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SOME REMARKS ON THE PRODUCT OF TWO ALMOST CONTACT MANIFOLDS

BY

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1. Introduction. Let M and M' be almost contact metric manifolds and let f be a positive function on the product manifold $\tilde{M}=M\times M'$. Let $\tilde{g}$ be the f-depending Riemannian metric on $\tilde{M}$, as defined in [5]. First we introduce an f-depending almost complex structure $\tilde{J}$ on $\tilde{M}$, which is a generalization of a result in Morimoto's paper [7]. Then $(\tilde{g}, \tilde{J})$ makes $\tilde{M}$ into an almost Hermitian manifold. By the assumption that f is constant on each component of $\tilde{M}$, we prove the following main result : the manifold $\tilde{M}$ is nearly Kählerian, almost Kählerian or Kählerian if and only if the manifolds M and M' are both closely cosymplectic , almost cosymplectic or cosymplectic respectively.

In the following all the manifolds, functions and tensor fields considered are of class C^∞ ; more we denote by $T(M)$ the Lie algebra of vector fields on a differentiable manifolds M.

2. Preliminaries. First we recall some definitions and properties of the contact geometry.

A $(2n+1)$-dimensional manifold M is said to have an *almost contact structure* (Φ, ξ, η) if there exist on M a tensor field Φ of type $(1, 1)$, a vector field ξ and a 1-form η satisfyng $\Phi^2+I=\eta\oplus\xi$, $\eta(\xi)=1$, where I is the identity transformation. It is well known that if M has such a structure, we can find a Riemannian metric g on M such that $g(X, Y)=g(\Phi X, \Phi Y)+n(X)\eta(Y)$, for any vector fields X and Y on M (see [8]): then (Φ, ξ, η, g) is called an *almost contact metric structure* on M. If (Φ, ξ, η) is an almost contact structure on M, define four tensors N_1, N_2, N_3 and N_4 by

$$N_1(X, Y)=[\Phi, \Phi](X, Y)+2d\eta(X, Y)\xi, \qquad N_3(X)=(L_\xi\Phi)X,$$
$$N_2(X, Y)=(L_{\Phi X}\eta)Y-(L_{\Phi Y}\eta)X, \qquad N_4(X)=(L_\xi\eta)X,$$

where L denotes Lie differentiation, while $[\Phi, \Phi]$ is the Nijenhuis torsion of the tensor field Φ. If $N_1=0$, then the almost contact structure (Φ, ξ, η) is said to be *normal* ; in particular $N_1=0$ implies $N_2=0$, $N_3=0$ and $N_4=0$ (see [9]).

Now let (Φ, ξ, η, g) be an almost contact metric structure on M: a tensor field H of type $(r, 1)$ on M is called a *Killing tensor field* if $(\nabla_X H)Y + (\nabla_Y H)X = 0$, where ∇ denotes the covariant differentiation with respect to the Riemannian connection defined by g (see [1]). If we denote by Ω the *fundamental 2-form defined by* $\Omega(X, Y) = g(X, \Phi Y)$, then the almost contact metric structure (Φ, ξ, η, g) is called *almost cosymplectic* if both Ω and η are closed and it is said to be *closely cosymplectic* if η is closed and Φ is Killing. An almost cosymplectic structure is said to be *cosymplectic* if it is normal.

Now we state two results which are required later.

Proposition 2.1. *Let (Φ, ξ, η, g) be an almost contact metric structure on M and denote by ∇ the covariant differentiation with respect to the Riemannian connection Γ defined by g.*

(1) *If Φ is Killing, then η is Killing.*

(2) *If η is Killing and if ξ is parallel with respect to Γ, then the form η is closed.*

Proof. (1). Since Φ is Killing, then $L_\xi g = 0$ (see [3], prop. 2.1) and consequently for any $X, Y \in T(M)$ we have

$$(\nabla_X \eta)Y + (\nabla_Y \eta)X = g(\nabla_X \xi, Y) + g(\nabla_Y \xi, X) =$$

$$= g(\nabla_\xi X, Y) + g([X, \xi], Y) + g(\nabla_\xi Y, X) + g([Y, \xi], X) = (L_\xi g)(X, Y) = 0.$$

(2). Since η is Killing, then

$$(2.1) \qquad d\eta(X, Y) = (\nabla_X \eta)Y$$

and consequently $d\eta(X, \xi) = 0$. Since ξ is parallel with respect to Γ, from (2.1) we have $d\eta(X, \Phi Y) = -\eta(\nabla_X \Phi Y) = -g(\nabla_X \Phi Y, \xi) = g(\Phi Y, \nabla_X \xi) - X(\eta(\Phi Y)) = 0$. Thus we have $d\eta = 0$. From this proposition we obtain

Proposition 2.2. *Let (Φ, ξ, η, g) be an almost contact metric structure on M. The following conditions are equivalent:*

(1) *the almost contact metric structure is closely cosymplectic;*

(2) *Φ is Killing and ξ is parallel with respect to the Riemannian connection defined by g.*

3. The main results. Let (Φ, ξ, η, g) and (Φ', ξ', η', g') be almost contact metric structures on the odd-dimensional manifolds M and M' respectively and let $\tilde{M}$ be the product manifold $M \times M'$; more for any $X \in T(M)$ and $X' \in T(M')$, denote by (X, X') the vector field on $\tilde{M}$ given by $(X, X')_{(p,q)} = (X_p, X'_q)$ for all $(p, q) \in \tilde{M}$. Now let f be a positive function on $\tilde{M}$ and set $h = 1/f$; then we consider the Riemannian metric $\tilde{g}$ on $\tilde{M}$ (see [5], p. 66) and we introduce the almost complex structure $\tilde{J}$ on $\tilde{M}$ such that

$$(\tilde{g}((X, X'), (Y, Y')))_{(p,q)} = (g(X, Y))_p + (f(p, q))^2 (g'(X', Y'))_q,$$

$$(\tilde{J}(X, X'))_{(p,q)} = ((\Phi X)_p - f(p, q)(\eta'(X'))_q \xi_p, (\Phi' X')_q + h(p, q)(\eta(X))_p \xi'_q)$$

for any $X, Y \in T(M)$, $X', Y' \in T(M')$ and $(p, q) \in \tilde{M}$. It is easy to prove that

Proposition 3.1. *$(\tilde{J}, \tilde{g})$ is an almost Hermitian structure on the manifold $\tilde{M}$.*

From now on, we suppose that the function f is constant on each component of $\tilde{M}$ and we consider each function on M (resp. M') as a function on $\tilde{M}$ as usual. Now we prove the following

Proposition 3.2. *The almost Hermitian structure $(\tilde{J}, \tilde{g})$ on $\tilde{M}$ is Hermitian if and only if the almost contact structures (Φ, ξ, η) and (Φ', ξ', η') are both normal.*

Proof. Denote by $\tilde{N}$ the Nijenhuis torsion of $\tilde{J}$, then the integrability condition of the structure $(\tilde{J}, \tilde{g})$ is equivalent to the following three conditions

$$(3.1) \qquad \tilde{N}((X, 0), (Y, 0)) = 0, \quad \tilde{N}((X, 0), (0, X')) = 0, \quad \tilde{N}((0, X'), (0, Y')) = 0$$

for any $X, Y \in T(M)$ and $X', Y' \in T(M')$. By a direct calculation we see

$$\tilde{N}((X, 0), (Y, 0)) = (N_1(X, Y), 0) + h(N_2(X, Y))(0, \xi'),$$

$$\tilde{N}((X, 0), (0, X')) = f(\eta'(X'))(N_3(X), 0) - \eta'(X)(N'_4(X'))(\xi, 0) +$$

$$+ h(\eta(X))(0, N'_3(X')) + \eta'(X')(N_4(X))(0, \xi'),$$

$$\tilde{N}((0, X'), (0, Y')) = (0, N'_1(X', Y')) - f(N'_2(X', Y'))(\xi, 0),$$

where N'_1, N'_2, N'_3, N'_4 are the tensor fields on M' defined in the same way as N_1, N_2, N_3 and N_4. Since the vanishing of N_1 (resp. N'_1) implies the vanishing of N_2, N_3, N_4 (resp. N'_2, N'_3, N'_4), then the conditions (3.1) are equivalent to $N_1 = 0$ and $N'_1 = 0$, namely the almost contact structures (Φ, ξ, η) and (Φ', ξ', η') are both normal.

Proposition 3.3. *The almost Hermitian structure $(\tilde{J}, \tilde{g})$ on $\tilde{M}$ is almost Kählerian if and only if the almost contact metric structure $\Sigma = (\Phi, \xi, \eta, g)$ and $\Sigma' = (\Phi', \xi', \eta', g')$ are both almost cosymplectic.*

Proof. Let $\tilde{\Omega}$ be the fundamental 2-form of the almost Hermitian structure $(\tilde{J}, g)$ and let Ω and Ω' be the fundamental 2-forms of the almost contact metric structures Σ and Σ' respectively. By a direct computation we find

$$3(d\tilde{\Omega})((X, X'), (Y, Y'), (Z, Z')) = 3(d\Omega)(X, Y, Z) + 3f^2(d\Omega')(X', Y', Z') -$$

$$(3.2) \qquad - 2f(\eta'(X'))(d\eta)(Y, Z) - 2f(\eta'(Y'))(d\eta)(Z, X) - 2f(\eta'(Z'))(d\eta)(X, Y) +$$

$$+ 2f(\eta(X))(d\eta')(Y', Z') + 2f(\eta(Y))(d\eta')(Z', X') + 2f(\eta(Z))(d\eta')(X', Y').$$

for any $X, Y, Z \in T(M)$ and $X', Y', Z' \in T(M')$. Now if $d\tilde{\Omega} = 0$, putting in (3.2) $Z = 0$, $X' = Y' = 0$, $Z' = \xi'$; $X = Y = 0$, $Z = \xi$, $Z' = 0$; $X' = Y' = Z' = 0$ or $X = Y = Z = 0$ we obtain $d\eta(X, Y) = 0$, $d\eta'(X', Y') = 0$, $d\Omega(X, Y, Z) = 0$, $d\Omega'(X', Y', Z') = 0$ for any $X, Y, Z \in T(M)$ and $X', Y', Z' \in T(M')$. Conversely, if the forms η, η', Ω, Ω' are all closed, from (3.2) it follows that $d\tilde{\Omega} = 0$. Thus, the structure $(\tilde{J}, \tilde{g})$ is almost Kählerian if and only if $d\eta = 0$, $d\Omega = 0$, $d\eta' = 0$, $d\Omega' = 0$ and so the proposition 3.3 is proved.

From propositions 3.2 and 3.3 we get

Proposition 3.4. *The almost Hermitian structure* $(\tilde{J}, g)$ *is Kählerian if and only if the almost contact metric structures* (Φ, ξ, η, g) *and* (Φ', ξ', η', g') *are both cosymplectic.*

Now let $\tilde{\nabla}$ be the covariant differentiation with respect to the Riemannian connection defined by $\tilde{g}$. It is easy to prove that for any vector fields $X, Y \in T(M)$ and $X', Y' \in T(M')$, we have $\nabla_{(X,X')}(Y, Y') = (\nabla_X Y, \nabla'_{X'} Y')$, where ∇ and ∇' are the covariant differentiations with respect to the Riemannian connections defined by g and g' respectively, and thus by a direct calculation we obtain

$$(3.3)\qquad (\tilde{\nabla}_{(X,X')}\tilde{J})(Y, Y') = ((\nabla_X\Phi)Y, (\nabla'_{X'}\Phi')Y') - f(\eta'(Y'))(\nabla_X\xi, 0) + \\ + h(\eta(Y))(0, \nabla'_{X'}\xi') - f((\nabla'_{X'}\eta')Y')(\xi, 0) + h((\nabla_X\eta)Y)(0, \xi').$$

Let $\tilde{R}$ be the tensor field of type (1, 2) on $\tilde{M}$ given by $\tilde{R}(\tilde{X}, \tilde{Y}) = (\tilde{\nabla}_{\tilde{X}}\tilde{J})\tilde{Y} + (\tilde{\nabla}_{\tilde{Y}}\tilde{J})\tilde{X}$ for any vector fields $\tilde{X}$ and $\tilde{Y}$ on $\tilde{M}$. It is well know that the almost Hermitian structure $(\tilde{J}, g)$ is a nearly Kählerian structure if and only if $\tilde{R} = 0$. Since $\tilde{R}$ is a tensor field, then its vanishing condition is equivalent to the following conditions

$$\tilde{R}((X, 0), (Y, 0)) = 0, \quad \tilde{R}((0, X'), (0, Y')) = 0, \quad \tilde{R}((X, 0), (0, X')) = 0$$

for any vector fields $X, Y \in T(M)$ and $X', Y' \in T(M')$. By means of (3.3), we see the above three conditions are equivalent to the following conditions

$$(3.4)\qquad (\nabla_X\Phi)Y + (\nabla_Y\Phi)X = 0, \qquad (3.7)\qquad (\nabla'_{X'}\eta')Y' + (\nabla'_{Y'}\eta')X' = 0,$$

$$(3.5)\qquad (\nabla_X\eta)Y + (\nabla_Y\eta)X = 0, \qquad (3.8)\qquad \eta'(X')\nabla_X\xi = 0,$$

$$(3.6)\qquad (\nabla'_{X'}\Phi')Y' + (\nabla'_{Y'}\Phi')X' = 0, \qquad (3.9)\qquad \eta(X)\nabla'_{X'}\xi' = 0.$$

Since (3.4) and (3.6) imply (3.5) and (3.7) respectively (see prop. 2.1), consequently by virtue of prop. 2.2 we see that (3.4)—(3.9) are equivalent to the statement that the almost contact metric structures (Φ, ξ, η, g) and (Φ', ξ', η', g') are both closely cosymplectic. Thus we have

Proposition 3.5. *The almost Hermitian structure* $(\tilde{J}, \tilde{g})$ *is nearly Kählerian if and only if the almost contact metric structures* (Φ, ξ, η, g) *and* (Φ', ξ', η', g') *are both closely cosymplectic.*

It is well known that a spehre of odd dimension has a Sasakian structure (see [4], p. 75) : in particular such a structure is a normal almost contact metric structure, but is it neither cosymplectic, nor almost cosymplectic, nor closely cosymplectic. By the assumption that f is the function with constant value 1 on $\tilde{M}$, from prop. 3.2, 3.3, 3.4 and 3.5 we obtain the following

Corollary. *The direct product of two spheres of add dimension has a Hermitian structure, which is neither Kählerian, nor almost Kählerian, nor nearly Kählerian.*

This result generalises a theorem of Calabi-Eckmann which says that the product of two odd-dimensional spheres admits a complex structure, which is not a Kählerian structure (see [4]).

Finally let $M' = R$ be the real line, $\Phi' = 0$, $\xi' = d/dt$, $\eta' = dt$, t being the coordinate in **R**, and let g' be the standard Riemannian metric on **R**; if, in addition, we assume that f is the function with constant value 1 on $M \times$ **R**, from prop. 3.3, 3.4 and 3.5 we obtain some results in paper [6].

REFERENCES

1. Blair D.E. — *Almost contact manifolds with Killing structures tensors*, Pacific J. of Math., 39 (1971), 285—292
2. Blair D.E. — *Contact manifolds in Riemannian geometry*. Lecture Notes in Mathematics, vol. 509 Springer-Verlag. Berlin, 1976
3. Blair D.E. Showers D.K. — *Almost contact manifolds with Killing structures tensors II*, Diff. Geom. 9 (1974), 577—582
4. Calabi E., Eckmann B. — *A class of compact, complex manifolds which are not algebraic*, Ann. of Math., 58 (1953), 494—500
5. Chen B.Y. — *Geometry of submanifolds and its applications*, Science University of Tokyo, 1981
6. Ianus S., Smaranda D. — *Some remarkable structures on the product of an almost contact metric manifold with the real line*. Lucrările colocviului național de Geometrie și Topologie din Timișoara. 26—28 aprilie 1977, 107—110
7. Morimoto A. — *On normal almost contact structures*, J. Math. Soc. Japan, 15 (1963), 420—436
8. Sasaki S. — *On differentiable manifolds with certain structures which are closely related to almost contact structures I* Tôhoku Math. J., 12 (1960), 459—476
9. Sasaki S., Hatakeyama Y. — *On differentiable manifolds with certain structures which are closely related to almost contact structures II*, Tôhoku Math. J., 13 (1961). 281—294

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