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GENERATION OF THE PROBABILISTIC QUASI-PSEUDO-METRIC SPACES *

BY

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1. Introduction. The concept of a probabilistic metric space (PM space) has been introduced in 1942 by K. Menger [8]. An important moment in the development of the PM space theory is B. Schweizer and A. Sklar's paper [10]. The introduction of an adequate notion of probabilistic topological space (PT space) was the object of preoccupation for F. Wagner [11] and more recently for M. J. Frank [3]. Various problems concerning probabilistic metric spaces have also been considered by several other authors.

In a PM space the distances between points are distribution functions. More exactly, with every pair of elements p, q of a set, one associates a distribution function F_{pq}, and for any positive number x, one interprets $F_{pq}(x)$ as the probability that the classical distance from p to q be less than x. It is our purpose to point out two structures, a probabilistic one and a classical one, corresponding to the two facets of this interpretation of $F_{pq}(x)$. Furthermore, given two such structures, they will generate a PM space and this generated space induces the given structures in a way which will be specified. The natural framework in which this problem is approached is that of the probabilistic quasi-pseudo-metric spaces (q-p-PM spaces). Probabilistic quasi-metric spaces (q-PM spaces) and, connected with them, bitopological spaces were utilized the first time by V. Istrăţescu in [5].

Let Δ be the set of all real-valued non-decreasing and left-continuous functions defined on the real line, having supremum one and value zero for $x \leqslant 0$. H will denote that member of Δ for which $H(x)=1$ for all positive x. Let S be an arbitrary set and F be a mapping from $S \times S$ into Δ relative to which we formulate the following conditions (we write $F(p, q)=F_{pq}$) :

I. $F_{pp}=H$ for every $p \in S$;

II. $F_{pq}=H \Rightarrow p=q$;

III. $F_{pq}=F_{qp}$ for every $p, q \in S$;

IV. $F_{pq}(x)=1$ and $F_{qr}(y)=1 \Rightarrow F_{pr}(x+y)=1$.

*) This paper presents some results of my doctoral thesis „Contribuţii la studiul spaţiilor bitopologice şi aplicaţii", University of Iaşi (Romania), 1976.

An ordered pair (S, F) is called a
- *q-p-PM space*, if F satisfies I, and IV ;
- *q-PM space*, if F satisfies I, II, and IV ;
- *p-PM space*, if F satisfies I, III, and IV ;
- *PM space*, if F satisfies the conditions I – IV.

If F is a *q-p-PM* distance on S, then the mapping F^* from $S \times S$ into Δ defined by $F^*_{pq} = F_{qp}$ is called the *q-p-PM conjugate* distance.

For each *q-p-PM* space (S, F), we associate a generalized bitopological space (S, P, Q), where P and Q are the generalized topologies induced by F and F^*, respectively (we utilize the term of generalized topology in the sense of [4, Definition 2.3]) ; that is, P is determined by (ε, λ)-neighborhoods relative to F defined by

$$N_p(\varepsilon, \lambda) = \{q ; F_{pq}(\varepsilon) > 1 - \lambda\},$$

and Q is determined by (ε, λ)-neighborhoods relative to F^* defined by

$$N^*_p(\varepsilon, \lambda) = \{q ; F^*_{pq}(\varepsilon) > 1 - \lambda\} = \{q ; F_{qp}(\varepsilon) > 1 - \lambda\}.$$

A *t-norm* is a function $T : I \times I \to I$, $I = [0,1]$, which is associative, commutative, non-decreasing in each place, and satisfies $T(a, 1) = a$ for each a in I. An ordered triple (S, F, T) is a *q-p-PM space of Menger type* if (S, F) is a *q-p-PM* space, T is a *t*-norm, and the following triangle inequality, due to K. Menger, holds :

IVM $$F_{pr}(x + y) \geqslant T(F_{pq}(x), F_{qr}(y)),$$

for all $p, q, r \in S$ and all $x, y \geqslant 0$.

The next result corresponds to Th. 7.2 in [10] and slightly completes the Th. 3.3 of [5].

Proposition 1. a) *If (S, F, T) is a q-p-PM space of Menger type satisfying the condition*

(*) $$\lim_{x \to 1} T(a, x) = a, \textit{ for every } a \in I,$$

then (S, P, Q) is a bitopological space.

b) *If (S, F, T) is a q-PM space of Menger type satisfying the condition* (*), *then (S, P, Q) is a pairwise Hausdorff bitopological space.*

The fundamental mathematical tool in obtaining the principal results of the paper is presented in Section 2. Some investigations concerning the triangle inequalities in *PT* spaces form the object of Section 3. The main results are presented in Section 4.

Bitopological terms not explained here may be found in [6].

2. Probabilistic distances and spheres. The concepts of probabilistic diameter of a set and probabilistic distance, in the Hausdorff sense, between two sets in *PM* spaces of Menger type were introduced and investigated by R. J. E g b e r t in [2]. In this Section the classical notions of distance from a point to a set and of generalized sphere are extended to *PM* spaces.

Let (S, F) be a *q-p-PM* space, let A be a nonempty subset of S and $p \in S$. The *probabilistic distance from p to A* (respectively, *probabilistic distance from A to p*) is the function $d(p, A)$ (respectively, $d(A, p)$) defined by

$$d(p, A)(x) = \sup_{q \in A} F_{pq}(x) \text{ (respectively, } d(A, p) = \sup_{q \in A} F_{qp}(x)).$$

We first note that $d(p, A) \in \Delta$. We only show that $d(p, A)$ is left-continuous. Let $x > 0$ and $\varepsilon > 0$ be given. There exists a point $q \in A$ such that $F_{pq}(x) > d(p, A)(x) - \varepsilon/2$. Since F_{pq} is left-continuous at x, it follows that there exists a real number $\delta > 0$ such that, for every $t \in (\delta, x]$, we have $F_{pq}(t) > F_{pq}(x) - \varepsilon/2$. Consequently,

$$d(p, A)(t) \geqslant F_{pq}(t) > F_{pq}(x) - \varepsilon/2 > d(p, A)(x) - \varepsilon$$

so that $d(p, A)$ is left-continuous at x.

Remarks. 1) $d(A, p) = d^*(p, A)$, where $d^*(p. A)$ is the probabilistic distance from p to A in the conjugate space (S, F^*).

2) The number $d(p, A)(x)$ may be interpreted as the probability that the classical distance from p to A be less than x.

Proposition 2. *In a q-p-PM space (S, F) the function $d(p, A)$ has the following properties :*

1) $d(p, A) \geqslant d(p, B)$, *if* $A \supset B \neq \varnothing$;

2) $d(p, \bigcup_{k \in K} A_k) = \sup_{k \in K} d(p, A_k)$, *for an arbitrary index set* K ;

3) $d(p, A) = H$ *if and only if* $p \in \bar{A}^P$, *that is* $\bar{A}^P = \{p ; d(p, A) = H\}$, *where* $\bar{A}^P$ *denotes the closure of* A *relative to the generalized topology* P. *Analogous properties hold for* $d(A, p)$.

Proposition 3. *In a q-p-PM space of Menger type (S, F, T), with the t-norm T satisfying* (*), *we have :*

$$d(p, A) = d(p, \bar{A}^P) \textit{ and } d(A, p) = d(\bar{A}^Q, p).$$

The following result may be established using similar arguments as in the proof of Theorem in [9].

Proposition 4. *Let (S, F, T) be a PM space of Menger type, the t-norm T satisfying the condition*

(**) $$\lim_{X \to 1} T(a, x) = a, \textit{ uniformly on } I.$$

Then the $d(., A) : S \to \Delta$ is uniformly continuous (on the set Δ it is considered the Lévy metric).

Let (S, F) be a *q-p-PM space, A a nonempty subset of S and $\varepsilon, \lambda > 0$ be given. The open (closed) probabilistic sphere with centre at A and radii ε and λ is the set*

$$S(A, \varepsilon, \lambda) = \{p ; d(A, p)(\varepsilon) > 1 - \lambda\}$$
$$\text{(respectively, } \bar{S}(A, \varepsilon, \lambda) = \{p ; d(A, p)(\varepsilon) \geqslant 1 - \lambda\}).$$

Remarks. 1) In the case of the closed probabilistic spheres λ may also the value zero.

2) We denote $S^*(A, \varepsilon, \lambda) = \{p ; d(p, A)(\varepsilon) > 1 - \lambda\}$ and $\bar{S}^*(A, \varepsilon, \lambda) = \{p ; d(p, A)(\varepsilon) \geqslant 1 - \lambda\}$; in fact these are the generalized spheres with respect to the conjugate probabilistic distance F^*.

Proposition 5. *In a q-p-PM space (S, F), we have*

1) $S(\{p\}, \varepsilon, \lambda) = N_p(\varepsilon, \lambda)$ *and* $\bar{S}(\{p\}, \varepsilon, \lambda) = \bar{N}_p(\varepsilon, \lambda)$ (*we denote* $\bar{N}_p(\varepsilon, \lambda) = \{p ; F_{pq}(\varepsilon) \geqslant 1 - \lambda\}$),

2) $A \subset S(A, \varepsilon, \lambda) \subset \bar{S}(A, \varepsilon, \lambda)$,

3) $\bar{S}(A,\varepsilon,\lambda)\subset S(A,\varepsilon,\mu)$ *if* $\lambda<\mu$.

4) $S(A,\varepsilon,\lambda)\subset S(B,\varepsilon,\lambda)$ *and* $\bar{S}(A,\varepsilon,\lambda)\subset\bar{S}(B,\varepsilon,\lambda)$ *if* $A\subset B$.

5) $S(A,\varepsilon,\lambda)\subset S(A,\eta,\mu)$ *and* $\bar{S}(A,\varepsilon,\lambda)\subset\bar{S}(A,\eta,\mu)$ *if* $\varepsilon\leqslant\eta$ *and* $\lambda\leqslant\mu$,

6) $S(A,\varepsilon,\lambda)=\bigcup\{S(A,\eta,\mu)\ ;\ \eta<\varepsilon,\ \mu<\lambda\}=\bigcup\{\bar{S}(A,\eta,\mu)\ ;\ \eta<\varepsilon,\ \mu<\lambda\}$;

7) $S(A,\varepsilon,\lambda)=\bigcap\limits_{\mu>\lambda}\bigcup\limits_{\eta<\varepsilon}S(A,\eta,\mu)=\bigcap\limits_{\mu>\lambda}\bigcup\limits_{\eta<\varepsilon}\bar{S}(A,\eta,\mu)$.

8) $S(\bigcup\limits_{k\in K}A_k,\varepsilon,\lambda)=\bigcup\limits_{k\in K}S(A_k,\varepsilon,\lambda)$ *and* $\bar{S}(A\cup B,\varepsilon,\lambda)=\bar{S}(A,\varepsilon,\lambda)\cup\bar{S}(B,\varepsilon,\lambda)$ (*K is an arbitrary index set*),

9) $S(A,\varepsilon,\lambda)=\bigcup\{N_p(\varepsilon,\lambda)\ ;\ p\in A\}$.

10) $S=\bigcup\limits_{\varepsilon>0}S(A,\varepsilon,\lambda)$.

11) $\bar{A}^Q=\bigcap\{S(A,\varepsilon,\lambda)\ ;\ \varepsilon>0,\ \lambda>0\}=\bigcap\{\bar{S}(A,\varepsilon,\lambda)\ ;\ \varepsilon>0,\ \lambda>0\}$.

Proposition 6. *Let (S,F,T) be a q-p-PM space of Menger type with the t-norm T satisfying (*). Then :*

1) $S(A,\varepsilon,\lambda)=S(\bar{A}^Q,\varepsilon,\lambda)$ *and* $\bar{S}(A,\varepsilon,\lambda)=\bar{S}(\bar{A}^Q,\varepsilon,\lambda)$;

2) $S(A,\varepsilon,\lambda)$ *is P-open ;*

3) *if, for every $q\in S$, F_{pq} are continuous at the point $\varepsilon>0$, then $\bar{N}_p(\varepsilon,\lambda)$ is Q-closed.*

Corollary. *If (S,F,T) satisfies the hypotheses of Prop. 6 and all F_{pq} are continuous on $R-\{0\}$, then the closed sphere $\bar{N}_p(\varepsilon,\lambda)$ is Q-closed for every $p\in S$, $\varepsilon>0$ and $\lambda\geqslant 0$.*

In general, the closed probabilistic spheres may not be closed sets. Indeed, let (R,ρ_0) be the set of real numbers endowed with the usual metric and let G be the member of Δ defined by $G(x)=x$ if $0<x\leqslant 1/2$, and $G(x)=1$ if $x>1/2$. In the simple space [10] generated by (R,ρ_0) and G we have $\bar{N}_0(\varepsilon,\lambda)=(-2\varepsilon,2\varepsilon)$ for $\lambda\in(0,1/2)$. Since the (ε,λ)-topology is the usual topology, $\bar{N}_0(\varepsilon,\lambda)$ is not closed.

3. Triangle inequalities in PT spaces. One of the natural conditions that must be satisfied by a structure of a probabilistic topological space is the following : A probabilistic metric should determine such a structure, just as a metric determines a topology. The purpose of Th. 1 in [3] is to verify that the concept of the probabilistic topological space introduced there satisfies the above requirement. But this theorem is false. An analysis in this direction points out two aspects : a) Th. 1 [3] is true if the t-norm T, which intervenes there, is left-continuous, b) in the probabilistic topological spaces we have to consider also other triangle inequalities besides $(C6)$ considered in [3].

Let (S,F) be a q-p-PM space. We define the mapping $\theta:\mathcal{P}(S)\times[0,1]\to\mathcal{P}(S)$ (we denote $\theta(A,\lambda)=A^\lambda$) by

$$A^\lambda=\begin{cases}\bigcap\limits_{\varepsilon>0}\bar{S}(A,\varepsilon,1-\lambda), & A\neq\varnothing \text{ and } \lambda\in[0,1],\\ \varnothing, & A=\varnothing \text{ and } \lambda\in(0,1],\\ S, & A=\varnothing \text{ and } \lambda=0.\end{cases}\tag{1}$$

In view of Prop. 5, we have (we write $\{p\}^\lambda=p^\lambda$) :

C1. $A^0=S$ C2. $\varnothing^\lambda=\varnothing$ for every $\lambda\in(0,1]$,

C3. $A\subset A^\lambda$ C4. $(A\cup B)^\lambda=A^\lambda\cup B^\lambda$.

C5. $A^\lambda\subset A^\mu$ if $\lambda\geqslant\mu$.

T. $q\in p^1$ and $r\in q^1\Rightarrow r\in p^1$,

LC. $A^\lambda=\bigcap\limits_{\mu<\lambda}A^\mu$ for every $\lambda\in(0,1]$.

Indeed, $C1$, $C2$, $C3$, and $C5$ follow immediately. To show $C4$ we utilize 5) and 8) of Prop. 5. We first have $(A\cup B)^\lambda\supset A^\lambda\cup B^\lambda$. On the other hand, if $q\in(A\cup B)^\lambda$, then $q\in\bigcap\limits_{\varepsilon>0}[\bar{S}(A,\varepsilon,1-\lambda)\cup S(B,\varepsilon,1-\lambda)]$, hence $q\in\bar{S}(A,\varepsilon,1-\lambda)\cup\bar{S}(B,\varepsilon,1-\lambda)$ for every $\varepsilon>0$. It is easy to verify that $q\in\bar{S}(A,\varepsilon,1-\lambda)$ for every $\varepsilon>0$ or $q\in S(B,\delta,1-\lambda)$ for every $\delta>0$. Consequently, $q\in[\bigcap\limits_{\varepsilon>0}\bar{S}(A,\varepsilon,1-\lambda)]\cup[\bigcap\limits_{\delta>0}\bar{S}(B,\delta,1-\lambda)]=A^\lambda\cup B^\lambda$. We now prove the „triangle inequality" T. Let ε, $\delta>0$ be given. Then :

$$q\in p^1,\ r\in q^1\Rightarrow q\in\bar{N}_p(\varepsilon,0),\ r\in\bar{N}_q(\delta,0)\Rightarrow$$

$$F_{pq}(\varepsilon)=1,\ F_{qr}(\delta)=1\Rightarrow F_{pr}(\varepsilon+\delta)=1.$$

Consequently, $r\in\bar{N}_p(\varepsilon+\delta,1)$ for every ε, $\delta>0$, that is $r\in p^1$. The relation LC results from Prop. 5, 7), as follows :

$$A^\lambda=\bigcap_{\varepsilon>0}\bar{S}(A,\varepsilon,1-\lambda)=\bigcap_{\varepsilon>0}\bigcap_{\mu<\lambda}\bigcup_{\eta<\varepsilon}\bar{S}(A,\varepsilon,1-\mu)=$$

$$\bigcap_{\mu<\lambda}\Big(\bigcap_{\varepsilon>0}\bigcup_{\eta<\varepsilon}\bar{S}(A,\eta,1-\mu)\Big)=\bigcap_{\mu<\lambda}\bigcap_{\varepsilon>0}\bar{S}(A,\varepsilon,1-\mu))=\bigcap_{\mu<\lambda}A^\mu.$$

Remarks. 1) In [3] a mapping $\theta:\mathcal{P}(S)\times I\to\mathcal{P}(S)$ satisfying conditions $C1-C3$, $C5$, and, instead of our $C4$, the less restrictive condition $A^\lambda\subset B^\lambda$ if $A\subset B$, is called a θ-*closure*. Also, a θ-closure is left-continuous if the condition LC is satisfied.

2) From (1) it follows an obvious geometrical interpretation of the set A^λ. Another interpretation may be given if we remark that for $A\neq\varnothing$ we have

$$A^\lambda=\{p\ ;\ d(A,p)\,(0+)\geqslant\lambda\},\tag{2}$$

and taking into account the indicated interpretation of $d(A,p)$; this is exactly the interpretation of A^λ given in [3, p. 69] : p is in A^λ if and only if the probability that p is in the Kuratowski closure of A is $\geqslant\lambda$. This coincidence is justified by the equality $d(A,p)\,(0+)=c(A,p)$ [3, Th.1].

Definition. *A probabilistic topological space (PT space) is a couple (S,θ), where S is an arbitrary set and θ is a mapping from $\mathcal{P}(S)\times I$ into $\mathcal{P}(S)$ satisfying the conditions $C1-C5$, T, and LC. A PT space of Menger type is an ordered triple (S,θ,T), where (S,θ) is a PT space and T is a t-norm such that the following „triangle inequality" :*

(TM) $\quad q \in p^{\lambda}$ and $r \in q^{\mu} \Rightarrow r \in p^{T(\lambda, \mu)}, \quad \lambda, \mu \in [0,1]$

is satisfied.

Now, the above results may be summarized in the

Proposition 7. *If (S, F) is a q-p-PM space, then the couple (S, θ) with θ defined by (1), is a PT space (we call it the induced PT space). If (S, F, T) is a q-p-PM space of Menger type, then the induced PT space is of Menger type.*

Proof. It only remains to show that the „triangle inequality" (TM) holds. Let $\alpha > 0$ be an arbitrary number. Then

$$q \in p^{\lambda}, \quad r \in q^{\mu} \Rightarrow F_{pq}(0+) \geqslant \lambda, \quad F_{qr}(0+) \geqslant \mu \Rightarrow F_{pq}\left(\frac{\alpha}{2}\right) \geqslant \lambda, \quad F_{qr}\left(\frac{\alpha}{2}\right) \geqslant \mu.$$

Hence, $F_{pr}(\alpha) \geqslant T(F_{pq}(\alpha/2), F_{qr}(\alpha/2)) \geqslant T(\lambda, \mu)$, and, consequently, $F_{pr}(0+) \geqslant T(\lambda, \mu)$, i.e., $r \in p^{T}(\lambda, \mu)$. Thus, the proof is complete.

Proposition 8. *Let (S, F, T) be a q-p-PM space of Menger type. We have :*

a) *if T is left-continuous in each place, then*

(3) $$(p^{\lambda})^{\mu} \subset p^{T(\lambda, \mu)}, \ p \in S \text{ and } \lambda, \mu \in [0,1] ;$$

b) *if T is a left-continuous t-norm, then*

(4) $$(A^{\lambda})^{\mu} \subset A^{T(\lambda, \mu)}, \ A \subset S \text{ and } \lambda, \mu \in [0,1].$$

Proof. We prove only the statement b). A similar argument works for the other assertion. Because T is left-continuous, we have :

$$\overline{S}(\overline{S}(A, \varepsilon, 1-\lambda), \delta, 1-\mu) \subset \overline{S}(A, \varepsilon+\delta, 1-T(\lambda, \mu)), \ \lambda, \mu \in [0,1].$$

We omit the routine proof of this inclusion. Then

$$(A^{\lambda})^{\mu} = \bigcap_{\delta>0} \overline{S}\left(\bigcap_{\varepsilon>0} \overline{S}(A, \varepsilon, 1-\lambda), \ \delta, 1-\mu\right) \subset \bigcap_{\delta>0} \bigcap_{\varepsilon>0} \overline{S}(\overline{S}(A, \varepsilon, 1-\lambda), \delta,$$
$$1-\mu) \subset \bigcap_{\delta>0} \bigcap_{\varepsilon>0} \overline{S}(A, \ \varepsilon+\delta, 1-T(\lambda, \mu)) = \bigcap_{\eta>0} \overline{S}(A, \eta, 1-T(\lambda, \mu)) = A^{T(\lambda, \mu)}$$

and the proof is complete.

The result of Th. 1 [3] is precised by Prop. 8. In a *PM* space of Menger type the „triangle inequality" (4) does not hold. Moreover, the next example shows that in a *PM* space of Menger type even relation (3) may not hold.

Example. Let T be the t-norm defined as follows : $T(a, b) = \min(a, b)$ for $a \geqslant 1/2$, and $b \geqslant 1/2$; $T(a, b) = a$ for $b = 1$; $T(a, b) = b$ for $a = 1$; $T(a, b) = 0$ otherwise. Now, let $S = N \cup \{p, q\}$, with $p, q \notin N$ and $p \neq q$. Suppose that $m, n \in N$, $m \neq n$, and $r, s \in S$. Let $F_{pn}(x)$ has the values : $1/2\ (1-e^{-nx})$ if $0 < x \leqslant 1$; and 1 if $x > 1$. Let $F_{qn}(x)$ has the values : $1/3\ (1-e^{-nx})$ if $0 < x \leqslant 1$; and 1 if $x > 1$. Let $F_{pq}(x)$ and $F_{mn}(x)$ have the values : 1/2 if $0 < x \leqslant 1$; and 1 if $x > 1$. Finally, $F_{rr} = H$ and $F_{rs} = F_{sr}$ for all r and s.

It is a routine calculation to verify that (S, F, T) is a *PM* space of Menger type. One observes that at the point (1/2, 1/2) the t-norm T is not left-continuous in either one of its variables. The inclusion $(1^{\frac{1}{2}})^{\frac{1}{2}} \subset 1^{T(\frac{1}{2}, \frac{1}{2})}$ is false. Indeed, $(1^{\frac{1}{2}})^{\frac{1}{2}} = N^{\frac{1}{2}} = N \cup \{p\}$ and $1^{T(\frac{1}{2}, \frac{1}{2})} = 1^{\frac{1}{2}} = N$.

4. Generation of q-p-PM spaces. Let (S, F) be a *q-p-PM* space. In Section 3 the induced *PT* space (S, θ) has been defined. We now associate to the given space (S, F) another structure of classical nature. Let ρ be the mapping from $S \times S$ into the set $\overline{R}_{+} = R_{+} \cup \{+\infty\}$ defined by

(5) $$\rho(p, q) = \inf \{\varepsilon ; F_{pq}(\varepsilon) = 1\}$$

(if $\{\varepsilon ; F_{pq}(\varepsilon) = 1\} = \varnothing$, then it is natural to consider $\rho(p, q) = +\infty$). One can easily verify that : a) $\rho(p, p) = 0$, $p \in S$, and b) $\rho(p, r) \leqslant \rho(p, q) + \rho(q, r)$, $p, q, r \in S$. The mapping ρ defined by (5) is called the *induced extended q-p-M* on S. (Here, we use terms as : extended metric, extended *p-M*, extended *q-M*, extended *q-p-M*, aiming to point out that $+\infty$ may be taken as a value. In [1 ; ch. IX, Definition 1] an extended *p-M* is called an *écart*). It is not difficult to verify that τ_{ρ}, i.e., the topology induced by ρ on S, is finer than P. At last, we observe that the spaces (S, θ) and (S, ρ), induced by the space (S, F), appear to be obtained „to the extremities", corresponding to the two facets of the mentioned interpretation for $F_{pq}(x)$.

In the sequel, for two spaces (S, θ) and (S, ρ) of the above types, we give some conditions in which they generate a *q-p-PM* space (S, F).

Theorem 1. *Let S be an arbitrary set, let $A \xrightarrow{\theta} A^{\lambda}$ be a mapping such that (S, θ) is a PT space and let ρ be an extended q-p-M on S. If the following compatibility conditions*

(I) $(S_{\rho}(p, \varepsilon))^{1} \subset \overline{S}_{\rho}(p, \varepsilon), \qquad p \in S, \ \varepsilon > 0$;

(II) $S_{\rho} = \bigcup_{\varepsilon>0} (S_{\rho}(p, \varepsilon))^{\lambda}, \qquad p \in S, \ \lambda \in [0,1)$;

(III) $(S_{\rho}(p, \varepsilon))^{\lambda} = \bigcap_{\mu>\lambda} \bigcup_{\eta<\varepsilon} (S_{\rho}(p, \eta))^{\mu}, \qquad p \in S, \ \varepsilon > 0, \ \lambda \in (0,1]$,

are satisfied ($S_{\rho}(p, \varepsilon)$ and $\overline{S}_{\rho}(p, \varepsilon)$ denote the open and closed spheres relative to ρ), and if $F : S \times S \to \Delta$ is the mapping defined by

(F) $$F_{pq}(\varepsilon) = \begin{cases} 0, & \varepsilon \leqslant 0, \\ \sup \{\lambda ; q \in (S_{\rho}(p, \varepsilon))^{\lambda}\}, & \varepsilon > 0, \end{cases}$$

then (S, F) is a q-p-PM space and we have

a) $\overline{N}_{p}(\varepsilon, 1-\lambda) = (S_{\rho}(p, \varepsilon))^{\lambda}$, *for every $p \in S$, $\varepsilon > 0$, and $\lambda \in [0,1]$;*

b) *the induced extended q-p-M coincides with ρ.*

Proof. We first show that $F_{pq} \in \Delta$. Indeed, it is clear from the definition that $F_{pq}(0) = 0$ and F_{pq} is non-decreasing. It results from the compatibility condition (II) that $\sup_{x} F_{pq}(x) = 1$. The fact that F_{pq} is left-continuous is a consequence of the compatibility condition (III). It suffices to prove this fact with respect to an arbitrary point ε in which $F_{pq}(\varepsilon) > 0$. In this case, for every $\alpha > 0$, there exists $\lambda > F_{pq}(\varepsilon) - \alpha$, $\lambda > 0$, such that $q \in (S_{\rho}(p, \varepsilon))^{\lambda}$. Making use of (III), it follows that for $\mu < \lambda$ there exists an $\eta < \varepsilon$ so that $q \in (S_{\rho}(p, \eta))^{\mu}$. Taking $\mu = F_{pq}(\varepsilon) - \alpha$, we can assert that there is an η so that for every δ, $\eta < \delta \leqslant \varepsilon$, we have $q \in (S_{\rho}(p, \delta))^{\mu}$. Consequently, for every $\alpha > 0$

there exists an $\eta > 0$ such that for every δ, $\eta < \delta \leqslant \varepsilon$, we have $F_{pq}(\delta) \geqslant \mu = F_{pq}(\varepsilon) - \alpha$, i.e., F_{pq} is left-continuous at ε.

From (1) and the fact that $S_\rho(\overline{S}_\rho(p, \varepsilon), \eta) \subset S_\rho(p, \varepsilon + \eta)$ we have:

$$[S_\rho((S_\rho(p, \varepsilon))^1, \eta)]^\mu \subset (S_\rho(p, \varepsilon + \eta))^\mu. \tag{6}$$

As an immediate consequence of (6) it follows that

$$q \in (S_\rho(p, \varepsilon))^1 \text{ and } r \in (S_\rho(q, \eta))^1 \Rightarrow r \in (S_\rho(p, \varepsilon + \eta))^1. \tag{7}$$

We now prove that

$$(S_\rho(p, \varepsilon))^\lambda = \{q \,;\, F_{pq}(\varepsilon) \geqslant \lambda\}, \tag{8}$$

or, equivalently,

$$q \in (S_\rho(p, \varepsilon))^\lambda \Leftrightarrow F_{pq}(\varepsilon) \geqslant \lambda. \tag{8'}$$

Indeed, $q \in (S_\rho(p, \varepsilon))^\lambda \Rightarrow \lambda \in \{\mu \,;\, q \in (S_\rho(p, \varepsilon))^\mu\} \Rightarrow F_{pq}(\varepsilon) \geqslant \lambda$, and $F_{pq}(\varepsilon) \geqslant \lambda \Rightarrow \sup \{\mu \,;\, q \in (S_\rho(p, \varepsilon))^\mu\} \geqslant \lambda \Rightarrow q \in (S_\rho(p, \varepsilon))^\nu$ for every ν, $0 \leqslant \nu < \lambda \Rightarrow q \in \bigcap_{\nu < \lambda} (S_\rho(p, \varepsilon))^\nu \Rightarrow q \in (S_\rho(p, \varepsilon))^\lambda$ (for $\lambda = 0$, the relation (8) is evident).

With these preliminaries, we now establish that (S, F) is a *q-p-PM* space having the properties a) and b). That $F_{pp} = H$, for every $p \in S$, is immediate. The „triangle inequality" IV follows from (8') and (7):

$$F_{pq}(\varepsilon) = 1 \text{ and } F_{qr}(\eta) = 1 \Rightarrow q \in (S_\rho(p, \varepsilon))^1 \text{ and } r \in (S_\rho(q, \eta))^1 \Rightarrow$$
$$\Rightarrow r \in (S_\rho(p, \varepsilon + \eta))^1 \Rightarrow F_{pr}(\varepsilon + \eta) = 1.$$

The relation a) is prescisely the equality (8). To prove the statement b), we use the compatibility condition (I) and (8'). We have

$$\{\varepsilon \,;\, \rho(p, q) < \varepsilon\} = \{\varepsilon \,;\, q \in S_\rho(p, \varepsilon)\} \subset \{\varepsilon \,;\, q \in (S_\rho(p, \varepsilon))^1\} =$$
$$= \{\varepsilon \,;\, F_{pq}(\varepsilon) = 1\} \subset \{\varepsilon \,;\, q \in \overline{S}_\rho(p, \varepsilon)\} = \{\varepsilon \,;\, \rho(p, q) \leqslant \varepsilon\}. \tag{9}$$

Consequently, we have $\inf \{\varepsilon \,;\, F_{pq}(\varepsilon) = 1\} = \rho(p, q)$, where $\rho(p, q)$ may be finite or not. Therefore, the induced extended *q-p-M* is exactly ρ. The proof of the theorem is complete.

We say that the *q-p-PM* space (S, F) constructed in the previous theorem is *generated by the spaces* (S, θ) *and* (S, ρ).

Remarks. 1) The space (S, F) generated by (S, θ) and (S, ρ) satisfies the stronger „triangle inequality":

$$F_{pq}(\varepsilon) = 1 \text{ and } F_{qr}(\delta) \geqslant \lambda \Rightarrow F_{pr}(\varepsilon + \delta) \geqslant \lambda. \tag{10}$$

2) The compatibility condition (I) may be interpreted as follows; the set of points which are almost certain in $\overline{S_\rho(p, \varepsilon)}$ (i.e., the closure of $S_\rho(p, \varepsilon)$ relative to τ_ρ) is included in $\overline{S}_\rho(p, \varepsilon)$. Particularly, we may consider $(S_\rho(p, \varepsilon))^1$ to be equal with $\overline{S}_\rho(p, \varepsilon)$ or $\overline{S_\rho(p, \varepsilon)}$. These two situations will appear in the formulation of compatibility conditions IV and V.

3) From (9), it follows that $\{\varepsilon \,;\, F_{pq}(\varepsilon) = 1\} = \varnothing \Leftrightarrow \rho(p, q) = +\infty$. Consequently, the graph of the function F_{pq} touches the straight line $y = 1$ (i.e., $F_{pq}(x) = 1$ for some $x \in R$) if and only if $\rho(p, q)$ is finite.

The next four theorems show that, under suitable conditions, the generated *q-p-PM* space may be of Menger type.

Theorem 2. *Let us suppose that on the set S are given: 1) a mapping* $A \xrightarrow{\theta} A^\lambda$ *such that* (S, θ, T) *is a PT space of Menger type verifying the „triangle inequality" (3), 2) an extended q-p-M* ρ. *If the compatibility conditions (I), (II), (III), and*

$$(S_\rho(A, \varepsilon))^\lambda = \overline{S}_\rho(A^\lambda, \varepsilon), \quad A \subset S, \ \varepsilon > 0, \ \lambda \in [0,1], \tag{IV}$$

are satisfied, then the space (S, F, T), *where the mapping F is defined by the relation (F), is a q-p-PM space of Menger type having the properties a) and b) of Th. 1.*

Proof. In view of Th. 1, it only remains to show that the triangle inequality IV M holds. To this end, we first show that

$$[S_\rho((S_\rho(p, \varepsilon))^\lambda, \eta)]^\mu \subset (S_\rho(p, \varepsilon + \eta))^{T(\lambda, \mu)}. \tag{11}$$

Indeed, with the aid of relations (IV) and (3), we obtain:

$$[S_\rho((S_\rho(p, \varepsilon))^\lambda, \eta)]^\mu = [(S_\rho(\overline{S}_\rho(p^\lambda, \varepsilon), \eta)_\eta]^\mu \subset (S_\rho(p^\lambda, \varepsilon + \eta))^\mu \subset$$
$$\subset \overline{S}_\rho((p^\lambda)^\mu, \varepsilon + \eta) \subset \overline{S}_\rho(p^{T(\lambda, \mu)}, \varepsilon + \eta) = (S_\rho(p, \varepsilon + \eta))^{T(\lambda, \mu)}.$$

It is clear from (11) that $q \in (S_\rho(p, \varepsilon))^\lambda$ and $r \in (S_\rho(q, \eta))^\mu \Rightarrow r \in (S_\rho(p, \varepsilon + \eta))^{T(\lambda, \mu)}$, or, in view of relation (8'), $F_{pq}(\varepsilon) \geqslant \lambda$ and $F_{qr}(\eta) \geqslant \mu \Rightarrow F_{pr}(\varepsilon + \eta) \geqslant T(\lambda, \mu)$. The first side of this implication is verified for $\lambda = F_{pq}(\varepsilon)$ and $\mu = F_{qr}(\eta)$. In this case, the second side is exactly the triangle inequality *IV M*.

Theorem 3. *Let* (S, θ, T) *and* (S, ρ) *be two spaces as in Th. 2. If the compatibility conditions (I), (II), (III), and*

$$(S_\rho(A, \varepsilon))^\lambda = \overline{S_\rho(A^\lambda, \varepsilon)}, \ A \subset S, \ \varepsilon > 0, \ \lambda \in [0,1], \tag{V}$$

are satisfied, then the conclusion of Th. 2 is true.

Proof. It suffices to show that under the hypotheses of the present thorem the relation (11) is again valid. We have

$$[S_\rho((S_\rho(p, \varepsilon))^\lambda, \eta)]^\mu = [S_\rho(\overline{S_\rho(p^\lambda, \varepsilon)}, \eta)]^\mu \subset (S_\rho(p^\lambda, \varepsilon + \mu))^\mu \subset$$
$$\subset \overline{S_\rho((p^\lambda)^\mu, \varepsilon + \eta)} \subset \overline{S_\rho(p^{T(\lambda, \mu)}, \varepsilon + \eta)} = (S_\rho(p, \varepsilon + \eta))^{T(\lambda, \mu)}.$$

Thus (11) holds and the proof may be continued as in Th. 2.

Remark. The conclusion of Th. 2 holds if we substitute the compatibility condition (IV) by the less restrictive conditions $(S_\rho(p, \varepsilon))^\lambda = \overline{S}_\rho(p^\lambda, \varepsilon)$ and $(S_\rho(A, \varepsilon))^\lambda \subset \overline{S}_\rho(A^\lambda, \varepsilon)$. We can formulate a similar remark regarding the compatibility condition (V) of Th. 3.

If the space (S, θ, T) satisfies the stronger „triangle inequality" (4), then, in the conditions (IV) and (V), A may be a singleton.

Theorem 4. *Suppose that on the set S are given: 1) a mapping* $A \xrightarrow{\theta} A^\lambda$ *such that* (S, θ, T) *is a PT space of Menger type satisfying the „triangle inequality" (4), 2) an extended q-p-M* ρ. *If the compatibility conditions (I), (II), (III), and*

$$(IV_p) \qquad (S_\rho(p, \varepsilon))^\lambda = \overline{S}_\rho(p^\lambda, \varepsilon),\ p \in S,\ \varepsilon > 0,\ \lambda \in [0,1],$$

are satisfied, then the conclusion of Th. 2 holds.

Proof. In our hypotheses the relation (11) is verified. Indeed,

$$[S_\rho((S_\rho(p, \varepsilon))^\lambda, \eta)]^\mu = [S_\rho(\overline{S}_\rho(p^\lambda, \varepsilon), \eta)]^\mu \subset (\overline{S}_\rho(p^\lambda, \varepsilon + \eta))^\mu =$$
$$= [(S_\rho(p, \varepsilon + \eta))^\lambda]^\mu \subset (S_\rho(p, \varepsilon + \eta))^{T(\lambda, \mu)}.$$

Theorem 5. *Let (S, θ, T) and (S, ρ) be two spaces as in Th. 4. If the compatibility conditions (I), (II), (III), and*

$$(S_\rho(p, \varepsilon))^\lambda = \overline{S_\rho(p^\lambda, \varepsilon)},\ p \in S,\ \varepsilon > 0,\ \lambda \in [0,1],$$

are verified, then the conclusion of Th. 2 is true.

It remains to approach the question of determining some conditions under which the space (S, F, T), generated by one of Th. 2–5, induces the initial spaces (S, θ, T) and (S, ρ). This question is the object of a subsequent paper.

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LE FIBRÉ DES TENSEURS DE TYPE (1,1) SUR UNE VARIÉTÉ DIFFÉRENTIABLE

PAR

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Parmi les fibrés associés d'une manière canonique à une variété différentiable, le fibré des tenseurs de type (1,1) présente un intérêt particulier. A côté des propriétés géométriques généralles des fibrés vectoriels [1,2] et tensoriels [5], le fibré des tenseurs de type (1,1) possède des propriétés spéciales, remarquables, déterminées par sa structure de fibré en algèbres associatives et par suite en algèbres de Lie, sur le champ R des réels.

Ce travail est le premier d'un cycle consacré à l'étude de ce fibré.

1. Introduction. Soient M une variété différentiable à n dimensions, de classe C^∞, E l'ensemble des tenseurs de type (1,1) sur M et $\pi : E \to M$ la projection canonique. En associant à chaque carte locale (U, φ, R^n) sur M, les triplets $(\pi^{-1}(U), \Phi, R^n \times R^{n^2})$ et $(U, \psi, gl(n, R))$, où $\Phi(t_p) = (x^i, y^i_j)$ avec $\varphi(p) = (x^i)$, $t_p = y^i_j \dfrac{\partial}{\partial x^i} \otimes dx^j$ et $\psi(t_p) = (p, y = \|y^i_j\|)$, on obtient une structure de variété différentiable de classe C^∞ sur E et une structure de fibré vectoriel (E, π, M) de classe C^∞ avec la fibre type $gl(n, R)$. En considérant l'application $\tau_p : gl(n, R) \to E_p$ donnée par

$$(1) \qquad \tau_p(y) = y^i_j \frac{\partial}{\partial x^i} \otimes dx^j,$$

on en obtient pour $p \in M$ et $y', y'' \in gl(n, R)$,

$$(2) \qquad \tau_p(y' \circ y'') = \tau_p(y') \circ \tau_p(y''), \qquad \tau_p([y', y'']) = [\tau_p(y'), \tau_p(y'')].$$

Par suite, (E, π, M) est un fibré en algèbres associatives et en algèbres de Lie, localement trivial. L'ensemble $\Gamma(E)$ des sections C^∞ sur E est une algèbre associative et une algèbre de Lie sur l'anneau $\mathcal{F}(M)$ des fonctions réelles C^∞ sur M. Le triplet (K, π_1, M), où K est l'ensemble des tenseurs de type (1,1) nonsinguliers sur M et $\pi_1 = \pi_{/K}$, est un fibré en groupes de Lie, localement trivial, avec la fibre type $GL(n, R)$. (E, π, M) est le fibré en algèbres de Lie correspondant et K est une sous-variété ouverte de E.

Un changement de coordonées locales sur E, induit par un changement sur M, est donné par

$$(3) \qquad x^{i'} = x^{i'}(x^i), \quad y^{i'}_{j'} = \frac{\partial x^{i'}}{\partial x^i} y^i_j \frac{\partial x^j}{\partial x^{j'}}.$$