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MORPHISMS OF H-CONES (II)

BY

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This paper continues the study of morphisms of H-cones, begun in [11]. Accordingly, the same notions and notations as in [11] will be used. Most of the results presented here were announced in [9].

§5 studies the morphisms of standard H-cones, which preserve a weak unit. Prop. 5.4 and 5.6 give the characterisation of H-maps and cor. 5.7 apply these results to isomorphisms of H-cones, getting thus a „Stone-Banach type" theorem. Prop. 5.8 gives a characterisation of quasi H-maps. Prop. 5.9 and 5.10 give compacity results for morphisms. Finally, prop. 5.11 and 5.12 give results about the extreme points in the set of morphisms of pointed H-cones.

A remarkable property of harmonic mappings is presented in prop. 6.2 and 6.5. This property justifies the introduction of a new class of morphisms (def. 6.6). The existence of the adjoint for such morphisms is proved in prop. 6.8. Prop. 6.9 and 6.10 show that such morphisms give rise to „localised" morphisms, between cones of the form C_u.

In §7 the morphisms of (pre-)sheaves of H-cones are studied.

§ 5. **Morphisms of pointed H-cones. Definition 5.1.** *Any pair (C, u), where C is a standard H-cone and $u \in C$ is a weak unit, will be called a pointed H-cone.*

We recall [2] that the set : $K(C, u) = \{\mu \in C^* \mid \mu(u) \leq 1\}$ is a convex, compact space in the natural topology. X will denote the set of non-zero extreme points of $K(C, u)$. C is then isomorphic to the standard H-cone of functions on the saturated space X_u, formed of the restrictions to X_u of the maps : $\mu \to \mu(s)$.

Definition 5.2. *Let (C, u), (C', u') be pointed H-cones. We call a morphism of pointed H-cones any $\varphi \in \mathrm{Hom}\,(C, C')$, such that $\varphi(u) = u'$.*

Clearly, any morphism of pointed H-cones is semi-finite, hence it has an adjoint $\varphi^* \in \mathrm{Hom}\,(C'^*, C^*)$. We denote $K(\varphi)$ the restriction of φ^* to $K(C, u)$. Obviously, $K(\varphi)$ is a map from $K(C, u)$ to $K(C', u')$. If C_c denotes the set of all $s \in C$, such that the map : $\mu \to \tilde{s}(\mu) = \mu(s)$ is naturally continuous on $K(C, u)$, then :

Proposition 5.3. *$K(\varphi)$ is an affine map. $K(\varphi)$ is fine-to-fine continuous and naturally measurable. $K(\varphi)$ is naturally continuous iff $\varphi(C_0) \subseteq C'_c$.*

Prof. All the continuity assertions follow from the formula : $\tilde{s}[K(\varphi)(\mu)] = \widetilde{\varphi(s)}(\mu)$.

Theorem 5.4. *Let φ be a morphism of pointed H-cones. The following are equivalent :*
1) *For each extreme point μ of $K(C', u')$, $K(\varphi)(\mu)$ is an extreme point of $K(C, u)$.*
2) $\varphi(s \wedge t)=\varphi(s) \wedge \varphi(t)$, $\forall s, t \in C$.
3) *There exists a (unique H-)map, denoted by $p(\varphi) : X'_{u'} \to X_u$, such that : $\varphi(s)=s \circ p(\varphi)$, $\forall s \in C$ (i.e. $p(\varphi)$ induces φ).*

Proof. Let us remark first, that if $\mu \neq 0$ is an extreme point in $K(C', u')$, then $K(\varphi)(\mu) \neq 0$, since : $K(\varphi)(\mu)(u)=\mu(\varphi(u))=\mu(u')=1$. $1 \Rightarrow 3 \Rightarrow 2$ are now obvious. $2 \Rightarrow 1$ Using the characterisation of extreme points of $K(C, u)$, as given in [2] we have, for any extreme point $\mu \in K(C', u')$:

$$K(\varphi)(\mu)(s \wedge t)=\mu(\varphi(s \wedge t))=\mu(\varphi(s) \wedge \varphi(t))=\min\{\mu(\varphi(s)), \mu(\varphi(t))\}=$$
$$=\min\{K(\varphi)(\mu)(s), K(\varphi)(\mu)(t)\}.$$

Corollary 5.5. *Let φ be a morphism of pointed H-cones, satisfying one of the equivalent conditions of the th. 5.4. Then :* a) *$p(\varphi)$ is a fine-to-fine and naturally measurable map.* b) [12] *If C satisfies the following „Global peak point" condition : „$\forall x \in X_u\ \exists s \in C$ such that, for any natural neighbourhood V of x we have : $\sup_{CV} s < s(x)$", then $p(\varphi)$ is naturally continuous.*

If we denote : $C^u=\{s \in C \mid \exists \alpha \geqslant 0,\ s \leqslant \alpha u\}$, it is known that $[C^u]$ is an algebra.

Proposition 5.6. *Let φ be a morphism of pointed H-cones. Then φ satisfies the equivalent conditions of the th. 5.4 iff: $\varphi(s \cdot t)=\varphi(s) \cdot \varphi(t)$, $\forall s, t \in C^u$.*

Proof. Let φ be a morphism of pointed H-cones such that $\varphi(s) \cdot \varphi(t)=\varphi(s \cdot t)$, $\forall s, t \in C^u$. Then :

$$[\varphi(|s-t|)]^2=\varphi(|s-t|^2)=\varphi((s-t)^2)=[\varphi(s-t)]^2=(\varphi(s)-\varphi(t))^2=$$
$$=(|\varphi(s)-\varphi(t)|)^2, \quad \forall s, t \in C^u.$$

Since $s \wedge t=(1/2)(s+t-|s-t|)$, it follows that φ satisfies th. 5.4.2. Conversely, if φ satisfies th. 5.4.3, then clearly $\varphi(s \cdot t)=\varphi(s) \cdot \varphi(t)$, $\forall s, t \in C^u$.

Remark. Since $u^2=u$, we can suppose only that $\varphi(u) \in C'^{u'}$, instead of φ being a morphism of pointed H-cones.

Proposition 5.7. *Let C, C' be isomorphic standard H-cones. Let φ be an isomorphism between C and C'. Then there exists a weak unit $u \in C$, such that $\varphi(u) \in C'$ is a weak unit. The saturated spaces X_u and $X_{\varphi(u)}$ are homeomorphic.*

Proof. Let $u' \in C'$ be a weak unit. Since φ is a bijection between C^{**} and C'^{**}, there exists $v \in C^{**}$ such that $\varphi(v)=u'$. Clearly, v is a weak unit. Let $u \in C$ be a weak unit. Then $u \wedge v \in C$ is a weak unit, and $\varphi(u \wedge v) \in C'$ is also a weak unit. φ becomes an isomorphism of pointed H-cones $(C, u \wedge v)$, $(C', \varphi(u \wedge v))$. Since $\varphi(s \wedge t)=\varphi(s) \wedge \varphi(t)$ and $\varphi(C_0) \subseteq C'_0$, it follows that $p(\varphi)$ is a homeomorphism between $X_{u \wedge v}$ and $X_{\varphi(u \wedge v)}$.

Remark. The above result can be considered as a „Stone-Banach type" theorem for standard H-cones [13].

We prove next the converse of prop. 4.6 from [11].

Proposition 5.8. *Let C, C' be standard H-cones of functions on nearly (resp. semi-)saturated spaces X, X'. Let φ be a morphism of pointed H-cones*

$(C, 1)$, $(C', 1)$ such that : $\varphi(s \wedge t)=\varphi(s) \wedge \varphi(t)$, $\forall s, t \in C$, $\forall \mu \in (C')^*_0$ (resp. universally bounded), $\varphi^*(\mu)$ is an H-measure on X.

Then there exists $A' \subseteq X$ semi-polar (resp. such that any compact set of A' is polar) and a (unique) map $p(\varphi) : X' \setminus A' \to X$, such that $\varphi(s)=s \circ p(\varphi)$ on $X' \setminus A'$, $\forall s \in C$.

Proof. From th. 5.4, $p(\varphi) : X'_1 \to X_1$ exists and is a H-map. Let us denote : $A'=\{x \in X'_1 \mid p(\varphi)(x) \notin X\}$. For any compact $K \subseteq A'$ and any $\mu \in (C')^*_0$ (resp. universally bounded) we have : $\mu(K) \leqslant \mu(p(\varphi)^{-1}(p(\varphi)(K))=$ $=\varphi^*(\mu)(p(\varphi)(K))=0$, since $\varphi^*(\mu)$ is a H-measure. Hence [1], A' has the desired properties.

Remark. The hypothesis that X, X' are nearly (resp. semi-)saturated must be added in prop. 4.6 (resp. the remark that follows this proposition) from [11].

We present now two natural-compacity results for sets of morphisms of H-cones.

Theorem 5.9. *Let C, C' be standard H-cones and $A \subseteq$ Hom (C, C'). The following are equivalent :*
1) *A is relatively compact in the natural topology* [5];
2) $\forall \mu \in C_0'^*$, $\forall s \in C_0$ *we have* $\sup_{\varphi \in A} \mu(\varphi(s)) < +\infty$;
3) $\exists \mu \in C'^*$, $\exists u \in C$ *weak units, such that* $\mu(\varphi(u)) \leqslant 1$, $\forall \varphi \in A$;

Proof. $1 \Rightarrow 2$ is clear and $3 \Rightarrow 1$ is prop. 1.8 from [5]. $2 \Rightarrow 3$. Let $\{\mu_m \mid m \in \mathbf{N}\} \subseteq C_0'^*$, $\{s_n \mid n \in \mathbf{N}\} \subseteq C_0$ be dense sets. If $\gamma_{mn}=\sup_{\varphi \in A} \mu_m(\varphi(s_n)) < +\infty$, then one can choose inductively : $\alpha_m > 0$, $\beta_n > 0$ such that : $\alpha_m \beta_n \gamma_{mn} \leqslant 1$, $\forall m, n \in \mathbf{N}$. From [5] it follows that there exists $0 < \alpha'_m \leqslant 1$, $0 < \beta'_n$ such that :

$$\mu=\sum_{m \in \mathbf{N}} 2^{-m} \alpha_m \alpha'_m \mu_m \in C'^*, \quad u=\sum_{n \in \mathbf{N}} 2^{-n} \beta'_n \beta_n s_n \in C^{**}.$$

μ and u are weak units and we have : $\mu(\varphi(u))=\sum_{m,n \in \mathbf{N}} 2^{-m} 2^{-n} \alpha_m \beta_n \alpha'_m \beta'_n \mu_m(\varphi(s_n)) \leqslant 1$.
Clearly, there exists a weak unit $u' \in C$, such that $u' \leqslant u$.

Remark. This results extends prop. 4.2.7 from [2].

Theorem 5.10. *Let (C, u), (C', u') be pointed H-cones. If u is nearly continuous, then the set of all morphisms of pointed H-cones is a convex, compact set in the natural topology.*

Proof. Since there exists a weak unit $\mu \in C'^*$, such that $\mu(u') \leqslant 1$, it remains to show only that $\varphi(u)=u'$, for any morphism φ, which is the limit of a sequence (φ_n) of morphisms of pointed H-cones. Let : $u=\sum_{n \in \mathbf{N}} s_n$, $\mu \in C_0'^*$ and $\varepsilon > 0$. There exists $n_0 \in \mathbf{N}$ such that : $\mu(\varphi(\sum_{n \geqslant n_0} s_n)) < \varepsilon$, hence :

$$\mu(\varphi(\sum_{n \in \mathbf{N}} s_n)) \leqslant \liminf_{m \to \infty} \mu(\varphi_m(u)) \leqslant \limsup_{m \to \infty} \mu(\varphi_m(u)) \leqslant \mu(\varphi(\sum_1^{n_0} s_n))+2\varepsilon.$$

It follows that there exists $\lim_{m \to \infty} \mu(\varphi_m(u))$ and that it equals $\mu(\varphi(u))$.

Remark. The same proof as above shows that, without any assumptions on u, the set of all morphisms $\varphi \in$ Hom (C, C') such that $\varphi(u) \leqslant u'$ is a convex, compact set in the natural topology.

We end with two results about the extreme points of the convex, compact set of morphisms of pointed H-cones.

Proposition 5.11. *Any H-map between the saturated spaces, induces an extreme element in the set of all morphisms φ, such that $\varphi(u)\leqslant u'$.*

Proof. Let φ be induced by a H-map, and suppose that: $\varphi=\alpha\psi+(1-\alpha)\,\xi$, $\alpha\in(0,1)$. It follows that: $u'=\varphi(u)=\alpha\psi(u)+(1-\alpha)\,\xi(u)\leqslant\alpha u'+(1-\alpha)\,u'=u'$, hence $\psi(u)=\xi(u)=u'$. Moreover, for any $s,t\in C$:

$$\varphi(s)\wedge\varphi(t)=\varphi(s\wedge t)=\alpha\psi(s\wedge t)+(1-\alpha)\,\xi(s\wedge t)\leqslant\alpha(\psi(s)\wedge\psi(t))+$$
$$+(1-\alpha)\,(\xi(s)\wedge\xi(t))\leqslant\varphi(s)\wedge\varphi(t).$$

Then: $\psi(s\wedge t)=\psi(s)\wedge\psi(t)$ and $\xi(s\wedge t)=\xi(s)\wedge\xi(t)$. Hence ψ and ξ are induced by H-maps (which will be denoted by the same letter). Hence, for any $x\in X$ and any $s\in C'$: $s(\varphi(x))=\alpha s(\psi(x))+(1-\alpha)\,s(\xi(x))$.
If we suppose that $\varphi(x)=\psi(x)$ and $\varphi(x)\neq\xi(x)$, then we can choose $s\in C$, which separates $\varphi(x)$ and $\xi(x)$, thus arriving at a contradiction. Suppose now that all three points $\varphi(x)$, $\psi(x)$, $\xi(x)$ are distinct. If there exists $s\in C$ such that $s(\varphi(x))>s(\psi(x))$ and $s(\varphi(x))\geqslant s(\xi(x))$, then we obtain again a contradiction. Hence, it remains to consider the case when: $s(\psi(x))>s(\varphi(x))>s(\xi(x))$, $\forall s\in C$. Let us denote: $f(s)=(s(\psi(x))-s(\xi(x)))\cdot[s(\varphi(x))-s(\xi(x))]^{-1}$. If $p\in C$ is a generator, we can choose a balayage B on C such that $f(p)\neq f(Bp)$, and this is again a contradiction.

Remark. In the case $C'=\mathbf{R}_+$, the converse is also true: any extreme H-integral is induced by a H-map. If this holds in general is an open question.

Proposition 5.12. *The set of all morphisms induced by H-maps is a G_δ-set in the natural topology.*

Proof. Let $\{s_n\}$ be a countable, dense subset of universally continuous elements of C. Let us denote:

$$A_{n,m,k}=\{\varphi\in\mathrm{Hom}\,(C,C')\mid\varphi(1)=1,\quad\varphi(s_n\wedge s_m)+k^{-1}>\varphi(s_n)\wedge\varphi(s_m)\}.$$

Every $A_{n,m,k}$ is a G_δ-set, and $\bigcap\limits_{n,m,k}A_{n,m,k}$ is exactly the set of all morphisms induced by H-maps.

§ 6. Localisable morphisms. In this part, we study a special type of morphisms of H-cones, whose definition is imposed by the property proved in prop. 6.2 and 6.5.

We begin by recalling:

Definition 6.1. *Let $(X,\mathcal{U})$, $(X',\mathcal{U}')$ be harmonic spaces [4]. A continuous map $\varphi:X'\to X$ is called a harmonic mapping if, for any open set $U\subseteq X$ and any hyperharmonic function $s\in\mathcal{U}(U)$, we have $s\circ\varphi\in\mathcal{U}'(\varphi^{-1}(U))$.*

We prove an important property of such mappings:

Proposition 6.2. *Let $(X,\mathcal{U})$, $(X',\mathcal{U}')$ be $\mathfrak{P}$-harmonic spaces, with countable base, and $\varphi:X'\to X$ a harmonic mapping. For any regular open set $U\subseteq X$ and any positive superharmonic $s\in\mathcal{U}(X)$, we have:*

$$B^{X'\setminus\varphi^{-1}(U)}\,(s\circ\varphi)=(B^{X\setminus U}s)\circ\varphi.$$

Proof. It suffices to prove the equality for continuous s. Since $B^{X\setminus U}s=s$ on $X\setminus U$, it follows that $B^{X'\setminus\varphi^{-1}(U)}\,(s\circ\varphi)\leqslant B^{X\setminus U}s\circ\varphi$. Moreover, both functions are harmonic on $\varphi^{-1}(U)$. Now: $p=s\circ\varphi-B^{X'\setminus\varphi^{-1}(U)}\,(s\circ\varphi)$ is a potential on $\varphi^{-1}(U)$, and dominates $p'=s\circ\varphi-(B^{X\setminus U}s)\circ\varphi$ on $\varphi^{-1}(U)$. The difference $p-p'$ being harmonic on $\varphi^{-1}(U)$, we conclude the equality.

The same proof, together with [6] gives:

Corollary 6.3. *Let $(X,\mathcal{U})$, $(X',\mathcal{U}')$ be $\mathfrak{P}$-harmonic spaces with countable base, and $\varphi:X'\to X$ a locally non-constant harmonic mapping. If $(X,\mathcal{U})$ satisfies the axiom of polarity, then: $B^{X'\setminus\varphi^{-1}(U)}\,(s\circ\varphi)=(B^{X\setminus U}s)\circ\varphi$, for any open $U\subseteq X$ and any positive superharmonic $s\in\mathcal{U}(X)$.*

Proposition 6.4. Let *C, C' be H-cones of functions on X, X'. Let τ, τ' be regular topologies on X, X', coarser than the fine topology. Suppose that $\varphi:(X',\tau')\to(X,\tau)$ is a continuous H-map, and that there exists a base $\mathcal{B}$ for τ, such that: $B^{X'\setminus\varphi^{-1}(U)}\,(s\circ\varphi)=(B^{X\setminus U}s)\circ\varphi$, for any $U\in\mathcal{B}$ and (any) $s\in C$. Then:* carr $(s\circ\varphi)\subseteq\varphi^{-1}($carr $s)$ *for (any) $s\in C$.*

Proof. Let $x\in X'$ be such that $y=\varphi(x)\notin$ carr s. There exists $U\in\mathcal{B}$ which is a neighbourhood for y, such that $B^{X\setminus U}s=s$. It follows that $B^{X'\setminus\varphi^{-1}(U)}(s\circ\varphi)=(B^{X\setminus U}s)\circ\varphi=s\circ\varphi$, which proves that $x\notin$ carr $(s\circ\varphi)$.

A partial converse holds, namely:

Proposition 6.5. *Let C, C' be H-cones of functions on X, X'. Let $\varphi:X'\to X$ be a H-map, such that:*

$$A\subseteq X\ \text{polar}\Rightarrow\varphi^{-1}(A)\subseteq X'\ \text{polar}.$$
$$\text{carr}\,(s\circ\varphi)\subseteq\varphi^{-1}\,(\text{carr}\ s),\quad\forall s\in C,$$

If $U\subseteq X$ is open and $(X\setminus U)\setminus b(X\setminus U)$ is polar, then:

$$B^{X'\setminus\varphi^{-1}(U)}(s\circ\varphi)=(B^{X\setminus U}s)\circ\varphi.$$

Proof. Since $s=B^{X\setminus U}s$ on $b(X\setminus U)$, it follows that: $s\circ\varphi=B^{X\setminus U}s\circ\varphi$ on $\varphi^{-1}(b(X\setminus U))$. The hypotheses show that $(X'\setminus\varphi^{-1}(U))\setminus\varphi^{-1}(b(X\setminus U))$ is polar, hence: $B^{X'\setminus\varphi^{-1}(U)}\,(s\circ\varphi)\leqslant(B^{X\setminus U}s)\circ\varphi$. Now, since φ preserves the carrier, $B^{X\setminus U}s\circ\varphi$ is invariant under any balayage which dominates $B^{X'\setminus\varphi^{-1}(U)}$, proving thus the equality.

The above results suggested the following definition:

Definition 6.6. *Let $(C,\mathcal{B})$, $(C',\mathcal{B}')$ be pairs, with C, C' standard H-cones, and $\mathcal{B}$, $\mathcal{B}'$ sets of balayages on C, C'. Between such pairs we define two types of morphisms, as pairs (φ,ψ), where $\varphi\in\mathrm{Hom}\,(C,C')$ and: a) for the first type: $\psi:\mathcal{B}'\to\mathcal{B}$ is a map, such that the following diagram is commutative, for any $B'\in\mathcal{B}'$:*

$$\begin{tikzcd}
\tilde{C} \arrow[r, "\varphi"] \arrow[d, "\psi(B')"'] & \tilde{C}' \arrow[d, "B'"] \\
\tilde{C} \arrow[r, "\varphi"] & \tilde{C}'
\end{tikzcd}$$

i.e. $\forall s\in\tilde{C}:B'(\varphi(s))=\varphi(\psi(B')(s))$. b) for the second type: $\psi:\mathcal{B}\to\mathcal{B}'$ is a map, such that the following diagram is commutative, for any $B\in\mathcal{B}$:

$$\begin{tikzcd}
\tilde{C} \arrow[r, "\varphi"] \arrow[d, "B"'] & \tilde{C}' \arrow[d, "\psi(B)"] \\
\tilde{C} \arrow[r, "\varphi"] & \tilde{C}'
\end{tikzcd}$$

i.e. $\forall s\in C:\varphi(Bs)=\psi(B)\,(\varphi(s))$.

It is clear that, with each type of morphisms, we can define a category.

Examples 6.7. For any ψ, $(0, \psi)$ is a morphism of both types. Also $(1, 1)$ is a morphism of both types. If $C = C'$ and φ is given, then ψ can be choosen for both types, as identity on those balayages which are greater than φ (this is the only case considered in [5]). Let us consider the case $C = \mathbf{R}_+$. If (φ, ψ) is a morphism of the first type, then ψ can be defined only on those balayages B' on C', for which : $B's = 0$ or $B's = s$, where $s = \varphi(1)$. However, if (φ, ψ) is of the second type, then for given φ, ψ can be generally chosen in an infinity of ways. Now, if $C' = \mathbf{R}_+$, then for each $\mu \in C^*$, there exists generally an infinity of maps ψ, such that (μ, ψ) is a morphism of the first type ; while, in order that (μ, ψ) be a morphism of the second type, ψ must be defined only on those balayages B which satisfies either :

$$\mu(s) = \mu(Bs) \text{ or } \mu(Bs) = 0, \quad \forall s \in C.$$

We show now how to define the adjoint of the morphisms introduced above. This construction will justify also the consideration of the two types of morphisms. We denote $\mathcal{B}^* = \{B^* \mid B \in \mathcal{B}\}$ and : $\psi^*(B) = [\psi(B^*)]^*$, where B is a balayage on C^*.

Proposition 6.8. *Let (φ, ψ) be a morphism of the first (resp. the second) type from $(C, \mathcal{B})$ to $(C', \mathcal{B}')$. If φ^* exists, then (φ^*, ψ^*) is a morphism of the second (resp. the first) type, from $(C'^*, \mathcal{B}'^*)$ to $(C^*, \mathcal{B}^*)$.*

Proof. Indeed, we have for any $B \in \mathcal{B}'^*$, $\mu \in C'^*$, $t \in C$:

$$\varphi^*(B\mu)(t) = B\mu(\varphi(t)) = \mu(B^*(\varphi(t))) = \mu(\varphi(\psi(B^*)(t)), \text{ and : } \psi^*(B)(\varphi^*(\mu))(t) = $$
$$= \varphi^*(\mu)([\psi^*(B)]^*(t) = \mu(\varphi(\psi(B^*))(t).$$

Respectively, for any $B \in \mathcal{B}^*$, $\mu \in C'^*$, $s \in C$:

$$B(\varphi^*(\mu))(s) = \varphi^*(\mu)(B^*s) = \mu(\varphi(B^*s)) = \mu(\psi(B^*))(\varphi(s)) \text{ and : } \varphi^*(\psi^*(B)\mu)(s) =$$
$$= \psi^*(B)[\mu(\varphi(s))] = \mu(\psi^*(B))^*(\varphi(s)) = \mu[\psi(B^*)](\varphi(s)).$$

Next, we prove that the morphisms considered above, can be localised. Let $(\varphi, \psi) : (C, \mathcal{B}) \to (C', \mathcal{B})$ be a morphism of the first type, where φ is semi-finite. For each $B' \in \mathcal{B}'$, we define $\varphi_{B'} \in \operatorname{Hom}(C_{\psi(B')}, C'_{B'})$ as follows : $\varphi_{B'}(s - \psi(B')s) = \varphi(s) - B'(\varphi(s))$, for each $s \in D(\varphi)$. (φ, ψ) being a morphism of the first type, it follows that $\varphi_{B'}$ is correctly defined. By [5], there exists a generator $p \in D(\varphi)$; hence $p \cdot \psi(B')p \in C_{\psi(B')}$ is a weak unit. Now, it is easy to see that $\varphi_{B'}$ is additive and increasing. Let $\{s_i - \psi(B')s_i\}$ be an increasing family in $C_{\psi(B')}$ such that : $s - \psi(B')s = V(s_i - \psi(B')s_i)$ Since we can suppose $s_i \wedge \psi(B')s_i = 0$, $\forall_i$, we have : $s - \psi(B')s = Vs_i - \psi(B')(Vs_i)$. This equality proves that $\varphi_{B'}$ is also continuous. Hence :

Proposition 6.9. *$\varphi_{B'}$ is a semi-finite morphism ; if φ is finite, then $\varphi_{B'}$ is also finite.*

Using the same arguments, one proves that, if (φ, ψ) is a morphism of the second type, then, for each $B \in \mathcal{B}$ we have a semifinite morphism : $\varphi_B : C_B \to C'_{\psi(B)}$, defined by : $\varphi_B(s - Bs) = \varphi(s) - \psi(B)(\varphi(s))$, $\forall s \in D(\varphi)$. If φ is finite, then φ_B is also finite.

Let again (φ, ψ) be a morphism of the first type. Let $B' \in \mathcal{B}'$ be fixed. We can apply the above considerations to $\varphi = B_1$, and $\psi = 1_{B'}$, where $B_1 \in \mathcal{B}'$

is such that $B_1 \geq B'$. Since $B'B_1 = B_1B' = B'$, we have a morphism, denoted still by $B_1 : C'_{B'} \to C'_{B'}$, defined by: $B_1(s - B's) = B_1s - B'(B_1s)$. It is easy to see that B_1 is a balayage on $C'_{B'}$. $\mathcal{B}'|_{B'}$ will denote the set of balayages on $C'_{B'}$ of this form.

If we suppose now that $\psi : \mathcal{B}' \to \mathcal{B}$ is monotone, then it induces a map $\psi_{B'} : \mathcal{B}'|_{B'} \to \mathcal{B}|_{\psi(B')}$. We have thus proved :

Proposition 6.10. *The couple $(\varphi_{B'}, \psi_{B'})$ is a morphism of the first type, from $(C_{\psi(B')}, \mathcal{B}|_{\psi(B')})$ to $(C'_{B'}, \mathcal{B}'|_{B'})$.*

An entirely analogous construction gives a morphism of the second type (φ_B, ψ_B). We omit the details.

We end by proving a commutativity property with the „restriction" morphisms. Let again $B' \in \mathcal{B}'$ be fixed, and $B_1 \in \mathcal{B}'$ be such that $B' \leq B_1$. We suppose also that $\psi : \mathcal{B}' \to \mathcal{B}$ is monotone. We define $\pi : C'_B \to C'_{B_1}$ by : $\pi(s - B's) = s - B_1s$. It is easy to see that π is correctly defined and additive ; π is also monotone and continuous with respect to the specific order. Now, we have a commutative diagram :

$$\begin{array}{ccc} C_{\psi(B')} & \xrightarrow{\varphi_{B'}} & C'_{B'} \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ C_{\psi(B_1)} & \xrightarrow{\varphi_{B_1}} & C'_{B_1} \end{array}$$

Moreover, the vertical arrows being defined in an obvious manner, we have also the commutative diagram :

$$\begin{array}{ccc} \mathcal{B}|_{\psi(B')} & \xleftarrow{\psi_{B'}} & \mathcal{B}'|_{B'} \\ \uparrow & & \uparrow \\ \mathcal{B}|_{\psi(B_1)} & \xleftarrow{\psi_{B_1}} & \mathcal{B}'|_{B_1} \end{array}$$

In the same way, we have a commutative diagram for the morphisms of the second type :

$$\begin{array}{ccc} C_B & \xrightarrow{\varphi_B} & C'_{\psi(B)} \\ {\scriptstyle\pi}\downarrow & & \downarrow \\ C_{B_1} & \xrightarrow{\varphi_{B_1}} & C'_{\psi(B_1)} \end{array}$$

Finally, we remark that, under the same conditions as above, we have another type of compatibility, namely : $(\varphi_{B'})_{B_1} = \varphi_{B_1}$ and $(\psi_{B'})_{B_1} = \psi_{B_1}$.

Indeed, we have : $C'_{B_1} = (C'_{B'})_{B_1}$ and $C_{\psi(B_1)} = [C_{\psi(B')}]_{\psi(B_1)}$. If $s \in C$, we have, if $t = s - \psi(B')s \in C_{\psi(B')}$:

$$(\varphi_{B'})_{B_1}(t - \psi(B_1)t) = \varphi_{B'}(s - \psi(B')s) - B_1(\varphi_{B_1}(s - \psi(B')s) =$$
$$= \varphi(s) - B'(\varphi(s)) - B_1(\varphi(s) - B'(\varphi(s))) = \varphi(s) - B_1(\varphi(s)) = \varphi_{B_1}(s - \psi(B_1)s)$$

and $t - \psi(B')t = s - \psi(B_1)s$.

Analogous relations hold for the morphisms of the second type.

§7. Morphisms of sheaves of H-cones. Let (X, τ), (X', τ') be topological spaces, and $\mathcal{F}$, $\mathcal{F}'$ (pre) sheaves on X, X'. We recall :

Definition 7.1. [14] *By a φ-morphism we mean a collection of morphisms $(\Phi_U)_{U \in \tau}$, such that, for each $U \in \tau$:*

$$\Phi_U : \mathcal{F}(U) \to \mathcal{F}'(\varphi^{-1}(U)),$$

and the commutativity with the restrictions holds: if V, $U \in \tau$, $V \subseteq U$, then the following diagram is commutative:

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{\Phi_U} & \mathcal{F}'(\varphi^{-1}(U)) \\ \rho \downarrow & & \downarrow \rho \\ \mathcal{F}(V) & \xrightarrow{\Phi_V} & \mathcal{F}'(\varphi^{-1}(V)) \end{array}$$

In what follows, we shall be concerned mainly with the following objects and morphisms:

Definition 7.2. *$(X, \mathcal{F})$ is called a (pre) general harmonic space (GHS for short) if: a) X is a topological space, b) $\mathcal{F}$ is a (pre) scheaf on X,*

c) for each open $U \subseteq X$, $\mathcal{F}(U)$ is an ordered convex cone of positive elements and inf-complete lattice

d) there exists a base $\mathcal{B}$ of open sets, such that, for each $U \in \mathcal{B}$, $\mathcal{F}(U)$ is an H-cone.

e) the restriction morphisms $\rho_V^U : \mathcal{F}(U) \to \mathcal{F}(V)$ satisfy:

$$\rho_V^U(x+y) = \rho_V^U(x) + \rho_V^U(y), \quad \forall x, y \in \mathcal{F}(U),$$
$$x \leqslant y \Rightarrow \rho_V^U(x) \leqslant \rho_V^U(y), \quad \forall x, y \in \mathcal{F}(U),$$
$$(x_i) \text{ increasing and } x = \bigvee x_i \Rightarrow \rho_V^U(x) = \bigvee \rho_V^U(x_i).$$

Remark. When $\mathcal{F}$ is a sheaf, the H-cones $\mathcal{F}(U)$, $U \in \mathcal{B}$ define completely $\mathcal{F}$.

Definition 7.3. *$(X, \mathcal{F})$ is called a (pre) GHS of functions if, for each open $U \subseteq X$, the elements of $\mathcal{F}(U)$ are l.s.c. functions $U \to \overline{\mathbf{R}}_+$, such that, $\mathcal{F}(U)$ being endowed with the pointwise operations and order, $(X, \mathcal{F})$ is a (pre) GHS and moreover: $1 \in \mathcal{F}(X)$; $(f \wedge g)(x) = \min\{f(x), g(x)\}$, $\forall x \in X$, $f, g \in \mathcal{F}(U)$; (f_i) increasing $\Rightarrow (\bigvee f_i)(x) = \sup f_i(x)$, $\forall x \in X$. the restriction morphisms are the usual function restrictions.*

Examples 7.4. a) If $(X, \mathcal{U})$ is a harmonic space [4], then $(X, \mathcal{U})$ is a GHS of functions. Moreover, the base $\mathcal{B}$ is hereditary, being formed of $\mathfrak{P}$-sets.

b) If S is a standard H-cone of functions on X, and τ is a topology coarser than the fine topology, then $(X, \mathcal{S})$ [2] or $(X, \mathcal{S}')$ [1] are preGHS. When τ is the fine topology or the natural topology, one can give various equivalent conditions on S, such that $(X, \mathcal{S})$ or $(X, \mathcal{S}')$ be GHS (see [2], [1], [8], [10]).

c) Let $(X, \mathcal{U})$ be a harmonic space, and $\mathcal{P}(U)$ be the convex cone of potentials on U, endowed with the specific order. We take as morphisms of restriction the following: if $p \in \mathcal{P}(U)$, then $p|_V$ is the potential on V, such that $p = p|_V + h$ on V, with h harmonic on V. $(X, \mathcal{P})$ is then a preGHS. $(X, \mathcal{P})$ is a GHS iff: $\forall U \in \tau$, $\forall$ open covering (U_i) of U and any family $p_i \in P(U_i)$ such that $p_i - p_j$ is harmonic on $U_i \cap U_j$, there exists $p \in \mathcal{P}(U)$ such that $p - p_i$ is harmonic on U_i. Such a formulation appears in [15].

Definition 7.5. *Let $(X, \mathcal{F})$, $(X', \mathcal{F}')$ be preGHS. We call a morphism of GHS, any pair (φ, ψ), where $\varphi : X' \to X$ is a continuous map, and $\psi = (\psi_U)_{U \in \tau}$ is a φ- morphism, such that:*

$$\psi_U(x+y) = \psi_U(x) + \psi_U(y), \ \forall x, y \in \mathcal{F}(U); \quad x \leqslant y \Rightarrow \psi_U(x) \leqslant \psi_U(y),$$
$$\forall x, y \in \mathcal{F}(U), \ (x_i) \text{ increasing and } x = \bigvee x_i \Rightarrow \psi_U(x) = \bigvee \psi_U(x_i).$$

The next proposition shows that, under „reasonable" assumptions, the morphisms of GHS are uniquely defined by the map $\varphi : X' \to X$.

Proposition 7.6. *Let $(X, \mathcal{F})$, $(X', \mathcal{F}')$ be preGHS of functions. Suppose that there exists a base $\mathcal{B}_0$ in X, such that $\forall U \in \mathcal{B}_0$ $\mathcal{F}_c(U)$ (i.e. the set of continuous real functions from $\mathcal{F}(U)$) is dense in $\mathcal{F}(U)$. If (φ, ψ) is a morphism of GHS, such that $\psi_U(1) = 1$, $\forall U$, then: $\psi_U(f) = f \circ \varphi$, $\forall f \in \mathcal{F}(U)$.*

Proof. Let $f \in \mathcal{F}_0(U)$ and $y \in \varphi^{-1}(U)$, if $f(\varphi(y)) > 0$, let α, β be such that: $f(\varphi(y)) \in (\alpha, \beta)$. If we denote: $U_{\alpha,\beta} = \{x \in U \mid \alpha < f(x) < \beta\}$, then $U_{\alpha,\beta}$ is open and:

$$\alpha = \psi_U(\alpha) < \psi_{U_{\alpha,\beta}}(f|_{U_{\alpha,\beta}}) = \psi_U(f)|_{\varphi^{-1}(U_{\alpha,\beta})} < \psi_U(\beta) = \beta,$$

which shows that $\psi_U(f)(y) \in (\alpha, \beta)$. The case $f(\varphi(y)) = 0$ is even simpler. It follows that $\psi_U(f) = f \circ \varphi$. The proof is ended, using the hypothesis that $\mathcal{F}_c(U)$ is dense in $\mathcal{F}(U)$.

Remark. In the above result, it suffices to suppose that ψ_U is monotone and positively homogeneous.

We give next conditions under which a H-map give rise to a morphism of GHS. Let S, T be standard H-cones of functions on X, Y. Let $\varphi : Y \to X$ be a H-map; hence φ is finely continuous. Let us suppose that:

$$(*) \qquad \varphi(B^{\beta(X \setminus G)} s) = B^{\beta(Y \setminus \varphi^{-1}(G))}(\varphi(s)),$$

$\forall G \subseteq X$ finely open and $s \in S$ finite.

Let $s \in S$ be finite, $G \subseteq X$ finely open and denote: $f = (s - B^{\beta(X \setminus G)} s)|_G$. We have $f \in \mathcal{S}(G)$ and:

$$f \circ \varphi = (\varphi(s) - \varphi(B^{\beta(X \setminus G)} s))|_{\varphi^{-1}(G)} = (\varphi(s) - B^{\beta(Y \setminus \varphi^{-1}(G))}(\varphi(s))|_{\varphi^{-1}(G)}$$

hence: $f \circ \varphi \in \mathcal{T}(\varphi^{-1}(G))$. It follows:

Theorem 7.7. *$\varphi|_{\varphi^{-1}(G)}$ is a H-map; it induces the morphism $\varphi_G : \mathcal{S}(G) \to \mathcal{T}(\varphi^{-1}(G))$. If, moreover, φ is finely open, then φ_G is finite.*

Remarks. a) If S and T satisfy the axiom of polarity, then we can drop „β" in (*).

b) Condition (*) can be considered as defining a morphism of the second type (φ, φ), provided: $\varphi(B^{\beta(X \setminus G)}) = B^{\beta(Y \setminus \varphi^{-1}(G))}$ makes sense.

Corollary 7.8. *$(\varphi, (\varphi_G))$ is a morphism of GHS, from $(X, \mathcal{S})$ to $(Y, \mathcal{T})$.*

In [1] a diferent presheaf $\mathcal{S}'$ is defined namely: $\mathcal{S}'(G)$ is the set of all functions $f : G \to \overline{\mathbf{R}}_+$ which are finite on a finely dense set and such that, for any finely open D for which $\overline{D} \subseteq G$, we have: $f|_D \in \mathcal{S}(D)$.

Corollary 7.9. *Suppose that φ is finely open and finely closed. Then $(\varphi, (\varphi_G))$ is a morphism of GHS from $(X, \mathcal{S}')$ to $(Y, \mathcal{T}')$.*

Using the fact that the semipolar sets with respect to S and $\mathcal{S}(G)$ are the same, we get:

Theorem 7.10. *Let S, T be standard H-cones of functions on semi-saturated spaces X, Y. Let φ be a quasi-H-map. If (*) holds, then φ induces a quasi-H-map $\varphi_G : \mathcal{S}(G) \to \mathcal{T}(\varphi^{-1}(G))$.*

Proof. Indeed, φ gives rise to a H-map between the saturated spaces Y_1, $X_1 \cdot \varphi^{-1}(G)$ makes sense and is a finely open set in Y_1.

Lemma 7.11. *There exists a base β for the natural topology on X, such that $B^{X\setminus U}$ and $B^{Y\setminus\varphi^{-1}(U)}$ are balayages, $\forall U \in \beta$.*

Proof. The reasoning is entirely analogous to the proof of th. 3 from [7].

Corollary 7.12. *Let S, T be standard H-cones of functions on X, Y. If $\varphi : Y \to X$ is a naturally continuous H-map and satisfies*

$$(^{**}) \qquad \varphi(B^{X\setminus U} s) = B^{Y\setminus\varphi^{-1}(U)}(\varphi(s)),$$

$\forall s \in S$, $U \in \beta$, *then $\varphi|_{\varphi^{-1}(U)}$ is a H-map from $\mathcal{S}(U)$ to $\mathcal{T}(\varphi^{-1}(U))$ $\forall U \in \beta$,*

Theorem 7.13. *Let S, T be standard H-cones of functions on X, Y. Let $\varphi : Y \to X$ be a naturally continuous H-map which satisfies $(^{**})$. If $\mathcal{T}$* (resp. *$\mathcal{T}'$) is a natural sheaf, then φ_U is a H-map from $\mathcal{S}(U)$ to $\mathcal{T}(\varphi^{-1}(U))$ (resp. $\mathcal{T}'(\varphi^{-1}(U))$) for any naturally open U.*

Proof. Let (U_α) be covering for U, with $U_\alpha \in \beta$. Let $s \in \mathcal{S}(U)$. Hence $s|_{U_\alpha} \in \mathcal{S}(U_\alpha)$. From cor. 7.12 we have $s \circ \varphi|_{\varphi^{-1}(U_\alpha)} \in \mathcal{T}(\varphi^{-1}(U_\alpha))$. $\mathcal{T}$ being a sheaf, it follows that $s \circ \varphi \in \mathcal{T}(\varphi^{-1}(U))$.

Remark. Since $(B^{X\setminus G} s) \circ \varphi \leqslant B^{Y\setminus\varphi^{-1}(G)}(s \circ \varphi)$ for any finely open G, it suffices that $(^{**})$ holds only for a generator $p \in S$.

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EXTENSION OF VECTOR AND GROUP-VALUED SUBMEASURES

BY

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The results of this paper are related to those from L. Drewnowski's article [4] and W. Herer's work [5]. In [4] the author obtains an extension procedure for a scalar submeasure; in [5] Herer gives an extension method for a group-valued measure.

In this article we define the concepts of vector submeasure and group-valued submeasure, the last being obviously more general than the previous one. Using Drewnowski's procedure, we give an extension procedure for a vector submeasure (Theorem 1.2), and next, taking into account Herer's method we obtain an extension method for a group-valued submeasure (Theorem 2.3). In the last section we compare the two extension methods described in Theorem 1.2 and Theorem 2.3 and we show that, under certain conditions, the two theorems mentioned above lead to the same extending σ-ring $\mathcal{R}_0$ and the same extending vector submeasure η_0 (see Theorem 3.2).

In the sequel we shall, generally, use the notations and terminology from [4].

§1. Let $(\mathcal{R}, \Delta, \cap)$ be a ring of subsets of a fixed set R and X a linear space ordered by the relation „$\leqslant$" induced by the convex cone P.

Definition 1.1. *A set function $\eta : \mathcal{R} \to X$ is subadditive if*

(1v) $\eta(\varnothing) = 0$;

(2v) $\eta(A \cup B) \leqslant \eta(A) + \eta(B)$ *for all* $A, B \in \mathcal{R}$ *with* $A \cap B = \varnothing$.

Definition 1.2. *We call vector submeasure a subadditive set function η defined on $\mathcal{R}$ with values in P which satisfies the condition:*

(3v) $\eta(A) \leqslant \eta(B)$ *for every* $A, B \in \mathcal{R}$ *with* $A \subseteq B$.

Example 1.1. Let $m : \mathcal{R} \to \mathbf{R}_+$ be a finite scalar submeasure and X a linear ordered space with the positive cone P. For every $p \in P$ we define the set function $\eta_p : \mathcal{R} \to P$ by $\eta_p(A) = m(A).p \in P$, for every $A \in \mathcal{R}$. It is obvious that η_p verifies the conditions (1v), (2v), (3v) so that η_p is a vector submeasure on $\mathcal{R}$.

Theorem 1.1. *If η is a vector submeasure on $\mathcal{R}$ then the family of classes $\mathcal{U}_p = \{A \in \mathcal{R} / \eta(A) \leqslant p\}$, $p \in P$, $p \neq 0$ is a normal base of neighbourhoods (ns.) of $\varnothing$ for a FN-topology $\Gamma(\eta)$.* (We refer to $\Gamma(\eta)$ as the Fréchet-Nikodym topology generated by η on $\mathcal{R}$).