II. (X, σ) is a sequentially complete uniform space ;
III. $f:(X, \sigma) \to (X, \tau)$ is continuous ;
IV. f is Q-contractive (in the sense of remark 1).
then f has a unique fixed point which can be obtained by successive approximations (starting from any element of X).

Proof. If one takes a sequence $\mathcal{S}$ of successive approximations, one proves (as in [1], [3]) that $\mathcal{S}$ is a Cauchy sequence in (X, τ). By I, $\mathcal{S}$ is also a Cauchy sequence in (X, σ), hence it is convergent (in vertue of II). The limit of this sequence is a fixed point of f (since f is continuous, as indicated by III). The uniqueness follows from IV as in [1], [3].

Remark 3. In the particular case in which both $\mathcal{P}$ and Q are made up by a single element, one obtains M a i a 's theorem.

Theorem 3. *Let X be a uniform space satisfying the properties of theorem 1 and let $f: X \to X$ be a function. If there exist real numbers λ_α (with $0 < \lambda_\alpha < 1/2$) such that*

$$\rho_\alpha(f(x), f(y)) \leqslant \lambda_\alpha[\rho_\alpha(x, f(x)) + \rho_\alpha(y, f(y))] \tag{3}$$

for all $\alpha \in A$ and $x, y \in X$, then f has a unique fixed point, which can be obtained via the method of successive approximations (starting from any element of X).

Proof. If one takes a sequence $\mathcal{S}$ of successive approximations, then by making use of (3) (as in [5]) one shows that $\mathcal{S}$ is a Cauchy sequence (for each fixed $\alpha \in A$), hence it is convergent to an element $x \in X$. It follows from (3) (as in [5]) that $\rho_\alpha(x, f(x)) = 0$ for each $\alpha \in A$ (i.e. $x = f(x)$) hence the existence of the fixed point is proved. The uniqueness can be proved as in [5] (observing that if x' and x'' are two fixed points then $\rho_\alpha(x', x'') = 0$ for each $\alpha \in A$).

Remark 4. In the case in which the uniform structure of X is given by a single distance one obtains Kannans theorem.

Remark 5. In general, with each problem related to fixed point theorems in metric spaces is associated a similar problem (related to fixed point theorems) in uniform spaces.

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DISTRIBUTIONS FOR SET FUNCTIONS

BY

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In the Schwartz theory of distributions, every distribution is a linear continuous functional on the strict inductive limit of the spaces $\mathcal{D}_K(\Omega)$, where Ω is a domain of R^n, K ranges over all compact subsets of Ω and $\mathcal{D}_K(\Omega)$ is the set of all functions $f \in C_0^\infty(\Omega)$ such that $\mathrm{supp}(f) \subseteq K$ (see [1], p. 19).

In this paper we extend the notion of distribution to set functions.

We shall consider $\Omega = (-\infty, 0)$, but we shall see that this assumption is not a restriction because our considerations are also valid when Ω is a non-void open set of R^n.

We denote by $C^\infty(\Omega)$ the linear space of all complex-valued functions defined on Ω which have continuous derivatives of every order. For any compact subset $Q \subset \Omega$ and any integer $n \geqslant 1$ we define (1)

$$p_{Q,n}(x) = \sup_{\substack{t \in Q \\ 0 \leqslant k \leqslant n}} |x^{(k)}(t)|, \quad x \in C^\infty(\Omega). \tag{1}$$

The family of the semi-norms of type (1) induces on $C^\infty(\Omega)$ a structure of Hausdorff locally convex space.

We denote by $\mathcal{P}(\Omega)$ the family of all subsets of Ω and for every compact $Q \subset \Omega$ let us consider

$$\mathcal{D}_Q = \left\{ F : \mathcal{P}(\Omega) \to \bar{R} : \textit{there exists } f \in C^\infty(\Omega) \textit{ with } \mathrm{supp}(f) \subseteq Q \right. \tag{2}$$
$$\left. \textit{such that } F(A) = \int_{\bar{A}} f d\mu, \ \forall A \in \mathcal{P}(\Omega) \right\} \text{ where } \mu \text{ is the Lebesgue measure.}$$

Every $\mathcal{D}_Q$ with the topology induced by the semi-norms

$$q_{Q,n}(F) = p_{Q,n}(f), \quad n = 0, 1, 2, \ldots \tag{3}$$

is a Hausdorff locally convex space.

The strict inductive limit of these spaces denoted by

$$\mathcal{D} = \bigcup_{Q \subset \Omega} \mathcal{D}_Q \qquad Q - \textit{a compact} \tag{4}$$

is the space of the test functions on which we shall define the distributions.

We remark that if we replace the family $\mathcal{P}(\Omega)$ by the family $\mathcal{A}$ of measurable subsets of Ω, then we take in (2) $F(A) = \int_A f d\mu$, $\forall A \in \mathcal{A}$.

We now define the basic operations on $\mathcal{D}$, which structures it as a test function space.

1. The derivation. If $F \in \mathcal{D}$ with $F(A)=\int_A f d\mu$, $\forall A \in \mathcal{P}(\Omega)$, then we define *the derivative* of F by

$$\frac{dF}{d\mu}(A)=\int_A f' d\mu, \quad \forall A \in \mathcal{P}(\Omega) \tag{5}$$

We remark that $dF/d\mu \in \mathcal{D}$.

2. The product. For every two functions F and $G \in \mathcal{D}$ with $F(A)=\int_A f d\mu$ and $G(A)=\int_A g d\mu$, $\forall A \in \mathcal{P}(\Omega)$ we define their product by

$$(FG)(A)=\int_A fg d\mu, \quad \forall A \in \mathcal{P}(\Omega). \tag{6}$$

3. The convolution. If $F, G \in \mathcal{D}$ with $F(A)=\int_A f d\mu$ and $G(A)=\int_A g d\mu$, $\forall A \in \mathcal{P}(\Omega)$, then the convolution (denoted by „$*$") of F and G is defined by

$$(F*G)(A)=\int_A f*g d\mu, \quad \forall A \in \mathcal{P}(\Omega). \tag{7}$$

We remark that from (7) and (5) it follows

$$\frac{d(F*G)}{d\mu}(A)=\int_A (f*g)' d\mu=\int_A f'*g d\mu=\int_A f*g' d\mu=\left(\frac{dF}{d\mu}*G\right)(A)=\left(F*\frac{dG}{d\mu}\right)(A)$$

4. The integration. If $f \in \mathcal{D}$ with $F(A)=\int_A f d\mu$, $\forall A \in \mathcal{P}(\Omega)$, we define *the integral* of F by

$$\int_A F' d\mu=\int_A\left(\int_{-\infty}^{t} f(\tau)d\tau\right)dt \tag{8}$$

We observe that

$$\int_A F' d\mu=\int_A\left(\int_{-\infty}^{t} f'(\tau)d\tau\right)dt=\int_A f dt=F(A) \text{ and}$$

$$\int_A FG' d\mu=\int_A\left(\int_{-\infty}^{t} (fg')(\tau)d\tau\right)dt=\int_A (fg)\Big|_{-\infty}^{t} dt-\int_A\left(\int_{-\infty}^{t} (g \cdot f')(\tau)d\tau\right)dt=$$

$$=(F.G)(A)-\int_A F'G d\mu.$$

In the dual space $\mathcal{D}'$ let us consider the following functionals generated by the functions of $H^{\infty}_{\mathrm{loc}}(-\infty, 0)$: if $f \in H^{\infty}_{\mathrm{loc}}(-\infty,0)$ then the functional

$$(\mathcal{F}, G)=\int_{-\infty}^{0}\left(\int_{-\infty}^{t} f(\tau)g(\tau)d\tau\right)dt, \quad \forall G \in \mathcal{D} \text{ with } G(A)=\int_A g d\mu \;\; g \in C_0^{\infty}(\Omega) \tag{9}$$

is a linear continuous functional on $\mathcal{D}$. Indeed, let Q be an arbitrary compact set of Ω. We have

$$\begin{aligned}|(\mathcal{F}, G)|&=\left|\int_{-\infty}^{0}\left(\int_{-\infty}^{t} f(\tau)g(\tau)d\tau\right)dt\right| \leqslant \int_{\inf Q}^{0}\int_{\inf Q}^{t} |f(\tau)g(\tau)d\tau dt \leqslant \\ &\leqslant q_{Q,0}(G)\int_{\inf Q}^{0}\left(\int_{\inf Q}^{0} |f(\tau)|d\tau\right)dt, \; \textit{for all } G \in \mathcal{D}_Q.\end{aligned} \tag{10}$$

Therefore $\mathcal{F}$ is a linear continuous functional on every $\mathcal{D}_Q$ and, because D is the inductive limit of the spaces $\mathcal{D}_Q$, it follows that $\mathcal{F}$ is a linear continuous functional on $\mathcal{D}$.

We call every functional of type (9) on $\mathcal{D}$, *distribution of function type* and we denote the set of these distributions by D'_f.

Two operations can be defined immediately in $\mathcal{D}'_f$.

1. The derivation. If $\mathcal{F} \in \mathcal{D}'_f$ is generated by $f \in H^{\infty}_{\mathrm{loc}}(\Omega)$, then we define the derivative of $\mathcal{F}$ (denoted by $d\mathcal{F}/d\mu$) by

$$\left(\frac{d\mathcal{F}}{d\mu}, G\right)=(\mathcal{F}, -G')+\int_{-\infty}^{0} f(\tau)g(\tau)d\tau \text{ for all } G \in \mathcal{D} \text{ whith} \tag{11}$$

$$G(A)=\int_A g d\mu, \quad \forall A \in \mathcal{P}(\Omega), \; g \in C_0^{\infty}(\Omega).$$

In a similar manner as in (10) we can prove that (11) is a linear continuous functional on $\mathcal{D}$ and we have

$$\begin{aligned}(\mathcal{F}, -G')+\int_{-\infty}^{0} f(\tau)g(\tau)d\tau&=-\int_{-\infty}^{0}\left(\int_{-\infty}^{t} f(\tau)g'(\tau)d\tau\right)dt+ \\ +\int_{-\infty}^{0} f(\tau)g(\tau)d\tau&=-\int_{-\infty}^{0} f(t)g(t)dt+\int_{-\infty}^{0}\left(\int_{-\infty}^{t} f'(\tau)g(\tau)d\tau\right)dt+ \\ +\int_{-\infty}^{0} f(\tau)g(\tau)d\tau&=\int_{-\infty}^{0}\left(\int_{-\infty}^{t} f'(\tau)g(\tau)d\tau\right)dt.\end{aligned} \tag{12}$$

2. The convolution. If $\mathcal{F}$ and $\mathcal{G}$ are two distributions of function type generated by f and g respectively, then, by definition, we call *the convolution* of $\mathcal{F}$ and $\mathcal{G}$ (denoted by $\mathcal{F}*\mathcal{G}$)

$$(13)\qquad \begin{aligned}(\mathcal{F}*\mathcal{G}, H) &= \int_{-\infty}^{0}\left(\int_{-\infty}^{t}(f*g)(\tau)h(\tau)\,d\tau\right)dt \quad \forall H\in\mathcal{D} \text{ with } H(A)=\\ &=\int_{A} h\,d\mu, \quad \forall A\in\mathcal{D}(\Omega),\ h\in C_0^\infty(\Omega).\end{aligned}$$

On $\mathcal{D}'_f$ we consider the topology induced by the family of semi-norms

$$(14)\qquad \rho_{Q,n}=\sup_{q_{Q,n}(G)\neq 0}\frac{\int_{-\infty}^{0}\left(\int_{-\infty}^{t}|f(\tau)g(\tau)|\,d\tau\right)dt}{q_{Q,n}(G)},$$

$$Q\subset\Omega,\ Q - \text{compact } n=0,1,2,\ldots$$

We remark that this topology is stronger than the topology induced by the topology of $\mathcal{D}'$ (considered as dual space of $\mathcal{D}$) on $\mathcal{D}'_f$. We call the completion of the space $\mathcal{D}'_f$ (considered with the topology generated by the semi-norms (14)) *the space of distributions* and we denote this space by $\tilde{\mathcal{D}}'_f$. This is the announced space of distributions. We observe that $\tilde{\mathcal{D}}'_f$ is continuousby embedded in $\mathcal{D}'$.

Remark 1. If $(\mathcal{F}_n)$ is a Cauchy sequence of distributions of function type (every $\mathcal{F}_n$ being generated by $f_n\in H_{\mathrm{loc}}(\Omega)$), then

$$\lim_{n,m\to\infty}\int_{-\infty}^{0}|f_n(\tau)-f_m(\tau)||g(\tau)|\,d\tau=0$$

for every $g\in C^\infty(\Omega)$ with compact support. Indeed, if $(\mathcal{F}_n)$ is a Cauchy sequence, then for every semi-norm $\rho_{Q,k}(\cdot)$ and every $\varepsilon>0$, there exists $n_\varepsilon\in N$ such that

$$(15)\qquad \int_{-\infty}^{0}\left(\int_{-\infty}^{t}|(f_n-f_m)(\tau)||g(\tau)|\,d\tau\right)dt\leqslant\varepsilon\sup_{\substack{t\in Q\\ 0\leqslant j\leqslant k}}|g^{(j)}(t)|$$

for every function $g\in C^\infty(\Omega)$ with compact support contained in Q, $n, m>n_\varepsilon$.

Let us suppose that there exists a number $\varepsilon_0>0$ and a function $g\in C^\infty(\Omega)$ with compact support such that for every $p\in N$, there exist $m,n\in N$ with $m,n>p$ satisfying the following inequality $\int_{-\infty}^{0}|f_n(\tau)-f_m(\tau)||g(\tau)|\,d\tau\geqslant$

$\geqslant\varepsilon_0$, which is equivalent with $\lim\limits_{\substack{s\to 0\\ s<0}}\int_{-\infty}^{s}|f_n(\tau)-f_m(\tau)||g(\tau)|\,d\tau\geqslant\varepsilon_0$. Hence there exists an interval $(-\eta,0)$ $(\eta>0)$ such that $\int_{-\infty}^{s}|f_n(\tau)-f_m(\tau)||g(\tau)|\,d\tau\geqslant\varepsilon_0$, $\forall s\in(-\eta,0)$.

Therefore $\int_{-\infty}^{-\eta}\left(\int_{-\infty}^{t}|f_n(\tau)-f_m(\tau)||g(\tau)|\,d\tau\right)dt+\int_{-\eta}^{0}\left(\int_{-\infty}^{t}|f_n(\tau)-f_m(\tau)||g(\tau)|\,d\tau\right)dt\geqslant$ $\geqslant\eta\varepsilon_0>0$ which contradicts (15).

We now define the derivation of distributions. If $\mathcal{F}\in\tilde{\mathcal{D}}'_f$ then there exists a Cauchy sequence $(\mathcal{F}_n)$ of distributions of function type converging to $\mathcal{F}$. We define *the derivative* of $\mathcal{F}$ by

$$(16)\qquad \left(\frac{d\mathcal{F}}{d\mu},G\right)=\lim_{n\to+\infty}(\mathcal{F}_n,-G')+\lim_{n\to+\infty}\int_{-\infty}^{0}f_n(\tau)g(\tau)\,d\tau,$$

where $G\in\mathcal{D}$, $G(A)=\int_A g\,d\mu$, $\forall A\in\mathcal{D}(\Omega)$ and $f_n\in H_{\mathrm{loc}}^\infty(\Omega)$ generates $\mathcal{F}_n$ for every $n\in N$.

We remark that the limits in (16) exist. $\lim\limits_{n\to+\infty}(\mathcal{F}_n,-G')$ exists because the sequence $(\mathcal{F}_n)$ converges to $\mathcal{F}$ and $\lim\limits_{n\to+\infty}\int_{-\infty}^{0}f_n(\tau)g(\tau)\,d\tau$ exists since, by virtue of Remark 1, the sequence $\int_{-\infty}^{0}f_n(\tau)g(\tau)\,d\tau$ is a Cauchy sequence of numbers.

Remark 2. The distributions of function type can be introduced as, being generated by functions from $H_{\mathrm{loc}}^k(\Omega)$, where k is the highest derivation order which appears in the applications. We considered the distributions of function type generated by the functions from $H_{\mathrm{loc}}^\infty(\Omega)$ to obtain distributions for which the derivative of every order exists.

Now let us consider an equation which does not admit classic solutions, but which has solutions in the distributional sense.

Consider the equation

$$(17)\qquad \int_A f'\,d\mu=c\int_A f\,d\mu,\quad \forall A\in\mathcal{D}(-\infty,0),\ c<0 \text{ is a constant.}$$

If we define the classic solution of (17) as a non-trivial function $f:(-\infty,0)\to R$ with $f,f'\in C(-\infty,0)\cap L^1(-\infty,0)$, then (17) does not admit classic solutions because it is equivalent with $f(t)=c_0e^{ct}$ where $t\in(-\infty,0)$

and $c_0 \neq 0$. The distribution of function type generated by the function e^{ct} is

$$(18)\quad (\mathcal{F}, G) = \int_{-\infty}^{0}\left(\int_{-\infty}^{t} e^{c\tau}g(\tau)d\tau\right)dt,\ \forall G \in \mathcal{D} \text{ with } G(A) = \int_{\bar{A}} g d\mu,\ \forall A \in \mathcal{D}(-\infty, 0)$$

and the corresponding distributional equation is

$$(19)\qquad \left(\frac{d\mathcal{F}}{d\mu}, G\right) = (c\mathcal{F}, G),\quad \forall G \in \mathcal{D} \text{ or}$$

$$(20)\qquad (\mathcal{F}, -G') + \int_{-\infty}^{0} f(\tau)g(\tau)d\tau = c\int_{-\infty}^{0}\left(\int_{-\infty}^{t} f(\tau)g(\tau)d\tau\right)dt$$

(if we want that the solution be a distribution of function type). The distribution (18) satisfies (20) : Indeed, we have

$$(\mathcal{F}, -G') + \int_{-\infty}^{0} f(\tau)g(\tau)d\tau = -\int_{-\infty}^{0}\left(\int_{-\infty}^{t} e^{c\tau}g'(\tau)d\tau\right)dt + \int_{-\infty}^{0} e^{c\tau}g(\tau)d\tau = -$$

$$-\int_{-\infty}^{0}\left(\int_{-\infty}^{t} e^{c\tau}d(g(\tau))d\tau\right)dt + \int_{-\infty}^{0} e^{c\tau}g(\tau)d\tau = -\int_{-\infty}^{0} e^{ct}g(t)dt +$$

$$+ c\int_{-\infty}^{0}\left(\int_{-\infty}^{t} e^{c\tau}g(\tau)d\tau\right)dt + \int_{-\infty}^{0} e^{c\tau}g(\tau)d\tau = c\int_{-\infty}^{0}\left(\int_{-\infty}^{t} e^{c\tau}g(\tau)d\tau\right)dt.$$

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SPECTRAL CHARACTERIZATION OF ABSTRACT FUNCTIONS

BY

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1. Introduction. Let G be locally compact Abelian group and Γ its dual group. Let X be locally convex complete space and X^* its dual space. Denote by $C_{ub}(G, X)$ the set of all bounded uniformly continuous functions defined on G with values in X. A function $\psi \in C_{ub}(G, X)$ is said to be almost periodic (a.p.) at a point $\gamma_0 \in \Gamma$ iff there exists $f \in L^1(G)$ such that $\hat{f}(\gamma_0) \neq 0$ and $\psi * f$ is a.p. A theorem of Loomis [16] (see also Reiter [18] and Levitan [15]) says that a function $\psi \in C_{ub}(G, C)$ is a.p. iff it is a.p. at each point $\gamma \in \Gamma$. In this paper, we show that Loomis theorem holds true for a class of functions $\Lambda \subset C_{ub}(G, X)$ with certain properties satisfied by many known classes, like *almost automorphic* (a.a.) [20] *absolutely recurrent* (a.r.) [2] and *distal* (d.) [10] functions. We construct a function $\xi \in C_{ub}(R, C)$ which is recurrent (r.) at each point of R, i.e. for each $c \in R$ there exists $f \in L^1(R)$ such that $\hat{f}(C) \neq 0$ and $f * \xi$ is r., but ξ itself is not r. Our investigations reveal that the linearity of the class is necessary for Loomis theorem to hold true. We notice that the class of r. functions is not linear [11], [12]. Nevertheless, we construct linear subclasses of recurrent functions of the type Λ for which Loomis theorem is true. These classes are helpful in the study of functional equations with recurrent right side and also explain the fact that most of the results which are true for a.p. and a.a. functions are also valid for recurrent functions [2], [3], [9]. We note that our results are the extension of many recent works in two different directions. First direction is the extension to locally convex spaces and second direction is the extension to the classes of type Λ.

This paper is divided into six sections. In section 2 we give extension of Loomis theorem to classes of type $\Lambda \subset C_{ub}(G, C)$. In section 3 we prove that Loomis theorem is not true for r. functions and construct classes of r. functions of type Λ. In section 4 we proved that Loomis theorem also holds true for the classes type $\Lambda \subset C_{ub}(G, X)$. We showed that if the triple $\{\psi, G, X\}$ satisfies the condition (K) (saying that either X is a Fréchet space not containing a subspace isomorphic to the Banach space c_0 and $\psi(G)$ is bounded or X is a complete locally convex space and the weak closure $\overline{\psi(G)}^W$ is weakly compact in X), then the classes of a.p., a.a., a.r., and d. functions are of the type Λ. The condition (K) arised in the paper of Kadets [14] in the study of the indefinite integrals of a.p. functions in Banach spaces

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