Proof. Since any $W_1 \oplus W_4$-manifold is a $W_1 \oplus W_2 \oplus W_4$-manifold [5], from Theorem 2.2 it follows

a) $\tilde{\delta}\tilde{\Phi}(X) = (n/r)\, \delta\Phi(X)$, $X \in \chi(M)$. b) $\delta\eta = (r/n)\, \tilde{\delta}\tilde{\Phi}(\tilde{U})$.

Now, using (1.23) and since $\tilde{U}$ is analytic and Killing, we deduce $0 = (\tilde{\nabla}_{\tilde{U}}\tilde{\Phi})(\tilde{U}, X) = -[1/(2n)]\,(\tilde{\delta}\tilde{\Phi}(X) - \eta(X)\, \tilde{\delta}\tilde{\Phi}(\xi))$. From this relation and a), we have

c) $\delta\Phi(\varphi X) = 0, \qquad U \in \chi(M)$.

Thus, from a), b), c), (1.12), (1.21), since $\tilde{U}$ is Killing and $\tilde{M}$ is a $W_1 \oplus W_4$-manifold, we have (1.7).

On the other hand, since $\tilde{M}$ is a $W_1 \oplus W_2 \oplus W_4$-manifold and a G_1-manifold [5], using (1.12), (1.22), (1.25), a), b), c), we have (1.8). Thus, we obtain that M is a nearly-trans-Sasakian manifold.

2.4. Semi-invariant Submanifolds of a $W_2 \oplus W_4$-Manifold.

Theorem 2.4. *If M is a $W_2 \oplus W_4$-manifold and if $\tilde{U}$ is analytic and Killing, then (φ, ξ, η, g) is an almost-trans-Sasakian structure on M.*

Proof. Since any $W_2 \oplus W_4$-manifold is a $W_1 \oplus W_2 \oplus W_4$-manifold [5], from Theorem 2.2 we have

a) $\tilde{\delta}\tilde{\Phi}(X) = (n/r)\, \delta\Phi(X)$, $X \in \chi M)$. b) $\delta\eta = (r/n)\, \tilde{\delta}\tilde{\Phi}(\tilde{U})$.

Thus, from a), b), (1.12), (1.13), (1.14) and since $\tilde{M}$ is a $W_2 \oplus W_4$-manifold, we have (1.9).

On the other hand, $3d\tilde{\Phi}(\tilde{U}, X, Y) = 2d\eta(X, Y) + (\mathcal{L}_{\tilde{U}}\tilde{\Phi})(X, Y)$ and since $\tilde{U}$ is analytic and Killing, from a), b), (1.12), (1.13), (1.20), we obtain (1.10). Therefore M is an almost-trans-Sasakian manifold.

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ON GENERALIZED FINSLER SPACES

BY

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Let M be a differentiable manifold and TM its tangent bundle. A coordinate system (x^i) in M induces a canonical coordinate system (x^i, y^i) in TM. Let $L(x, y)$ be a positive-valued differentiable function defined in $TM - \{0\}$, and we assume that L is positively homogeneous of degree 1, and the matrix (a_{ij}) is regular, where $a_{ij} = (1/2)\partial^2 L^2/\partial y^i \partial y^j$. If the length s of a differentiable curve $x^i = x^i(t)$ in M is measured by $s = \int L(x, dx/dt)dt$, we call (M, L) a *Finsler space*. Then a_{ij} is called the *Finsler metric*.

A Finsler metric is sometimes adopted as a basic concept in the theoretical physics. However, the fact that the metric is provided from a function L may be not always desirable for physicists. For example in his paper [2], H. Ishikawa modified the Finsler metric a_{ij} in a Finsler space (M, L), and introduced the metric $g_{ij} = a_{ij} - 2l_i l_j$, where $l_i = \partial L/\partial y^i$. If (a_{ij}) is positive-definite, any vector $y(\neq 0)$ is time-like: $g_{ij}y^i y^j = -a_{ij}y^i y^j < 0$. It seems that this is the reason why he had been interested in such a metric. It has also the another interesting property that the modification is projective. But it is not a Finsler metric.

Another example is given by a conformal change of a Finsler metric a_{ij}: $g_{ij} = e^{2\rho(x,y)}a_{ij}$ (R. Miron - M. Hashiguchi [7]). g_{ij} is not a Finsler metric, if ρ depends on direction. Interestingly, the tensor $2(a_{ij} - 2l_i l_j)/L^2$ is expressed as $\partial^2 \log L^2/\partial y^i \partial y^j$, and appears in the conformal theory of Finsler metrics as an important conformal invariant (M. Hashiguchi [1]).

In his paper [6], R. Miron generalized the notion of Finsler metric to the metric g_{ij} given as a differentiable tensor field of type (0,2) which is symmetric and non-degenerate, and he gave an interesting theory. We shall call this g_{ij} a *generalized Finsler metric* and (M, g_{ij}) a *generalized Finsler space*.

In this note we shall consider such a generalized Finsler space from the standpoint of variational calculus. Then we hope to refer to a significance of axioms for Finsler connections.

The terminology and notations are referred to M. Matsumoto [5].

1. We shall first investigate what conditions should be imposed on a generalized Finsler space (M, g_{ij}). Putting $L = (g_{ij}y^j y^i)^{1/2}$, the length s

of a curve $x^i = x^i(t)$ in M is defined in the same formula as in the case of Finsler space. In order that the length s is independent on the parameter t, it is desirable to be assumed that g_{ij} is positively homogeneous of degree 0.

Now, let a curve $C : x^i = x^i(t)$ join two points $A(t_0)$ and $B(t_1)$. Given an infinitesimal ε and differentiable functions $v^i(t)$, the curve $\overline{C} : \overline{x}^i = x^i(t) + \varepsilon v^i(t)$ is nearby C and passing through $\overline{A}(t_0)$ and $\overline{B}(t_1)$ which are nearby A and B respectively.

The first variation δs of s is given by

$$\delta s = \varepsilon \int_{t_0}^{t_1} (\partial L/\partial x^j - d(\partial L/\partial y^j)/dt) v^j dt + \varepsilon [(\partial L/\partial y^j) v^j]_{t_0}^{t_1}.$$

The curves satisfying the Euler equation are called the *geodesics* of (M, g_{ij}). If a curve C is a geodesic, the direction v^j satisfying $(\partial L/\partial y^j) v^j = 0$ is called to be *transversal* to C. If we translate the geodesic C such that its extreme points lie on the direction transversal to C, the variation δs vanishes.

Contrary to the case of Finsler metric, the transversality $(\partial L/\partial y^j) v^j = 0$ does not in general coincide with the orthogonality $g_{ij} y^i v^j = 0$, because we have $\partial L/\partial y^j = g_{ij} y^i/L + (\partial g_{ik}/\partial y^j) y^i y^k/2L$. According to one of the axioms of E. Cartan for the Finsler metric, we hope to make these equivalent. It is enough if we assume

(1) $$(\partial g_{ik}/\partial y^j) y^i y^k = 0.$$

Then, if we express the equations of a geodesic by its arc-length s, they become from the condition (1)

$$A^i_j (d^2 x^j/ds^2) + \gamma^i_{jk} (dx^j/ds)(dx^k/ds) = 0,$$

where $A^i_j = \delta^i_j + g^{im}(\partial g_{hm}/\partial y^j) y^h$, $\gamma^i_{jk} = g^{im}(\partial g_{jm}/\partial x^k + \partial g_{km}/\partial x^j - \partial g_{jk}/\partial x^m)/2$, and $(g^{im}) = (g_{ij})^{-1}$. If we assume further

(2) The matrix (A^i_j) is regular, the equations of the geodesic have the form $d^2 x^i/ds^2 + B^i_h \gamma^h_{jk} (dx^j/ds)(dx^k/ds) = 0$, where $(B^i_h) = (A^i_j)^{-1}$. Putting $G^i = (1/2) B^i_h \gamma^h_{jk} y^j y^k$, we have as the equations of the geodesics

(3) $$d^2 x^i/ds^2 + 2G^i(x, dx/ds) = 0.$$

The conditions (1) and (2) are just the regularity conditions given by R. Miron [6], which we shall assume in the following.

2. Next, we shall consider what Finsler connections are reasonable for a generalized Finsler space (M, g_{ij}). We have the equations (3) of the geodesics. Corresponding to the Berwald connection $B\Gamma$ in a Finsler space, we have a Finsler connection $MB\Gamma = (G^i_{jk}, G^i_k, 0)$, where $G^i_k = \partial G^i/\partial y^k$, $G^i_{jk} = \partial G^i_k/\partial y^j$. This was communicated by R. Miron to the author in Iași. And we can recognize that such a G^i_k is the most reasonable as the non-linear connection.

Now, we hope to determine $MB\Gamma$ by a system of axioms. It is noted that $L = (g_{ij} y^i y^j)^{1/2}$ is positively homogeneous of degree 1. If we put $\overline{g}_{ij} = (1/2)\partial^2 L^2/\partial y^i \partial y^j$, we have from (1): $\overline{g}_{ij} = g_{ij} + (\partial g_{hi}/\partial y^j) y^h = g_{im} A^m_j$. The

assumption (2) assures that the matrix $(\overline{g}_{ij})$ is regular. Thus we have a Finsler space (M, L).

In these two spaces (M, g_{ij}) and (M, L), the arc-length is measured by the same formula. Therefore, the equations (3) of the geodesics of (M, g_{ij}) are nothing but the ones of the Finsler space (M, L). Thus the $MB\Gamma$ is just the Berwald connection $B\overline{\Gamma}$ of the Finsler space (M, L). $MB\Gamma$ is defined in terms of local coordinates, but is it really a Finsler connection. We need not check this fact by calculation.

For a Finsler space (M, L) we have a system of axioms which determines uniquely the Berwald connection. In 1982 T. Okada [8] showed that the Berwald connection $B\Gamma = (F^i_{jk}, N^i_k, 0)$ is uniquely determined by the conditions:

(a) $L_{|k} = \partial L/\partial x^k - N^j_k(\partial L/\partial y^j) = 0$, (b) $D^i_k = y^j F^i_{jk} - N^i_k = 0$,

(c) $P^i_{jk} = \partial N^i_j/\partial y^k - F^i_{kj} = 0$, (d) $T^i_{jk} = F^i_{jk} - F^i_{kj} = 0$.

These conditions form a system of axioms which determines uniquely $MB\Gamma$ in the generalized Finsler space (M, g_{ij}). We shall call $MB\Gamma$ the *Miron-Berwald connection* of (M, g_{ij}).

We have derived a canonical Finsler connection $MB\Gamma$ for (M, g_{ij}), but this is not necessarily metrical. R. Miron [6] gave a canonical metrical Finsler connection $MC\Gamma = (F^{*i}_{jk}, G^i_k, C^{*i}_{jk})$, where putting $\delta/\delta x^k = \partial/\partial x^k - G^i_k(\partial/\partial y^i)$,

$$F^{*i}_{jk} = g^{im}(\delta g_{jm}/\delta x^k + \delta g_{km}/\delta x^j - \delta g_{jk}/\delta x^m)/2,$$

$$C^{*i}_{jk} = g^{im}(\partial g_{jm}/\partial y^k + \partial g_{km}/\partial y^j - \partial g_{jk}/\partial y^m)/2,$$

and he showed that this is uniquely determined as a Finsler connection $(F^i_{jk}, N^i_k, C^i_{jk})$ satisfying the conditions

(α) $g_{ij|k} = 0$, $g_{ij}|_k = 0$, (β) $N^i_k = G^i_k$,

(γ) $T^i_{jk} = F^i_{jk} - F^i_{kj} = 0$, ($\delta$) $S^i_{jk} = C^i_{jk} - C^i_{kj} = 0$.

We shall call $MC\Gamma$ the *Miron-Cartan connection* of (M, g_{ij}).

Contrary to the case of $MB\Gamma$, we have a priori assumed that $N^i_k = G^i_k$. Can we determine $MC\Gamma$ including G^i_k from some system of axioms?

In the case of Finsler space we have a system of axioms which determines uniquely the Cartan connection $C\Gamma$. In 1966, M. Matsumoto [4] showed that the Cartan connection $C\Gamma = (F^i_{jk}, N^i_k, C^i_{jk})$ is uniquely determined by the conditions (α), (b), (γ), (δ), where g_{ij} is the Finsler metric. Suggested by S. Watanabe-F. Ikeda [9] we should further discuss Finsler connections satisfying these conditions in the case of generalized Finsler space (M, g_{ij}).

On the other hand, we have the Cartan connection $C\overline{\Gamma}$ of the Finsler space (M, L) derived from a generalized Finsler space (M, g_{ij}). What relation is there between $MC\Gamma$ and $C\overline{\Gamma}$?

3. Finally, we shall refer to a significance of axioms for Finsler connections. In a Finsler space and also in a generalized Finsler space we have

a method by which we can get a metrical Finsler connection $F\Gamma=(F^i_{jk}, N^i_k, C^i_{jk})$ starting from an arbitrary Finsler connection $F\overset{\circ}{\Gamma}=(\overset{\circ}{F}{}^i_{jk}, \overset{\circ}{N}{}^i_k, \overset{\circ}{C}{}^i_{jk})$. In fact, the set of all metrical Finsler connections is obtained in the form

$$N^i_k=\overset{\circ}{N}{}^i_k-X^i_k,\ F^i_{jk}=\overset{\circ}{F}{}^i_{jk}+\overset{\circ}{C}{}^i_{jm}X^m_k+g^{im}(g_{mj\overset{\circ}{|}k}+g_{mj}\overset{\circ}{|}_p X^p_k)/2+\Omega^{ir}_{sj}X^s_{rk},$$

$$C^i_{jk}=\overset{\circ}{C}{}^i_{jk}+g^{im}g_{mj}\overset{\circ}{|}_k/2+\Omega^{ir}_{sj}Y^s_{rk},$$

where $\Omega^{ir}_{sj}=(\delta^i_s\delta^r_j-g_{sj}g^{ir})/2$, and X^i_k, X^i_{jk}, Y^i_{jk} are arbitrary tensors (R. M i r o n [6]). If we put $X^i_k=X^i_{jk}=Y^i_{jk}=0$, we have the well-known *Kawaguchi method* (A. K a w a g u c h i [3]). But $MC\Gamma$ does not follow from $MB\Gamma$ in the case of generalized Finsler spaces.

So we put $X^i_k=0$, $X^i_{jk}=-g^{im}g_{jl\overset{\circ}{|}m}$, $Y^i_{jk}=-g^{im}g_{jl}\overset{\circ}{|}_m$. Then we have a *new Kawaguchi* method:

$$N^i_k=\overset{\circ}{N}{}^i_k,\ F^i_{jk}=\overset{\circ}{F}{}^i_{jk}+g^{im}(g_{mj\overset{\circ}{|}k}+g_{mk\overset{\circ}{|}j}-g_{jk\overset{\circ}{|}m})/2, \tag{4}$$

$$C^i_{jk}=\overset{\circ}{C}{}^i_{jk}+g^{im}(g_{mj}\overset{\circ}{|}_k+g_{mk}\overset{\circ}{|}_j-g_{jk}\overset{\circ}{|}_m)/2.$$

Thus we have

Theorem 1. *In a generalized Finsler space* (M, g_{ij}), *let* $F\overset{\circ}{\Gamma}=(\overset{\circ}{F}{}^i_{jk}, \overset{\circ}{N}{}^i_k, \overset{\circ}{C}{}^i_{jk})$ *be a given Finsler connection. Then the Finsler connection* $F\Gamma=(F^i_{jk}, N^i_k, C^i_{jk})$ *given by* (4) *is metrical*: $g_{ij|k}=g_{ij}|_k=0$.

If $\overset{\circ}{N}{}^i_k=G^i_k$, *and* $\overset{\circ}{F}{}^i_{jk}$, $\overset{\circ}{C}{}^i_{jk}$ *are symmetric, then* $F\Gamma$ *is* $MC\Gamma$ *of* (M, g_{ij}).

The proof of the latter half is immediately obtained by checking the axioms for $MC\Gamma$. It is noted that we need not any calculation. And if we put $F\overset{\circ}{\Gamma}=MB\Gamma$, we have a relation between $MC\Gamma$ and $MB\Gamma$.

On the other hand, we shall call a generalized Finsler space (M, g_{ij}) a *Berwald space*, if G^i_{jk} of $MB\Gamma$ depend on position alone, that is, the corresponding Finsler space (M, L) is a Berwald space. Then we have

Theorem 2. *In a generalized Finsler space* (M, g_{ij}), *let* $F^i_{jk}(x)$ *be a given symmetric linear connection in* M. *If a Finsler connection* $F\Gamma=(F^i_{jk}(x), y^jF^i_{jk}(x), 0)$ *satisfies* $L_{|k}=0$, *where* $L=(g_{ij}y^iy^j)^{1/2}$, *the space* (M, g_{ij}) *is a Berwald space.*

The proof is also immediately obtained by showing that $F\Gamma$ satisfies the axioms for $MB\Gamma$.

For example, if a Finsler space (M, a_{ij}) is a Berwald space, the generalized Finsler space (M, g_{ij}) given by H. I s h i k a w a [2] is also Berwald space. Because the Berwald connection of (M, a_{ij}) is nothing but the Miron-Berwald connection of (M, g_{ij}).

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