and $c_0 \neq 0$. The distribution of function type generated by the function e^{ct} is

$$(18)\quad (\mathcal{F}, G) = \int_{-\infty}^{0} \left(\int_{-\infty}^{t} e^{c\tau} g(\tau) d\tau \right) dt, \ \forall G \in \mathcal{D} \text{ with } G(A) = \int_{\bar{A}} g d\mu, \forall A \in \mathcal{D}(-\infty, 0)$$

and the corresponding distributional equation is

$$(19)\quad \left(\frac{d\mathcal{F}}{d\mu}, G \right) = (c\mathcal{F}, G), \ \forall G \in \mathcal{D} \text{ or}$$

$$(20)\quad (\mathcal{F}, -G') + \int_{-\infty}^{0} f(\tau) g(\tau) d\tau = c \int_{-\infty}^{0} \left(\int_{-\infty}^{t} f(\tau) g(\tau) d\tau \right) dt$$

(if we want that the solution be a distribution of function type). The distribution (18) satisfies (20) : Indeed, we have

$$(\mathcal{F}, -G') + \int_{-\infty}^{0} f(\tau) g(\tau) d\tau = - \int_{-\infty}^{0} \left(\int_{-\infty}^{t} e^{c\tau} g'(\tau) d\tau \right) dt + \int_{-\infty}^{0} e^{c\tau} g(\tau) d\tau = -$$

$$- \int_{-\infty}^{0} \left(\int_{-\infty}^{t} e^{c\tau} d(g(\tau)) d\tau \right) dt + \int_{-\infty}^{0} e^{c\tau} g(\tau) d\tau = - \int_{-\infty}^{0} e^{c\tau} g(t) dt +$$

$$+ c \int_{-\infty}^{0} \left(\int_{-\infty}^{t} e^{c\tau} g(\tau) d\tau \right) dt + \int_{-\infty}^{0} e^{c\tau} g(\tau) d\tau = c \int_{-\infty}^{0} \left(\int_{-\infty}^{t} e^{c\tau} g(\tau) d\tau \right) dt.$$

Acknowledgement. We want to thank to Professor O. Costinescu for helpful discussions.

REFERENCES

1. Adams R. A. — *Sobolev spaces*. Academic Press, New York, San Francisco, London, 1975.
2. Haimovici A. — *Ecuaţii diferenţiale cu funcţii de mulţime ca necunoscute*. Ed. Acad R.S.R., Bucureşti, 1976.
3. Hörmander L. — *Linear partial differential operators*. Springer Verlag, Berlin, Gottingen, Heidelberg, 1963.
4. Köthe G. — *Topological vector spaces* I. Springer Verlag, Berlin, 1969.
5. Schwartz L. — *Théorie des distributions*. Hermann, Paris, 1967.
6. Yoshida K. — *Functional analysis*. Springer Verlag, Berlin, 1965.

Received 10.XII.1980

Faculty of Mathematics
University of Iaşi
R. S. Romania

Analele ştiinţifice ale Universităţii „Al. I. Cuza" din Iaşi
Tomul XXVIII, s. I a, 1982, f. 1

SPECTRAL CHARACTERIZATION OF ABSTRACT FUNCTIONS

BY

BOLIS BASIT

1. Introduction. Let G be locally compact Abelian group and Γ its dual group. Let X be locally convex complete space and X^* its dual space. Denote by $C_{ub}(G, X)$ the set of all bounded uniformly continuous functions defined on G with values in X. A function $\psi \in C_{ub}(G, X)$ is said to be almost periodic (a.p.) at a point $\gamma_0 \in \Gamma$ iff there exists $f \in L^1(G)$ such that $\hat{f}(\gamma_0) \neq 0$ and $\psi * f$ is a.p. A theorem of Loomis [16] (see also Reiter [18] and Levitan [15]) says that a function $\psi \in C_{ub}(G, C)$ is a.p. iff it is a.p. at each point $\gamma \in \Gamma$. In this paper, we show that Loomis theorem holds true for a class of functions $\Lambda \subset C_{ub}(G, X)$ with certain properties satisfied by many known classes, like *almost automorphic* (a.a.) [20] *absolutely recurrent* (a.r.) [2] and *distal* (d.) [10] functions. We construct a function $\xi \in C_{ub}(R, C)$ which is recurrent (r.) at each point of R, i.e. for each $c \in R$ there exists $f \in L^1(R)$ such that $\hat{f}(C) \neq 0$ and $f * \xi$ is r., but ξ itself is not r. Our investigations reveal that the linearity of the class is necessary for Loomis theorem to hold true. We notice that the class of r. functions is not linear [11], [12]. Nevertheless, we construct linear subclasses of recurrent functions of the type Λ for which Loomis theorem is true. These classes are helpful in the study of functional equations with recurrent right side and also explain the fact that most of the results which are true for a.p. and a.a. functions are also valid for recurrent functions [2], [3], [9]. We note that our results are the extension of many recent works in two different directions. First direction is the extension to locally convex spaces and second direction is the extension to the classes of type Λ.

This paper is divided into six sections. In section 2 we give extension of Loomis theorem to classes of type $\Lambda \subset C_{ub}(G, C)$. In section 3 we prove that Loomis theorem is not true for r. functions and construct classes of r. functions of type Λ. In section 4 we proved that Loomis theorem also holds true for the classes type $\Lambda \subset C_{ub}(G, X)$. We showed that if the triple $\{\psi, G, X\}$ satisfies the condition (K) (saying that either X is a Fréchet space not containing a subspace isomorphic to the Banach space c_0 and $\psi(G)$ is bounded or X is a complete locally convex space and the weak closure $\overline{\psi(G)}^W$ is weakly compact in X), then the classes of a.p., a.a., a.r., and d. functions are of the type Λ. The condition (K) arised in the paper of Kadets [14] in the study of the indefinite integrals of a.p. functions in Banach spaces

* On leave of absence from Cairo Universitiy , Faculty of Science, Egipt.

(see also [1], [2], [5], [7], [13]). Finally, in sections 5 and 6 we give applications of the obtained results to convolution and operator differential equations. The results of sections 5 and 6 are the extension of [3], [4], [6], [8], [22], [23].

2. Extension to Scalar Functions. We extend in this section Loomis theorem to the classes $\Lambda \subset C_{ub}(G, C)$ having the following properties :

P.1. Λ is a linear space.

P.2. Λ is closed with respect to uniform convergence, i.e., if a sequence or a net $\{\psi_i\} \subset \Lambda$ uniformly converges to ψ, then $\psi \in \Lambda$.

P.3. $\gamma\psi \in \Lambda$ for each $\gamma \in \Gamma$, $\psi \in \Lambda$.

P.4. If the difference $\psi - \psi_s \in \Lambda$ for each $s \in G$ and ψ is bounded, then $\psi \in \Lambda$ $(\psi_s(t) = \psi(t+s))$.

We notice that Λ is a Banach space with the sup. norm. We shall denote by $AP(G)$, $AA(G)$, $AR(G)$ and $D(G)$ respectively the sets of all a.p., a.a., a.r. and d. functions from $C_{ub}(G, C)$. One can verify that the $AP(G)$, $AA(G)$, $AR(G)$ and $D(G)$ have the properties P.1-P.4. To verify P.4 one can use [2] for a.p. and a.a. functions and deduce it similarly for a.r. and d. functions.

We prove the following needed property of the classes type Λ :

P.2.1. $f*\psi \in \Lambda$ for each $\psi \in \Lambda$, $f \in L^1(G)$.

Proof. Let $\psi \in \Lambda$. Then for each compact subset $K \subset G$ the function $\psi(t) = \int_K \psi(t+s)\, ds$ can be approximated uniformly by finite linear combinations of the form $\sum_i c_i \psi(t+s_i)$. Hence by P.1 – P.2, the function $\psi \in \Lambda$. Now let $f \in L^1(G)$. There exists a sequence of step functions $\{h_n\} \subset L^1(G)$ each of which has compact support such that $\|h_n - f\|_1 \to 0$, $n \to \infty$. This implies that $\|(h_n - f)*\psi\|_\infty \to 0$, $n \to \infty$. Each function $h_n*\psi$ is a finite linear combination of functions type ψ. Hence $h_n*\psi \in \Lambda$, $n \in N$. Using property P.2, we get that $f*\psi \in \Lambda$.

We remark that by using property P.4 we get that Λ contains the constant functions and by P.3 the class Λ contains every character. Hence by properties P.1 – P.2, Λ contains every a.p. function, i.e. $AP(G) \subset \Lambda$. This implies that $AP(G)$ is the minimal class of functions having properties P.1 – P.4.

To extend Loomis theorem, we need the following :

Definition 2.1. *For each* $\psi \in C_{ub}(G, C)$ *denote by*

$$I_\Lambda(\psi) = \{f \in L^1(G) : f*\psi \in \Lambda\}.$$

One can easily verify that $I_\Lambda(\psi)$ is a closed ideal of $L^1(G)$. The Λ-spectrum $\sigma_\Lambda(\psi)$ of ψ is defined by $\sigma_\Lambda(\psi) = \text{hull}\,(I_\Lambda(\psi)) = \{\gamma \in \Gamma : \hat{f}(\gamma) = 0,\ f \in I_\Lambda(\psi)\}$. Put $\Lambda = \{0\}$, we obtain that the Beurling spectrum $\sigma(\psi) \supset \sigma_\Lambda(\psi)$ (see [19], Ch. 7). One can also show for each $f \in L^1(G)$ that $\sigma_\Lambda(\psi*f) \subset \sigma_\Lambda(\psi) \cap \text{supp}\,\hat{f}$. We prove the following extension of Loomis theorem obtained in the thesis of our student E m a m [9] (see also [6]).

Theorem 2.1. *A function* $\psi \in C_{ub}(G, C)$ *is an element of* Λ *iff* $\sigma_\Lambda(\psi) = \varnothing$ *(empty set).*

Proof. If $\psi \in \Lambda$, then by P.2.1, we get $I_\Lambda(\psi) = L^1(G)$ so that $\sigma_\Lambda(\psi) = \varnothing$. Now let $\sigma_\Lambda(\psi) = \varnothing$. Using Lemma 2 of [16] we choose a sequence $\{h_n\} \subset L^1(G)$ with the following properties :

(i) $\psi_n = h_n*\psi \rightrightarrows \psi$. (ii) $\sigma\{\psi_n\}$ is compact.

Using ([19] 2.6.2) one can find $f_n \in L^1(G)$ such that $f_n*\psi_n = \psi_n$. Since $\sigma_\Lambda(\psi) = \varnothing$, we get $\sigma_\Lambda(\psi_n) = \varnothing$. By W i e n e r s theorem ([19] 7.2.6) we have $I_\Lambda(\psi_n) = L^1(G)$. This means that $\psi_n = \psi_n*f_n \in \Lambda$. Using P.2, we get $\psi \in \Lambda$.

In fact, using similar method as in [16], the following more general result is proved in [9] and [6].

Theorem 2.2. *A function* $\psi \in C_{ub}(G, C)$ *is an element of* Λ *iff* $\sigma_\Lambda(\psi)$ *is residual (i.e., contains no non-empty perfect sets).*

Proof. We show that $\sigma_\Lambda(\psi)$ is always perfect. Assuming the contrary, let $\alpha_0 \in \sigma_\Lambda(\psi)$ be an isolated point. Choose a compact neighbourhood $U(\alpha_0)$ of α_0 and a function $f \in L^1(G)$ such that $U(\alpha_0) \cap \sigma_\Lambda(\psi) = \{\alpha_0\}$, $\hat{f}(\alpha_0) \neq 0$ and $\hat{f}(\alpha) = 0$, $\notin U(\alpha_0)$. We have $\sigma_\Lambda(\psi*f) = \{\alpha_0\}$ and $\sigma_\Lambda(\psi) = \{e\}$, where $\psi = \bar{\alpha}_0(\psi*f)$. By W i e n e r s theorem ([19] Ch. 7.2.6) we have :

$$I_\Lambda(\psi) = \{f \in L^1(G) : \hat{f}(e) = 0\}.$$

Using P.4, we deduce that if $f \in L^1(G)$ has the property that $f - f_s \in I_s(\psi)$, $s \in G$, then $f \in I_\Lambda(\psi)$.

Since $(f_s - \hat{f})\,(e) = 0$, $s \in G$, $f \in L^1(G)$, we get that

$$I_\Lambda(\psi) = L^1(G).$$

This proves that $\sigma_\Lambda(\varphi) = \varnothing$, a contradiction which shows that $\sigma_\Lambda(\psi)$ cannot contain isolated points. Hence $\sigma_\Lambda(\psi)$ is perfect. Since $\sigma_\Lambda(\psi)$ is residua', then it is empty. Therefore, by theorem 2.1 the element $\psi \in \Lambda$.

Remark 2.1. The class Λ is invariant under translation, i.e. if $\psi \in \Lambda$, then $\psi_s \in \Lambda$, $s \in G$.

Proof. Let $\psi \in \Lambda$, $f \in L^1(G)$. Using P.2.1 we get that : $\psi_s*f = \psi*f_s \in \Lambda$, for $f_s \in L^1(G)$, $s \in G$. This implies that for each $\alpha_0 \in G$ there exists $f \in L^1(G)$ such that $\hat{f}(\alpha_0) \neq 0$ and $f*\psi_s \in \Lambda$, $s \in G$ and means that $\sigma_\Lambda(\psi_s) = \varnothing$. Using theorem 3.1. we get $\psi_s \in \Lambda$, $s \in G$.

3. Recurrent functions. In this section we study r. functions from $C_{ub}(R, C)$ and $C_{ub}(Z, C)$.

Definition 3.1. *A function* $\psi \in C_{ub}(R, C)$ *is* r. *iff for each* $\varepsilon > 0$ *and* $L > 0$ *the set*

$$E(\tau, L, \psi) = \{\tau \in R : |\psi(t+\tau) - \psi(t)| < \varepsilon, |t| < L\}$$

is relatively dense in R [17].

The class of r. functions from $C_{ub}(R, C)$ will be denoted by $r(R)$. Similarly, one can define r. functions from $C_{ub}(Z, C)$. The following Lemma is needed and gives a connection between r. functions on Z and R :

Lemma 3.1. *Let* $\psi \in C_{ub}(Z, R)$ *be* r. *Then* $\bar{\psi}(t) = \bar{\psi}(0 + m) = 0\psi(m) + (1-0)\psi(m+1)$, $m \in Z$, $0 \leqslant 0 \leqslant 1$, $t \in R$ *is an element of* $r(R)$.

The proof follows easily from definition 3.1.

A sequence $\{a_n\} \subset R$ is regular for $\psi \in C_{ub}(R, C)$ iff $\lim\limits_{n\to\infty} \psi(t+a_n)$ exists for each $t \in R$.

The following is equivalent to definition 3.1 (See [17]).

Definition 3.2. *A function $\psi \in C_{ub}(R, C)$ is r. iff for each regular sequence $\{a_n\} \subset R$, for ψ there exists a sequence $\{b_n\} \subset R$ such that*

$$\lim_{m\to\infty} \lim_{n\to\infty} \psi\,(t+a_n+b_m) = \psi(t).$$

Using Lemma 3.1 and [11] or [12], one can construct two real valued functions $\varphi, \psi \in r(R)$ such that $\varphi+\psi \notin r(R)$, for which we show :

Lemma 3.2. *Let $\varphi, \psi \in r(R)$ and $\varphi+\psi \notin r(R)$. Then for each character $\lambda(t) = \exp\,(ict)$, the function $\lambda\varphi+\psi \notin r(R)$, $(c \in R)$.*

Proof. Assuming that $\lambda_0(t) = \exp\,(ic_0t)$ has the property that $\lambda_0\varphi+\psi \in r(R)$, for each sequence $\{a_n\} \subset R$ regular for λ_0, φ, ψ there exists a sequence $\{b_n\} \subset R$ such that :

$$L\lambda_0(t) = \lim_{m\to\infty} \lim_{n\to\infty} \lambda_0(t+a_n+b_m) = \lambda_0(t),$$

$$L(\lambda_0\varphi+\psi)(t) = \lambda_0(t)L\varphi(t) + L\psi(t) = \lambda_0(t)\varphi(t) + \psi(t). \tag{3.1}$$

This is possible for the vector $\{\lambda_0, \lambda_0\varphi+\psi\}$ is r. [12]. From (3.1) we have $\lambda_0(L\varphi-\varphi) = L\psi-\psi$. Since $\lambda_0(t)$ is complex valued except in isolated points of R, we get

$$L\varphi-\varphi = L\psi-\psi = 0.$$

This implies that $L(\varphi+\psi) = \varphi+\psi$ and means that $\varphi+\psi \in r(R)$, a contradiction which proves the Lemma.

Using Lemma 3.2, we construct the following :

Counter example 3.1. *There exists a function $\xi(t)$, $\xi \in C_{ub}(R, C)$ which is r. at each point c of R but $\xi \notin r(R)$.*

Proof. Let φ, ψ be two real valued functions such that $\varphi, \psi \in r(R)$ but $\varphi+\psi \notin r(R)$. Let $\{h_n\} \subset L^1(R)$ be a sequence of approximating identity and supp. $(\hat{h}_n) \subset [-n, n]$. Then, there exists $n_0 \in N$ such that $(\varphi+\psi)*h_{n_0} \notin r(R)$, otherwise $(\varphi+\psi)*h_n \in r(R)$, $n \in N$ and hence $\varphi+\psi = \lim\limits_{n\to\infty} (\varphi+\psi)*h_n \in r(R)$ contradicting the choice of φ, ψ. Now, choose a character $\lambda_0(t) = \exp\,(ic_0t)$ such that $\sigma(\lambda_0(\varphi*h_{n_0}) \cap \sigma(\psi*h_{n_0}) = \varnothing$. We show that the function : $\xi = \lambda_0(\varphi*h_{n_0}) + \psi*h_{n_0}$ is r. at each point of R, but $\xi \notin r(R)$. Indeed, since $F_1 = \sigma(\lambda_0(\varphi*h_{n_0})$ and $F_2 = \sigma(\psi*h_{n_0})$ are disjoint compact subsets of R, one can choose $f_1, f_2 \in L^1(R)$ such that $\hat{f}_i(c) = 1$ on F_i, $\hat{f}_i(c) = 0$ on F_j, $i \neq j$, $i, j = 1, 2$. We have $f_1*\xi = \lambda_0(\varphi*h_{n_0})$, $f_2*\xi = \psi*h_{n_0} \in r(R)$. Hence ξ is r. at each point of $F_1 \cup F_2$. If $\bar{c} \notin F_1 \cup F_2$, then there exists $f \in L^1(R)$ such that $\hat{f}(\bar{c}) \neq 0$ and $\hat{f}(c) = 0$, $c \notin F_1 \cup F_2$. Hence, $f*\xi = 0$, which implies that ξ is r. at $\bar{c}$. This proves that ξ is r. at each point of R. On the other side, we have $\varphi*h_{n_0}, \psi*h_{n_0} \in r(R)$ but $\varphi*h_{n_0}+\psi*h_{n_0} \notin r(R)$. Hence by Lemma 3.2, the function $\xi = \lambda_0(\varphi*h_{n_0}) + \psi*h_{n_0} \notin r(R)$.

In the following we show an extension of Loomis theorem to r. functions. For that, we construct linear subclasses of $r(R)$ of type Λ. Indeed, let $\varphi \in r(R)$. Consider the set $L(\varphi)$ of functions generated by the uniform limits of finite linear combinations of functions of the form $\lambda\varphi_s*f$ and 1, where λ is a character of R, $f \in L^1(R)$ and $\varphi_s(t) = \varphi(t-s)$, $s \in R$. Let $\Lambda(\varphi)$ be the maximal Banach space of $r(R)$ containing $L(\varphi)$. Then $\Lambda(\varphi)$ is a linear class of $r(R)$ having properties P.1 – P.4. To verify P.4, one can use [2]. We have the following :

Theorem 3.1. *A function $\psi \in C_{ub}(R, C)$ is an element of $\Lambda(\varphi)$ iff $\sigma_{\Lambda(\varphi)}(\psi)$ is residual.*

4. Extension of Loomis Theorem to Functions from. $C_{ub}(G, X)$. A function $\varphi : G \to X$ is said to be integrable (measurable) if $x^*(\varphi)$ is Lebesgue integrable (measurable) for each $x^* \in X^*$. The bounded measurable functions from G to X will be denoted by $L^\infty(G, X)$, we have always $C_{ub}(G, X) \subset L^\infty(G X)$. If $\varphi \in L^\infty(G, X)$ and $f \in L^1(G)$, then the convolution $f*\varphi$ defined by $x^*(f*\varphi) = f*x^*(\varphi)$, $x^* \in X^*$ is an element of $C_{ub}(G, X)$. We shall consider linear classes $\Lambda \subset C_{ub}(G, X)$ having the same properties P.1 – P.4 as in section 2. The Beurling spectrum $\sigma(\varphi)$ and Λ-spectrum $\sigma_\Lambda(\varphi)$ can be defined exactly as in the scalar case for $\varphi \in L^\infty(G, X)$. We note that property P.2.1 is also true for the classes $\Lambda \subset C_{ub}(G, X)$.

In the following theorem the unions and the intersections will be taken by all elements from X^* unless explicitly otherwise mentioned.

Theorem 4.1. $\sigma(\varphi) = \overline{\bigcup \sigma(x^*\varphi)}$, $\varphi \in L^\infty(G, X)$.

Proof. Denote by $I_0(\varphi) = \{f \in L^1(G) : f*\varphi = 0\}$. First we show that $I_0(\varphi) = \bigcap I_0(x^*(\varphi))$. Indeed, if $f \in I_0(\varphi)$, then $f*\varphi = 0$. Therefore, $f*x^*(\varphi) = 0$, $x^* \in X^*$, which means that $f \in I_0(x^*(\varphi))$ and proves that $I_0(\varphi) \subset \bigcap I_0(x^*(\varphi))$.

If $f \in \bigcap I_0(x^*(\varphi))$, then $f*x^*(\varphi) = 0$, $x^* \in X^*$. This implies that $f*\varphi = 0$ and means that $f \in I_0(\varphi)$, which proves that $I_0(\varphi) = \bigcap I_0(x^*(\varphi))$. Now, let $\alpha \in \sigma(x^*(\varphi))$. Then $\hat{f}(\alpha) = 0$, $f \in I_0(\varphi) = \bigcap I_0(x^*(\varphi))$. This implies that $\alpha \in \sigma(\varphi) = $ hull $(I_0(\varphi))$ and proves that $\bigcup \sigma(x^*(\varphi)) \subset \sigma(\varphi)$. Since $\sigma(\varphi)$ is closed, $\overline{\bigcup \sigma(x^*(\varphi))} \subset \sigma(\varphi)$.

On the other side, if $\alpha_0 \in \overline{\bigcup \sigma(x^*(\varphi))}$, then there exists a compact neighbourhood $W(\alpha_0)$ and a function $f \in L^1(G)$ such that $\hat{f}(\alpha_0) \neq 0$, $\hat{f}(\alpha) = 0$, $\alpha \notin W(\alpha_0)$ and the intersection of $W(\alpha_0)$ with $\overline{\bigcup \sigma(x^*(\varphi))}$ is void. This implies that $x^*(\varphi)*f = 0$, $x^* \in X^*$. Hence $\varphi*f = 0$, which means that $f \in I_0(\varphi)$. Since $\hat{f}(\alpha_0) \neq 0$, we get that $\alpha_0 \in \sigma(\varphi)$. This proves that $\sigma(\varphi) \subset \overline{\bigcup \sigma(x^*(\varphi))}$. Hence $\sigma(\varphi) = \overline{\bigcup \sigma(x^*(\varphi))}$.

Using theorem 4.1, we show :

Lemma 4.1. *(i) If $\varphi \in L^\infty(G, X)$, $f, h \in L^1(G)$ are such that $\hat{f} = 1$ and $\hat{h} = 0$ on a neighbourhood of $\sigma(\varphi)$, then $f*\varphi = \varphi$, $\varphi*h = 0$. If $\sigma(\varphi)$ is compact, then such f exists. (ii) If $\varphi \in L^\infty(G, X)$, $\sigma(\varphi)$ is compact, then $\varphi \in C_{ub}(G, X)$ (φ = a uniformly continuous bounded function almost everywhere) and if $\varphi \in C_{ub}(G, X)$, then φ can be approximated uniformly by functions $\varphi*v$ with compact supp. $\hat{v}$.*

Proof. (i) By Lemma 1 and Lemma 2 of [16], we get $f*x(\varphi) = x^*(\varphi)$, $h*x^*(\varphi) = 0$, $x* \in X^*$, and hence $f*\varphi = \varphi$, $\varphi*h = 0$. If $\sigma(\varphi)$ is compact, then using theorem 4.1, there exists $f \in L^1(G)$ such that $f*x^*(\varphi) = x^*(\varphi)$, $x^* \in X^*$ and hence $f*\varphi = \varphi$. (ii) Since $\sigma(\varphi)$ is compact, there exists $f \in L^1(G)$ such $f*\varphi = \varphi$. Let P be the family of semi-norms of X. Then for each $p \in P$, we have $p(\varphi*(f-f_s)) \leqslant$

$\leqslant \sup p(\varphi)$. $\|f-f_s\|_1$ which proves the uniform continuity of φ. The second part of (ii) also can be proved similar to Lemma 2 of [16].

Using Lemma 4.1 one can prove as in the scalar case the following :

Theorem 4.2. *A function $\psi\in C_{ub}(G, X)$ is an element of Λ iff $\sigma_\Lambda(\psi)$ is residual.*

Now, we give examples of classes type Λ. For that we recall the following definitions :

Definition 4.1. *A continuous function $\psi: G\to X$ is said to be a.p. iff each net $\{\alpha_{i_r}\}_{r\in T'}$ of elements of G has a subset $\{\alpha_i\}_{i\in T}$ such that $\{\psi(\alpha_i t)\}_{i\in T}$ is uniformly convergent on G.*

One can prove that $\psi\in C_{ub}(G, X)$ and that the range of ψ is totally bounded in X. The class of a.p. functions will be denoted by $AP(G, X)$. It is easy to verify that $AP(G, X)$ satisfies the properties P.1 – P.3. As in the case when $X = B$ is a Banach space property P.4 is not always true. If the triple $\{\psi, G, X\}$ satisfies the condition (K), then by [5] partially by [7] $AP(G, X)$ satisfies P.4. In this case, we shall denote it by $AP(G, X, K)$. Hence we have :

Theorem 4.3. *$AP(G, X, K)$ satisfies the properties P.1 – P.4.*

In the following, we give :

Definition 4.2. *A continuous function $\psi: G\to X$ is said to be a.a. if every set $\{\alpha_i\}_{i\in T'}$, has a subset $\{a_i\}_{i\in T'}$ such that*

$$\lim_{i,j\in T} \psi(s\alpha_i\alpha_j^{-1}t)=\psi(st),$$

pointwise for each $s, t\in G$. This means that for each neighbourhood $U(0)$ of 0 of X there exists $i_0\in T$ such that

$$\{\psi(s\alpha_i\alpha_j^{-1}t)-\psi(st)\}\in U(0),\quad i, j\geqslant i_0,\quad i, j\in T.$$

Functions satisfying definition 4.2 are not necessary uniformly continuous or bounded. We shall denote by $AA(G, X)$ the a.a. functions from $C_{ub}(G, X)$. Then from definition 4.2, one can easily show that $AA(G, X)$ satisfies properties P.1 – P.3. If the triple $\{\psi, G, X\}$ satisfies the condition (K) then $AA(G, X)$ will be denoted by $AA(G, X, K)$ and by [5] partially by [7] it satisfies properly P.4. Hence we have :

Theorem 4.4. *$AA(G, X, K)$ satisfies the properties P.1 – P.4.*

Similarly, one can define the uniformly continuous bounded absolutely recurrent and distal functions. In this case, $AR(G, X, K)$ and $D(G, X, K)$ satisfy properties P.1 – P.4. Finally, if $\Lambda(\varphi, R, X, K)$ is the class of recurrent functions generated by the bounded uniformly continuous and recurrent function $\varphi: R\to X$ and the triple $\{\varphi, R, X\}$ satisfies (K) then $\Lambda(\varphi, R, X, K)$ satisfies the properties P.1 – P.4.

As a consequence of theorems 4.1 – 4.4, we get :

Theorem 4.5. *A function $\psi\in C_{ub}(G, X)$ is a.p., a.a., a.r., or d. iff its $AP(G, X, K)$ respectively $AA(G, X, K)$, $AR(G, X, K)$ or $D(G, X, K)$, spectrum is residual. Similarly, a function $\psi\in C_{ub}(R, X)$ is r. iff its $\Lambda(\varphi, R, X, K)$-spectrum is residual.*

5. Application to Convolution Equations. In this section, we consider equations of the form :

$$k*\psi+\lambda\psi=\varphi, \tag{5.1}$$

where $\lambda\in C$, $k\in L^1(G)$ and $\varphi\in\Lambda\subset C_{ub}(G, X)$.

We prove the following :

Theorem 5.1. *If the set $Z(k)=\{\alpha\in\Gamma: \hat{k}(\alpha)+\lambda=0\}$ is residual, then every solution of (5.1) from $C_{ub}(G, X)$ belongs to Λ.*

Proof. Let $\alpha_0\notin Z_\lambda(k)$. There exists $f\in L^1(G)$ such that $\hat{f}(\alpha_0)\neq 0$. Consider the equation

$$(f*k+\lambda f)*\psi=f*\varphi.$$

Since $f*\varphi\in\Lambda$, we obtain $\sigma_\Lambda(\psi)\subset\{\alpha\in\Gamma: \hat{f}(\alpha)\,[\hat{k}(\alpha)+\lambda]=0\}$.

Since $\hat{f}(\alpha)_0\neq 0$ and $\hat{k}(\alpha_0)+\lambda\neq 0$, we get $\alpha_0\notin\sigma_\Lambda(\psi)$. Hence $\sigma_\Lambda(\psi)\subset Z_\lambda(k)$ which implies that $\sigma_\Lambda(\psi)$ is residual. By theorem 4.2, $\psi\in\Lambda$.

As a consequence using theorems 4.3, 4.4 and 4.5 we can state the above theorem for the classes $AP(G, X, K)$, $AA(G, X, K)$, $AR(G, X, K)$ $D(G, X, K)$ and $\Lambda(\varphi, R, X, K)$.

In the case $\lambda=0$, $\Lambda=AP(R, C)$ and $\hat{k}$ is analytic, theorem 5.1 is due to L e v i t a n [15].

By the same argument as in theorem 5.1, we show :

Theorem 5.2. *Let $k\in L^1(R)$ with analytic Fourier transform $\hat{k}$ and let $a_{mj}\in C, w_{mj}\in R$, $m=0,1,\dots,p, j=0,1,\dots,q$ be constant numbers. Let $\psi\in\Lambda\ C_{ub}(R, X)$. Then every solution $\psi\in C_{ub}(R, X)$ of the equation*

$$(k*\psi)(t)+\sum_{m=0}^{p}\sum_{j=0}^{q} a_{mj}\psi^{(m)}(t-w_{mj})=\psi \tag{5.2}$$

belongs to Λ.

Proof. By the same method of 5.1, one can show that :

$$\sigma_\Lambda(\psi)\subset\{\alpha\in R: \hat{k}(\alpha)+\sum_{m,j=0}^{p,q} a_{mj}(-i\alpha)^m e^{i\alpha w_{mj}}=0\}=Z(k).$$

If $\hat{k}$ is analytic, then $Z(K)$ is residual. Hence by theorem 4.2, $\psi\in\Lambda$.

In the case $\Lambda=AP(R, C)$ theorem 5.2 is due to D o s s [8]. In the case $\Lambda=AP(R, B, K)$, $\Lambda=AA(R, B, K)$ and $\Lambda=\Lambda(\varphi, R, B, K)$, where B is a Banach space, theorem 5.2 is proved in [3].

6. Application to Operator Differential Equations in Banach Spaces. In this section we study the solutions of equations of the form :

$$\psi'=A\psi+\varphi, \tag{6.1}$$

where $\varphi\in\Lambda\subset C_{ub}(R, B)$, B is a Banach space and A is the infinitesimal generator of strongly continuous semigroup $U(t)$, $(t\geqslant 0)$. We are interested in the solutions of (6.1) from $C_{ub}(R, B)$. By definition the function $\psi\in C_{ub}(R,B)$ is a solution of (6.1) iff

$$\psi(t)=U(t-t_0)\psi(t_0)+\int_{t_0}^{t} U(t-s)\varphi(s)\,ds,\quad (t\geqslant t_0).$$

Denote by $L(B, B)$ the Banach space of linear bounded operators from B to B. In what follows, we shall assume that the class Λ has the property

P.5 $$T\psi \in \Lambda, \ \psi \in \Lambda, \ T \in L(B, B).$$

This property is satisfied by $AP(R, B)$, $AA(R, B)$, $AR(R, B)$, $D(R, B)$ and Λ (φ, R, B) where φ is a recurrent function.

Let $H : R \to L(B, B)$ be a continuous function such that $\int_{-\infty}^{\infty} \|H(t)\| \, dt < \infty$.

By the same method of the proof of property P.2.1 and using P.5 one can show the following:

Lemma 6.1. $H*\psi(t) = \int_{-\infty}^{\infty} H(s)\psi(t-s)ds \in \Lambda$ *for each* $\psi \in \Lambda$.

Denote by $\sigma_0(A)$ the intersection of the spectrum of A with the imaginary axis. We notice that $\sigma_0(A)$ is closed and if $\lambda_0 \in R$, $i\lambda_0 \notin \sigma_0(A)$, then the point $i\lambda_0$ is regular with respect to A.

We show the following:

Theorem 6.1. If $\psi \in C_{ub}(R, B)$ is a solution of (6.1), then $\sigma_\Lambda(\psi) \subset \sigma_0(A)$.

Proof. Let $c_0 \in R$, $ic_0 \notin \sigma_0(A)$. Consider the equation:

(6.2) $$\psi' - ic_0\psi = A\psi - ic_0\psi + \psi = (A - ic_0 I)\psi + \varphi$$

Applying the resolvent operator $Rc_0 = (A - ic_0 I)^{-1}$ to both sides of (6.2), we get:

$$\psi = R_{c_0}(\psi' - ic_0\psi) - R_{c_0}\varphi.$$

There exists $a > 0$ such that $R_c = (A - icI)^{-1}$ is analytic on the interval $(c_0 - -2a, \ c_0 + 2a)$. Choose infinitely smooth function $f : R \to R$ with compact support such that the support of its Fourier transform supp $\hat{f} \subset (c_0 - a, c_0 + a)$. Consider the equations:

(6.3) $$\psi*f = R_c\psi*(f' - icf) - R_c\varphi*f, \ \ c \in (c_0 - a, \ c_0 + a)$$

We have $\psi*f(t) = V(t, c) + W(t, c)$, $c \in (c_0 - a, \ c_0 + a)$, $t \in R$, where $V(t, c) = R_c\psi*(f' - icf)$, $W(t, c) = -R_c\varphi*f$. For each fixed $c \in (c_0 - a, c_0 + a)$, we consider the Fourier transform in the sense of distributions of both functions $V(t, c)$ and $W(t, c)$ ([21], Ch. VI). We notice that the function $\hat{H}(c) = \hat{f}(c)R_c : R \to \to L(B, B)$ is infinitely smooth and has compact support (supp $\hat{H} \subset (c_0 - a, c_0 + a)$). Hence $\hat{H}(c)$ is a Fourier transform of a continuous function $H : R \to \to L(B, B)$ and moreover $\int_{-\infty}^{\infty} \|H(t)\| \, dt < \infty$. This means that:

$$\hat{H}(c) = \int_{-\infty}^{\infty} H(t)e^{ict} \, dt,$$

and implies that the inverse Fourier transform $W(t)$ of the function $\hat{W}(c, c)$ can be written in the form:

$$\hat{\hat{W}} = H*\varphi.$$

We notice that the function $\hat{H}_1(c, c) = (f' \hat{-} icf)R_c \equiv 0$, $c \in (c_0 - a, c_0 + a)$. Hence $\hat{V}(c, c) \equiv 0$ and this implies that $\psi\hat{*}f(c) = \hat{W}(c, c)$, $c \in (c_0 - a, c_0 + a)$ and this means that $\psi*f = H*\psi$. By Lemma 6.1, we conclude that $\psi*f = H*\varphi \in \Lambda$. Hence $c_0 \notin \sigma_\Lambda(\psi)$ and this proves the theorem.

Corollary 6.1. *If* $\psi \in C_{ub}(R, B)$ *is a solution of the homogeneous equation of* (6.1), *then the Beurling spectrum* $\sigma(\psi) \subset \sigma_0(A)$.

Indeed, let $ic_0 \notin \sigma_0(A)$. Then arguing as in theorem (6.1), we get $\psi\hat{*}f(c) = \hat{V}(c, c) \equiv 0$, $c \in (c_0 - a, \ c_0 + a)$. Hence $c_0 \notin \sigma(\psi)$ which proves that $\sigma(\psi) \subset \sigma_0(A)$.

Remark 6.1. Theorem 6.1. is true for classes $\Lambda \subset C_{ub}(R, X)$ under certain restrictions on the operator A.

Theorem 6.2. *If* $\sigma_0(A)$ *is residual, then each solution* $\psi \in C_{ub}(R, B)$ *of the equation* (6.1) *belongs to* Λ. *Particularly, each solution* $\psi \in C_{ub}(R, B)$ *of the homogeneous equation of* (6.1) *is almost periodic.*

The proof is a direct consequence from theorems 4.1, 6.1 and corollary 6.1.

The following is the extension of the results of [22].

Theorem 6.3. *If* $d(\sigma(\psi), \ \sigma_0(A)) = \inf |c - s| = d_0 > 0$, $c \in \sigma(\psi)$, $s \in \sigma_0(A)$, *and if every solution* $\psi \in C_{ub}(R, \ B)$ *of the homogeneous equation of* (6.1) *belongs to* Λ, *then every solution* $\psi \in C_{ub}(R, B)$ *of the equation* (6.1) *belongs to* Λ.

Proof. Let $ic_0 \in \sigma_0(A)$. Using the condition of the theorem, there exists $f \in L^1(R)$ such that $\hat{f}(c_0) \neq 0$, supp $\hat{f} \cap \sigma(\varphi) = \varnothing$. This implies that $f*\varphi \equiv 0$, and $\psi'*f = A(\psi*f)$ which means that $f*\psi$ is a solution of the homogeneous equation of (6.1). Hence $\psi*f \in \Lambda$ and $c_0 \notin \sigma_\Lambda(\psi)$. Using theorem (6.1), we get: $\sigma_\Lambda(\psi) = \varnothing$. Hence by theorem 4.1. we get $\psi \in \Lambda$.

REFERENCES

1. Amerio L. and Prouse G. — *Almost periodic functions and functional equations.* Van Nostrand R.C. (1971).

2. Basit B. — *Generalization of two theorems of Kadets on the indefinite integral of abstract almost periodic functions.* Math. Notes 9, No. 3, 311—321 (1971).

3. Basit B. and Zikov V. V. — *Almost periodic solutions of integro-differential equations in Banach spaces.* Vestnik Mosc. State Univ. 2, 29—33 (1971).

4. Basit B. and Tsend L. — *The generalized Bohr-Neugebauer theorem.* Differential equations and its applications 8, 1031—1035 (1974).

5. Basit B. and Emam M. — *Differences of functions in locally convex spaces and applications to almost periodic and almost automorphic functions.* To appear in Annales Polonici Math.

6. Baskakov, A. G. — *Spectral criteria of almost periodicity of solutions of functional equations.* Math. Notes 24, No 2, 195—205 (1978).

7. Dimitrov, D. V. — *Almost periodic and N almost periodic solutions of difference equations*. Translated from Ukrainski, Math. Zhurnal, 25, No. 2, 193—198 (1973).
8. Doss, R. — *On almost periodic solutions of a class of integro-differential-difference equations*. Ann. Math. 81, No. 1, 117—123 (1965).
9. Emam, M. — *Almost periodic and almost automorphic solutions in locally convex spaces*. Thesis of Ph. D., Cairo Univ., (1977).
10. Ellis, R. — *Distal transformation groups*. Pacific J. Math. 8, 401—405 (1958).
11. Flor, P. — *Fastperiodisch Folgen, die nur endlich veile verschiedene Werte annehmen*, Monatschefte fur Mathematik 67, 385—400 (1963).
12. Flor, P. — *Rhythmische Abbildungen abelscher Gruppen II*, Z. Wahrescheinlichkeitstheorie verw. Geb. 7, 17—28 (1967).
13. Gunzler, M. — *Integration of almost periodic functions*, Math. Zeitschr. 102, 253—287 (1967).
14. Kadets — *On the integral of almost periodic functions in Banach spaces*. Functional analysis and its applications, 3, 71—74 (1969) (Russian).
15. Levitan, B. M. — *On an integral equation with almost periodic solutions*. Bull. Amer. Math. Soc. 43, 677—679 (1973).
16. Loomis, L. H. — *Spectral characterization of almost periodic functions*. Ann. Math. 72 362—368 (1960).
17. Nemytskii, V. V. and Stepanov, V. V. — *Qualitative Theory of Differential Equations* (Princeton Math. Series 22), Princeton Univ. Press, Princeton N. J., (1960).
18. Reiter, H. J. — *On a certain class of ideals in the L^1-Algebra of a locally compact abelian group*. Trans. Amer. Math. Soc. 75, 505—509 (1953).
19. Rudin, W. — *Fourier analysis on groups*. Inter. Pub. a division of John Wiley and Sons, New York (1962).
20. Veech, W. A. — *Almost automorphic functions on groups*. Amer. J. Math. 87, 719—751 (1965).
21. Yoshida, K. — *Functional analysis*. Springer-Verlag, (1965).
22. Zaidman, S. — *A remark on integration of almost periodic functions*. Canadian Math. Bull., 13, No. 2, 249—251, (1970).
23. Zikov, V. V. and Levitan, B. M. — *Favard's theory*. Yspechi Math. Nauk 32, No. 2, 123—171 (1977).

Received 26. III. 1981

Sana'a University
Yemen Arab Republic

Analele ştiinţifice ale Universităţii „Al. I. Cuza" din Iaşi
Tomul XXVIII, s. I a, 1982, f. 1

GENERATION OF THE PROBABILISTIC QUASI-PSEUDO-METRIC SPACES *

BY

T. BÎRSAN

1. Introduction. The concept of a probabilistic metric space (PM space) has been introduced in 1942 by K. Menger [8]. An important moment in the development of the PM space theory is B. Schweizer and A. Sklar's paper [10]. The introduction of an adequate notion of probabilistic topological space (PT space) was the object of preoccupation for F. Wagner [11] and more recently for M. J. Frank [3]. Various problems concerning probabilistic metric spaces have also been considered by several other authors.

In a PM space the distances between points are distribution functions. More exactly, with every pair of elements p, q of a set, one associates a distribution function F_{pq}, and for any positive number x, one interprets $F_{pq}(x)$ as the probability that the classical distance from p to q be less than x. It is our purpose to point out two structures, a probabilistic one and a classical one, corresponding to the two facets of this interpretation of $F_{pq}(x)$. Furthermore, given two such structures, they will generate a PM space and this generated space induces the given structures in a way which will be specified. The natural framework in which this problem is approached is that of the probabilistic quasi-pseudo-metric spaces (q-p-PM spaces). Probabilistic quasi-metric spaces (q-PM spaces) and, connected with them, bitopological spaces were utilized the first time by V. Istrăţescu in [5].

Let Δ be the set of all real-valued non-decreasing and left-continuous functions defined on the real line, having supremum one and value zero for $x \leqslant 0$. H will denote that member of Δ for which $H(x) = 1$ for all positive x. Let S be an arbitrary set and F be a mapping from $S \times S$ into Δ relative to which we formulate the following conditions (we write $F(p, q) = F_{pq}$):

I. $F_{pp} = H$ for every $p \in S$;

II. $F_{pq} = H \Rightarrow p = q$;

III. $F_{pq} = F_{qp}$ for every $p, q \in S$;

IV. $F_{pq}(x) = 1$ and $F_{qr}(y) = 1 \Rightarrow F_{pr}(x + y) = 1$.

---

*) This paper presents some results of my doctoral thesis „Contribuţii la studiul spaţiilor bitopologice şi aplicaţii", University of Iaşi (Romania), 1976.