\ x, y\in X. \tag{3}$$

Firstly, by (iii), any monotone Cauchy sequence in X is, necessarily, bounded above that is, (ii) holds. Furthermore, $(x_n ; n\in N)$ being a monotone sequence in X, $(\varphi(x_n) ; n\in N)$ is, by (iv), a monotone sequence in $\varphi(X)$ hence, by (v), a quasi-asymptotic one i.e., (i) holds too. Consequently, Theorem 1 applies and this will complete the proof. Q.E.D.

Under the same general conventions, let $\mathfrak{U}$ be a uniformity on X and call an element $z\in X$, *$\mathfrak{U}$-maximal* if and only if $z\leqslant y$ implies $(z, y)\in U$ for all $U\in\mathfrak{U}$. Then, as a second variant of Theorem 1 we have

Theorem 2'. *Suppose conditions* (iv)+(v) *as well as*

$(\mathrm{vi})_1$ *$X(x, \leqslant)$ is $\mathfrak{U}$-complete (modulo $\leqslant$) for every $x\in X$,*

$(\mathrm{vi})_2$ *for any $U\in\mathfrak{U}$ there exists $\varepsilon>0$ such that $x\leqslant y$ and*

$$e(\varphi(x), \varphi(y))<\varepsilon \textit{ imply } (x, y)\in U$$

hold. Then, for every $x\in X$ there is a $\mathfrak{U}$-maximal element $z\in X$ with $x\leqslant z$.

Proof. Let the quasi-metric d on X be defined as in Theorem 2. It is clear by (iv)+(v) that any monotone sequence in X is a quasi-asymptotic one, that is, (i) holds. On the other hand, let $(x_n ; n\in N)$ be a monotone Cauchy sequence in X (in the sense of our quasi-metric). Taking any $U\in\mathfrak{U}$ and letting $\varepsilon>0$ be defined by $(\mathrm{vi})_2$ there is, by the choice of our sequence, a $n=n(\varepsilon)\in N$ such that $d(x_p, x_q)<\varepsilon$, all $p, q\in N$, $n\leqslant p\leqslant q$ that is, $e(\varphi(x_p), \varphi(x_q))<\varepsilon$, all $p, q\in N$, $n\leqslant p\leqslant q$, which implies by $(\mathrm{vi})_2$ again, $(x_p, x_q)\in U$, all $p, q\in N$, $n\leqslant p\leqslant q$. So, $(x_n; n\in N)$ is a monotone Cauchy sequence (in the sense of our uniformity) and by $(\mathrm{vi})_1$, $x_n\to x$ for some $x\in X$ which satisfies also $x_n\leqslant x$, all $n\in N$. Hence, (ii) holds too and the main result applies. Q.E.D.

Now, let us take in the above results $Y=R$, $e=$the usual distance in R, and $\leqslant$ the usual dual ordering in R then, clearly, a necessary and sufficient condition guaranteeing the validity of (v) is

(v)' *$\varphi(X)$ is bounded below in R (i.e., φ is bounded bellow on X).*

In which case, Theorem 2 reduces to Brézis-Browder's ordering principle quoted in the introductory part of the present note, while Theorem 2' to Brøndsted's maximality principle [6]. Moreover, it was shown in Brézis-Browder's as well as Brøndsted's papers that their fundamental result may be regarded as a considerable extension of an important variational type result due to I. Ekeland [14], [15] (see also J. P. Aubin and J. Siegel [1] as well as E. Bishop and R. R. Phelps [3]) or, equivalently – after a pattern discovered by N. Bourbaki [4] and refined by A. Brøndsted [7] and I. Ekeland [16] – of a fixed point result known as Caristi-Kirk theorem [10], [20] (see also F. E. Browder [9], as well as C. S. Wong [33]) and therefore, the above theorems may be also considered as extending all these maximality results.

A close analysis of the main result just presented shows the boundedness property imposed in (ii) is, in fact, not intimately related to the notion of Cauchy sequence so that, it seems to be natural to replace it by a more appropriate property such as convergence. Before doing that, we need, however, a number of new notational conventions. Let X, $\leqslant$, and d be as before. A subset Y of X will be termed *order-closed* iff for any monotone sequence $(x_n ; n\in N)$ in Y and any $x\in X$ with $x_n\to x$ we have $x\in Y$; in this context, the considered quasi-ordering $\leqslant$ on X is said to be *self-closed* iff $X(x, \leqslant)$ is order-closed for all $x\in X$. Also, the ambient quasi-metric space (X, d) will be termed *order-complete* iff any monotone Cauchy sequence in X is a convergent one. Now, as another useful variant of the main result, we have

Theorem 3. *Suppose the elements X, $\leqslant$ and d are such that* (i) *plus*

(vii) *$\leqslant$ is a self-closed quasi-ordering,*

(viii) *(X, d) is order complete*

hold. Then, for every $x\in X$ there is a d-maximal element $z\in X$ with $x\leqslant z$.

Proof. Let $(x_n ; n\in N)$ be a monotone Cauchy sequence in X. By (viii), $x_n\to x$ for some $x\in X$ which gives, by (vii), $x_n\leqslant x$, all $n\in N$ proving x is an upper bound of this sequence. Consequently, the main result applies and the proof is complete. Q.E.D.

A partial indication concerning the power of the above maximality principle follows from the considerations below. Firstly, under the same context, let us call a sequence $(x_n ; n\in N)$ in X *asymptotic* iff $d(x_n, x_{n+1})\to 0$ and the ambient quasi-ordering $\leqslant$ *semi-closed* in Nachbin's sense [23, p. 100] iff $X(x,\leqslant)$ is closed for all $x\in X$. In such a situation, if we suppose condition (i) is replaced by the stronger one

(i)′ *any monotone sequence in X is an asymptotic one*,
and/or condition (vii) by
(vii)′ *is a semi-closed ordering on X.*

Then, the corresponding variant of Theorem 3 appears as a quasi-metric extension of a number of similar author's maximality principles [27], [28], [30]. Secondly, X, $\leqslant$, and d being the same as before, a function $f: X \to R$ is said to be *order-lsc* (*order-usc*) iff for every $t \in R$, the subset of all elements $x \in X$ with $f(x) \leqslant t$ ($f(x) \geqslant t$) is order-closed. Of course, any *lsc* (*usc*) function is, in the same time, order-lsc (order-usc); on the other hand, if we suppose $\leqslant$ is a self-closed quasi-ordering on X and f *decreasing* (*increasing*) on X then, necessarily f is order-lsc (order usc), showing the notion of order-lsc (order-usc) function is effectively more general than the corresponding notion of lsc (usc) function. Now, as an interesting application of Theorem 3, the following result holds.

Theorem 4. *Under the same conditions* (i) + (vii) + (viii) *of Theorem 3, suppose f and g are functions from X into R satisfying*

(ix) *f is order-lsc and g is order-usc*

(x) *the subset $X(f, g)$ of all $x \in X$ with $f(x) \leqslant g(x)$ is not empty.*

Then, for every $x \in X(f, g)$ there is an element $z \in X(f, g)$ with $x \leqslant z$ and, moreover, for every $y \in X$ with $d(z, y) > 0$, either $z \leqslant y$ does not hold or $f(y) > g(y)$ in case $z \leqslant y$.

Proof. Let γ denote the characteristic function of the subset $X(f, g)$. Define a new quasi-ordering $\leqq$ on X by the convention

$$x \leqq y \text{ iff } x \leqslant y \text{ and } \gamma(x) \leqslant \gamma(y). \tag{4}$$

Since $\leqq$ is finer than $\leqslant$ (that is, $x \leqq y$ implies $x \leqslant y$) conditions (i) + (viii) are automatically fulfilled with $\leqslant$ replaced by $\leqq$. Moreover, let the elements $x, y \in X$ and the sequence $(y_n;\ n \in N)$ in X be such that

$$x \leqq y_n,\ n \in N\ ;\ y_n \leqq y_m,\ n \leqslant m\ ;\ y_n \to y \tag{5}$$

As a first consequence of (5) $x \leqslant y_n$, $n \in N$, $y_n \leqslant y_m$, $n \leqslant m$, $y_n \to y$ so that, by (vii), $x \leqslant y$. In the same time, a second consequence of (5) is, evidently, $\gamma(x) \leqslant \gamma(y_n)$, $n \in N$, $\gamma(y_n) \leqslant \gamma(y_m)$, $n \leqslant m$. Without loss of generality suppose $\gamma(x) = 1$ then, by the above relations, $\gamma(y_n) = 1$, all $n \in N$ so, by (ix) combined with the last part of (5), $\gamma(y) = 1$ and this immediately gives $x \leqq y$ proving (vii) also holds, when $\leqslant$ is replaced by $\leqq$. Hence Theorem 3 is applicable and this ends the proof. Q.E.D.

Concerning the above particularization of Theorem 3, it should be noted that it may be considered as a general result about abstract inequalities, largely used in the theory of differential inequalities. Finally, X, $\leqslant$, and d being endowed with the same meaning, let (X', d') be another quasi-metric space. A mapping $T: X \to X'$ is said to be *order-closed* provided that, for any monotone sequence $(x_n;\ n \in N)$ in X and any couple $x \in X$, $x' \in X'$ with $x_n \to x$ and $Tx_n \to x'$ we have $Tx = x'$. Evidently, any *closed* mapping $T: X \to X'$ (in a sense precised, e.g., by D. D o w n i n g and W. A. K i r k [13]) is necessarily order-closed in the above sense; on the other hand, if we suppose (X', d') is a metric space and the mapping $T: X \to X'$ is „*left-continuous*" on X (that is, for any monotone sequence $(x_n;\ n \in N)$ in X and any $x \in X$ with $x_n \to x$ we have $Tx_n \to Tx$ then, evidently, T is order-closed, proving the notion of order-closed mapping is effectively more general than the notion of closed mapping. In this case, as another interesting application of Theorem 3, we have

Theorem 5. *Under the same basic hypotheses* (i) + (vii) + (viii), *suppose the complete quasi-metric space (X', d'), the mapping $T: X \to X'$ and the function $f: X \to R$ are such that*

(xi) *T is order closed on X,*

(xii) *f is order-lsc and bounded below on X.*

Then, for every $x \in X$ there is an element $z \in X$ with $x \leqslant z$, $d'(Tx, Tz) \leqslant f(x) - f(z)$ and, moreover, for any $y \in X$ with $d(z, y) > 0$, either $z \leqslant y$ does not hold or $d'(Tz, Ty) > f(z) - f(y)$ in case $z \leqslant y$.

Proof. Let us define a new quasi-ordering $\leqq$ on X by

$$x \leqq y \text{ iff } x \leqslant y \text{ and } d'(Tx, Ty) \leqslant f(x) - f(y) \tag{6}$$

As in the preceding theorem, we have only to verify condition (vii) holds when $\leqslant$ is replaced by $\leqq$. To this end, let the elements $x, y \in X$ and the sequence $(y_n;\ n \in N)$ in X be such that (5) holds with $\leqq$ defined by (6). Clearly, $x \leqslant y$. On the other hand, again by (5), we derive

$$d'(Tx, Ty_n) \leqslant f(x) - f(y_n),\ n \in N,\ d'(Ty_n, Ty_m) \leqslant f(y_n) - f(y_m),\ n \leqslant m \tag{7}$$

From the second part of (7), $(f(y_n);\ n \in N)$ is a decreasing sequence in R hence, by the second part of (xii), a Cauchy sequence. It follows at once $(Ty_n;\ n \in N)$ is also a Cauchy sequence in X' so that, by completeness, $Ty_n \to y'$ for some $y' \in X'$ which in turn implies (by (xi) combined with (5)) $Ty = y'$ that is, $Ty_n \to Ty$. Now, taking into account the first part of (xii) plus (7) we get, by a limit process, $d'(Tx, Ty) \leqslant f(x) - f(y)$ and therefore, $x \leqslant y$, proving our initial assertion. So, Theorem 3 again applies. Q.E.D.

Concerning the elements involved in the above result, it should be noted that in case (X, d) is a complete metric space, an important class of quasi-orderings $\leqq$ on X satisfying (i) + (vii) is that offered by the choice $x \leqslant y$ iff $d(x, y) \leqslant g(x) - g(y)$, g being a function from X into R, lsc and bounded from below in which case, the corresponding variant of Theorem 5 extends a similar author's result [31] or — after the above quoted E k e l a n d-B r ø n d s t e d's pattern — the D o w n i n g-K i r k's fixed point result [13] (see also D. D o w n i n g [12]). Moreover, in case $X = X'$, $d = d'$, $T = I$ (the identity) from the same corresponding variant, we get B r ø n d s t e d's maximality results [6], [8] (see also M. T u r i n i c i [26] and J.D. W e s t o n [32]) respectively the above quoted C a r i s t i-K i r k's fixed point theorems (see also S. K a s a h a r a [18], L. P a s i c k i [24] as well as J. S i e g e l [25] for a number of interesting new points of view in this direction) so, the above theorem appears also as a common extension of all these results.

As a final remark, it should be noted that such a maximality principle seems to be an appropriate instrument in deriving mean value theorems, differential inequalities, mapping theorems in K i r k-C a r i s t i's sense [21], as well as lipschitzianness tests in K i r k-R a y's sense [22] (see also S.A. H u s a i n and V.M. S e h g a l [17]); these aspects will be treated in a forthcoming paper under preparation.

REFERENCES

1. A u b i n J. P. and S i e g e l J. — *Fixed points and stationary points of dissipative multivalued maps*, Proc. Amer. Math. Soc., 78 (1980), 391—398.

2. Bebernes J. and Meisters G. H. — *An elementary proof of some differential inequalities*, J. Math. Anal. Appl., 17 (1967), 92—98.
3. Bishop E. and Phelps R. R. — *The support functionals of a convex set*, Proc. Symp. Pure Math., vol. VII (Convexity), pp. 27—36, Amer. Math. Soc., 1963.
4. Bourbaki N. — *Sur le théorème de Zorn*, Archiv der Math., 2 (1949/50), 434—437.
5. Brézis H. and Browder F. E. — *A general principle on ordered sets in nonlinear functional analysis*, Advances in Math., 21 (1976), 355—364.
6. Brondsted A. — *On a lemma of Bishop and Phelps*, Pacific J. Math., 55 (1974), 335—341.
7. Brondsted A. — *Fixed points and partial orders*, Proc. Amer. Math. Soc., 60 (1976), 365—366.
8. Brondsted A. — *Common fixed points and partial orders*, Proc. Amer. Math. Soc., 77 (1979), 365—368.
9. Browder F. E. — *On a theorem of Caristi and Kirk*, *Fixed Point Theory and its Applications*, pp. 23—27, Academic Press, 1976.
10. Caristi J. — *Fixed point theorems for mappings satisfying inwardness conditions*, Trans. Amer. Math. Soc., 215 (1976), 241—251.
11. Dieudonné J. — *Foundations of modern analysis*, Academic Press, 1960.
12. Downing D. — *Nonlinear mapping theory and the continuation of solutions of equations under homotopy*, J. Nonlinear Analysis, TMA, 4 (1980), 27—30.
13. Downing D. and Kirk W. A. — *A generalization of Caristi's theorem with application to nonlinear mapping theory*, Pacific J. Math., 69 (1977), 339—346.
14. Ekeland I. — *Sur les problèmes variationelles*, C. R. Acad. Sci. Paris, Sér. A, 275 (1972), 1057—1059.
15. Ekeland I. — *On the variational principle*, J. Math. Anal. Appl., 47 (1974), 324—353.
16. Ekeland I. — *Nonconvex minimization problems*, Bull. Amer. Math. Soc., (N.S.), 1 (1979), 443—474.
17. Husain S. A. and Sehgal V. M. — *A remark on a fixed point theorem of Caristi*, Math. Japonica, 25 (1980), 27—30.
18. Kasahara S. — *On fixed points in partially ordered sets and Kirk-Caristi theorem*, Math. Sem. Notes Kobe Univ., 3 (1975), 229—232.
19. Kelley J. — *General topology*, Springer Verlag, 1975.
20. Kirk W. A. — *Caristi's fixed point theorem and metric convexity*, Colloq. Math., 36 (1976), 81—86.
21. Kirk W. A. and Caristi J. — *Mapping theorems in metric and Banach spaces*, Bull. Acad. Pol. Sci., 23 (1975), 891—894.
22. Kirk W. A. and Ray W. O. — *A remark on directional contractions*, Proc. Amer. Math. Soc., 66 (1977), 279—283.
23. Nachbin L. — *Topology and order*, Van Nostrand, 1965.
24. Pasicki L. — *A short proof of the Caristi theorem*, Comm. Math., 20 (1978), 427—428.
25. Siegel J. — *A new proof of Caristi's fixed point theorem*, Proc. Amer. Math. Soc., 66 (1977), 54—56.
26. Turinici M. — *Maximal elements in partially ordered topological spaces and applications*, An. şt. Univ. Iaşi, 24 (1978), 259—264.
27. Turinici M. — *Maximal elements in topological and generalized metric spaces*, Periodica Math. Hung., to appear.
28. Turinici M. — *Maximal elements in ordered topological spaces*, Bull. Greek Math. Soc., 20 (1979), 141—148.
29. Turinici M. — *Maximal elements in a class of order complete metric spaces*, Math. Japonica, 25 (1980), 511—517.
30. Turinici M. — *Quasi-maximal elements in topological spaces*, Bull. Math. Soc. Sci. Math. Roum., to appear.
31. Turinici M. — *Local and global lipschitzian mappings on ordered metric spaces*, Math. Nachrichten, 101 (1981), 101—106.
32. Weston J. D. — *A characterization of metric completeness*, Proc. Amer. Math. Soc., 64 (1977), 186—188.
33. Wong C. S. — *On a fixed point theorem of contractive type*, Proc. Amer. Math. Soc., 57 (1976), 283—284.

Received 23.XII.1980

Seminarul Matematic „Al. Myller"
University of Iaşi
6600—Iaşi, R. S. România

Analele ştiinţifice ale Universităţii „Al. I. Cuza" din Iaşi
Tomul XXVIII, s. I a, 1982, f. 1

FIXED POINT THEOREMS IN UNIFORM SPACES

BY

N. GHEORGHIU

It is the purpose of this paper to point out the fact that in many cases with a fixed point theorem in metric spaces is (rather often) associated, by generalization, a fixed point theorem in uniform spaces.

To illustrate the assertion above, we recall our result given by theorem 1 (see [1], [2], [3]), which represents the natural generalization to uniform spaces of the famous Banach theorem on contraction mappings in metric spaces.

Furthermore, a generalization of Maia's theorem ([4], [3], Theorem 2) in uniform spaces endowed with two uniform structures is given. Finally, by theorem 3 we generalize the theorem of Kannan from metric spaces [5], [3]. This chapter of fixed point theorem in uniform spaces is little developed. It is important since any structure of a locally convex space (or even of topological vector space) is a uniform structure.

Theorem 1. *Let X be a sequentially complete, Hausdorff separated, uniform space. Let also $\{\rho_\alpha\}_{\alpha\in A}$ be a separated, saturated set of pseudometrics defining the uniform structure of X. If the function $f: X\to X$ is contractive with respect to the set $\{\rho_\alpha\}_{\alpha\in A}$, then f has a unique fixed point which can be found by the method of successive approximations (starting from any element of X).*

Remark 1. By f-contraction with respect to $\{\rho_\alpha\}_{\alpha\in A}$, we mean that there exist a function $\varphi: A\to A$ and the positive numbers $\lambda_\alpha>0$ such that

$$\rho_\alpha(f(x), f(y))\leqslant\lambda_\alpha\rho_{\varphi(\alpha)}(x, y),\quad \forall\alpha\in A \text{ and } x,y\in X \tag{1}$$

and the series

$$\rho_\alpha(x, y)+\sum_{n=1}^{\infty}\lambda_\alpha\lambda_{\varphi(\alpha)}\dots\lambda_{\varphi^{n-1}(\alpha)}\rho_{\varphi^n(\alpha)}(x, y) \tag{2}$$

are convergent for each $\alpha\in A$ and x, $y\in X$. Here φ^n is the n-th iteration of φ.

Remark 2. The proof of theorem 1 as well as some applications can be found in [1], [2].

Theorem 2. *Let X be a set endowed with two Hausdorff separated uniform structures σ, τ. Let $\mathcal{P}=\{p_\alpha\}_{\alpha\in A}$, $Q=\{q_\beta\}_{\beta\in B})$ be a separated saturated set of pseudo-matrics which defines the structure $\sigma(\tau$, respectively).*

If $f: X\to X$ is a function satisfying the conditions:

I. *There is a function $\varphi: A\to B$ and the real numbers $m_\alpha>0$ such that*

$$p_\alpha(x, y)\leqslant m_\alpha q_{\varphi(\alpha)}(x, y),\quad \forall\alpha\in A,\ x, y\in X\ ;$$