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BOUNDARY-VALUE STOCHASTYC PROBLEMS (I)

BY

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§ 0. Introduction. We are concerned here with the existence and uniqueness of a nonanticipative stochastic process u, strong solution of the nonlinear boundary-value stochastic problem :

$$(0.1)\qquad \begin{cases} du + A\,u\,dt = f(\omega, t, x)dt + dM(\omega, t, x) & \text{in } \Omega\times]0, T[\times D \\ \dfrac{\partial u}{\partial \nu} + \beta(u) \ni g(\omega, t, x) & \text{in } \Omega]0, T[\times\Gamma \\ u(\omega, 0, x) = u_0(\omega, x) & \text{in } \Omega\times D, \end{cases}$$

where D is a bounded and open subset of the Euclidean space R^N with boundary Γ, $T\in]0, +\infty[$ is fixed. Ω is an abstract space of random events, A is a second order and symmetric differential operator on D, β is a maximal monotone graph (in general multivalued) in $R\times R$, M is a martingale and f, g, u_0 are fixed stochastic processes.

System (0.1) may be write in the form

$$(0.2)\qquad \begin{aligned} &du + A\,u\,dt + \partial\varphi(u)dt = \bar{f}dt + dM(t),\\ &u(0) = u_0, \end{aligned}$$

where $A + \partial\varphi$ is an unbounded operator from a Hilbert space V to V^* (the dual space of V). Generally we cannot find strong stochastic solutions for the system (0.2). Therefore in [15] and [16] we introduced a concept of weak solutions for (0.2). In this paper the particular form of the problem permits to find strong (Itô) solutions.

System (0.1) originates from some physical problems (see for example [2] and [10]) : the Signorini-problem if

$$\beta(r) = \begin{cases} 0, & \text{for } r>0 \\ ]-\infty, 0], & \text{for } r=0 \\ \Phi, & \text{for } r<0\,; \end{cases}$$

and the enzyme diffusion (the Michaelis-Menten law) if

$$\beta(r) = \begin{cases} \dfrac{r}{r+m}, & \text{for } r>0 \\ ]-\infty, 0], & \text{for } r=0 \\ \Phi, & \text{for } r<0. \end{cases}$$

§ 1. **Preliminaries.** *A*1. Let $(\Omega, \mathcal{F}, P)$ be a fixed complete probability space and $(\mathcal{F}_t)_{t\geqslant 0}$ be an increasing and right-continuous family of sub-σ-algebras of $\mathcal{F}$, such that a $A \in \mathcal{F}_0$ if $P(A)=0$.

*A*2. If $(X, \|.\|)$ is a reflexive Banach space, we shall denote by $E(\xi | \mathcal{F}_t) \in L^p(\Omega, \mathcal{F}_t, P; X)$ the conditional expectation of $\xi \in L^p(\Omega, \mathcal{F}, P; X)$, $1 \leqslant p \leqslant +\infty$, with respect to $\mathcal{F}_t$.

Let $L^p_{ad}(\Omega \times]0, T[; X)$, $1 \leqslant p < +\infty$, the space of stochastic processes $f \in L^p(\Omega \times]0, T[; \mathcal{F} \times \mathcal{L}, P \times dt; X) = L^p(\Omega \times]0, T[; X)$ such that $\omega \to f(\omega, t)$ is $\mathcal{F}_t$-measurable a.e. $t \in]0, T[$ (t—a.e.). We have denoted by $\mathcal{L}$ the σ-algebras of Lebesgue measurable sets on $]0, T[$ and with dt the Lebesgue measure. The space $L^p_{ad}(\Omega \times]0, T[; X)$ is a linear closed subspace of $L^p(\Omega \times]0, T[; X)$.

We shall denote by $L^p_{ad}(\Omega; C(0, T; X))$, $1 \leqslant p < +\infty$, the space of the stochastic processes $f \in L^p(\Omega, \mathcal{F}, P; C([0, T]; X)) = L^p(\Omega; C(0, T; X))$ with the property: $\omega \to f(\omega, t)$ is $\mathcal{F}_t$-measurable for every $t \in [0, T]$ (f is called adapted process). The space $L^p_{ad}(\Omega; C(0, T; X))$ is a linear closed subspace of $L^p(\Omega; C(0, T; X))$.

*A*3. Let $(H, |\cdot|)$ be a real separable Hilbert space. By $\mathcal{M}^p(0, T; H)$, $1 \leqslant p < \infty$ we shall denote the space of the stochastic processes $M \in L^p_{ad}(\Omega; C(0, T; H))$ such that $M(\omega, 0) = 0$ a.s. $\omega \in \Omega$ (ω—a.s.) and $E(M(t) | \mathcal{F}_s) = M(s)$, ω—a.s., if $0 \leqslant s \leqslant t \leqslant T$. The elements of this space are called continuous p-martingales. The space $\mathcal{M}^2(0, T; H)$ with the inner product $((M_1, M_2)) = E(M_1(T), M_2(T))$ is a Hilbert space.

A 4. Let $\mathcal{P}$ the σ-algebras of the predictable sets i.e. σ-algebras generated by $\mathcal{R} = \{F_1 \times]s, t], F_2 \times \{0\} : 0 \leqslant s < t, F_1 \in \mathcal{F}_s, F_2 \in \mathcal{F}_0\}$.

Let G be a separable real Hilber space. A process $f : \Omega \times [0, T] \to \mathcal{L}(H, G)$ will be called point-predictable iff for any $h \in H$, $f(h)$ is $(\mathcal{P}, \mathcal{B}(G))$-measurable (strongly $\approx$ weakly since G is separable), and f is point measurable iff $f(h)$ is $(\mathcal{F} \times \mathcal{L}, \mathcal{B}(G))$-measurable. We have denoted by $\mathcal{B}(G)$ the σ-algebras Borel on G.

*A*4. We shall denote by $\mathcal{L}_2(H, G) \subset \mathcal{L}(H, G)$ the space of Hilbert-Schmidt operators, with the Hilbert-Schmidt norm denoted by $\|.\|_2$. Let us recall that $B \in \mathcal{L}(H, G)$ is in $\mathcal{L}_2(H, G)$ if $\|B\|_2^2 = \sum_i |Be_i|_G^2 < +\infty$ for an orthonormal basis $\{e_i\}$ of H.

*A*6. We shall recall some facts from [13] and [4].

1°. Let $M \in \mathcal{M}^2(0, T; H)$ and λ be Doleans measure of $|M|^2$ (i.e. a measure on $\mathcal{P}$ such that $\lambda(F \times]s, t]) = E[1_F(|M(t)|^2 - |M(s)|^2)]$ on $\mathcal{R}$). Then there exists an unique $\mathcal{L}(H, H)$-valued point-predictable process K such that

a) $K(\omega, s)$ is a nuclear, positive, symmetric operator with $\mathrm{Tr}K(\omega, s) = 1$, λ—a.e.;

b) $E[1_F(M(t) - M(s)) \otimes (M(t) - M(s))] = \int_{F \times]s, t]} K(\omega, \tau) d\lambda$ for every $F \in \mathcal{F}_s$, $0 \leqslant s < t \leqslant T$, (if $h \in H$, $g \in G$ then $h \otimes g \in \mathcal{L}(H, G)$ and is defined by $(h \otimes g)(x) = (h, x)_H g$). K is called the covariance process of M.

We denote $\Lambda_T(M; H, G) = \{f(.,.) : f$ is an $\mathcal{L}(H, G)$-valued, point-predictable process such that

$$\|f\|_{\Lambda_T} \overset{\text{def}}{=} \left(\int_{\Omega \times]0, T]} \|fK^{1/2}\|_2^2 d\lambda \right)^{1/2} < +\infty.$$

2°. If $f \in \Lambda_T$ then $\int_0^t f(\omega, s) dM(s)$ is a well-defined G-valued process such that

a) $$(\omega, t) \to \int_0^t f(\omega, s) dM(s) \text{ is in } \mathcal{M}^2(0, T; G),$$

b) $$E\left|\int_0^T f dM(s)\right|^2 = \|f\|_{\Lambda_T}^2.$$

Remark 1.1 (see for example [14] I-Partie).

For any $f \in L^2_{ad}(\Omega; C(0, T; H))$ and $M \in \mathcal{M}^2(0, T; H)$ the stochastic integral $\int_0^t (f(s), dM(s))$ is well defined, $\int_0^{\cdot} (f(s), dM(s)) \in \mathcal{M}^1(0, T; R)$ and

$$E \sup_{s \in [0, t]} \left| \int_0^s (f(\tau), dM(\tau)) \right| \leqslant 3E \sup_{s \in [0, t]} \|f(s)\| \sqrt{<M>_t},$$

where $<M> \in L^1_{ad}(\Omega; C(0, T; R))$ is an unique stochastic process defined by M such that $t \to <M>(\omega, t)$ is increasing ω-a.s. and $|M|^2 - <M> \in \mathcal{M}^1(0, T; R)$. If M is a real Wiener process then $<M>_t = t$.

3°. Let $(W(t))_{t \geqslant 0}$ be a Wiener process. Then $\lambda = P \times dt$, $K(\omega, s) = K = \sum_{i=1}^{\infty} \alpha_i e_i \otimes e_i$, $W(t) = \sum_i \sqrt{\alpha_i} b_i(t) e_i$, $\|f\|_{\Lambda_T}^2 = \sum_i \alpha_i \int_0^T |f(t) e_i|^2 dt$, $<W>_t = t \sum_i \alpha_i$,

where $\alpha_i \geqslant 0$, $\sum \alpha_i < +\infty$, $\{e_i\}$ is an orthonormal basis in H, b_i, are independent real-valued Wiener processes. In this case

$$\int_0^T f(s) dW(s) = \sum_{i=1}^{\infty} \alpha_i \int_0^T f(s) e_i db_i(s),$$

where this series is convergent in $L^2(\Omega, G)$, and

$$< \int_0^{\cdot} f(s) dW(s) >_t = \int_0^t Tr(f(s) K f^*(s)) ds.$$

*A*7. Let $S(t) \in \mathcal{L}(H, H)$, $t \geqslant 0$, be a C_0-semigroup of operators and A be its infinitesimal generator. The domain $(D(A))$ of A with the graph norm $\|h\|_{D(A)}^2 = |h|^2 + |Ah|^2$ is a separable Hilbert space continuously embedded in H and $A \in \mathcal{L}(D(A), H)$.

Proposition 1.1. *Let $M \in \mathcal{M}^2(0, T; H)$ and K its covariance process. If A generates an analytic semigroup $S(t)$ on H and*

$$\sup_{t \in [0, T]} \int_{\Omega \times [0, t]} \|A S(t-s) K^{1/2}\|_2^2 \, d\lambda < +\infty, \tag{1.1}$$

then

$$u(t) = \int_0^t S(t-s) dM(s), \quad t \in [0, T] \tag{1.2}$$

is unique strong solution of the equation

$$du(t) = A u dt + dM(t), \qquad u(0) = 0, \tag{1.3}$$

i.e.

$$u \in L^2_{ad}(\Omega \times]0, T[; D(A)) \cap L^2_{ad}(\Omega; C(0, T; H)) \tag{1.4}$$

and

$$u(t) = \int_0^t A u(s) ds + M(t) \tag{1.5}$$

for all $t \in [0, T]$; ω-*a.s.*

Proof. Since

$$\psi_t(\omega, s) = \begin{cases} 0, & \text{if } s \in [t, T] \\ S(t-s), & \text{if } s \in [0, t[\end{cases}$$

belongs to the space $\Lambda_t(M; H, D(A))$ we have that

$$u(t) = \int_0^t S(t-s) dM(s) \in D(A) \text{ and } A u(t) = \int_0^t A S(t-s) dM(s).$$

Moreover

$$u \in L^2_{ad}(\Omega \times]0, T[; D(A)) \cap C([0, T]; L^2(\Omega; H)). \tag{1.6}$$

From [4] the „mild" solution (1.2) is also „weakened" solution, i.e.

$u(t) = A \int_0^t u(s) ds + M(t)$ and by (1.6) it follows (1.5). Now condition (1.4) is quite easy to obtain

Remark 1.2. From [5] and [6] we have, for Eq. (1.3), the following theorem:

If A is the infinitesimal generator of the strongly continuous semigroup $S(t)$ on H and $dM(t) = D(t) dW(t)$, where $D \in L^2_{ad}(\Omega \times]0, T[; \mathcal{L}(H, G))$ and W is a Wiener process (the form is given in this Section, point A6—3°) then the conditions

$$S(t-s) D(s) e_i \in D(A), \ \omega\text{-a.s.} \tag{1.7}$$

$$\Sigma \alpha_i E \int_0^t \|A S(t-s) D(s) e_i\|^2 ds < +\infty. \tag{1.8}$$

are sufficient in order that $u(t)$ from (1.2) be a strong solution for Eq. (1.3).

B. Let V and H be two real separable Hilbert spaces such that V is densely and continuously embedded in H. Hence $V \subset H \subset V^*$ algebraically and topologically, where H is identified with its own dual, and V^* is the dual of V. The norms in V, H and V^* are denoted by $\|.\|$, $|.|$ and $\|\cdot\|_*$ respectively. By $< w, v >$ we shall denote the value of $w \in V^*$ at $v \in V$; if $v, w \in H$ then $< w, v > = (w, v)$, where $(.,.)$ denotes the inner product in H.

Energy equality. *If* $u \in L^2(\Omega \times]0, T[; V)$, $u_0 \in L^2(\Omega, \mathcal{F}_0, P; H)$, $f \in L^2_{ad}(\Omega \times]0, T[; V^*)$, $M \in \mathcal{M}^2(0, T; H)$ *and*

$$u(t) = u_0 + \int_0^t f(s) ds + M(t) \tag{1.9}$$

then $u \in L^2_{ad}(\Omega; C(0, T; H)) \cap L^2_{ad}(\Omega \times]0, T[; V)$ *and*

$$|u(t)|^2 = |u_0|^2 + 2 \int_0^t < f(s), u(s) > ds + 2 \int_0^t (u(s), dM(s)) + \tag{1.10}$$

$$+ < M >_t, \ \textit{for all } t \in [0, T]; \ \omega\text{-a.s.}$$

The proof may be found in [14] (Chap. II)

We put

$$W(0, T; V) = \left\{ u : u \in L^2(0, T; V), \ \frac{du}{dt} = u' \in L^2(0, T; V^*) \right\}, \tag{1.11}$$

where du/dt is the derivative of $u(t)$ in the sense of V^*-valued distributions on $]0, T[$. $W(0, T; V)$ is Hilbert space with the natural inner product

$$(u, v)_{W(0, T; V)} = \int_0^T (u, v)_V dt + \int_0^T (u', v')_{V^*} dt \tag{1.12}$$

and it is well known that $W(0, T; V) \subset C([0, T]; H)$ algebraically and topologically.

We denote

$$W_{ad}(\Omega \times]0, T[; V) = \{u : u \in L^2(\Omega; W(0, T; V)) \text{ such that } \tag{1.13}$$

$$u \in L^2_{ad}(\Omega \times]0, T[; V) \text{ and } u' (= du/dt) \in L^2_{ad}(\Omega \times]0, T[; V^*)\}$$

$W_{ad}(\Omega \times]0, T[; V)$ is a linear closed subspace of the Hilbert space $L^2(\Omega; W(0, T; H))$ and is embedded algebraically and topologically in $L^2_{ad}(\Omega; C(0, T; H))$.

Proposition 1.2. *If* $u \in L^2_{ad}(\Omega \times]0, T[; V) \cap L^2_{ad}(\Omega; C(0, T; H))$, $u_0 \in L^2(\Omega, \mathcal{F}_0, P; H)$, $f \in L^2_{ad}(\Omega \times]0, T[; V^*)$, $M \in \mathcal{M}^2(0, T; H)$ *then the following assertions are equivalent:*

(*i*) $u(t) = u_0 + \int_0^t f(s) ds + M(t)$ *for all* $t \in [0, T]$; ω-a.s.;

(ii) $(v(t), u(t)) = (v_0, u_0) + \int_0^t \left[<f, v> + <\frac{dv}{dt}, u> \right] ds + \int_0^t (v, dM)$ *for all* $t \in [0, T]$; ω-*a.s.* ; *for all* $v \in W_{ad}(\Omega \times]0, T[; V)$

(iii) $\int_0^T \left[<f, v> + <\frac{dv}{dt}, u> \right] ds + \int_0^T (v, dM) + (u_0, v_0) = 0$ ω-a.s. ; *for all* $v \in W_{ad}(\Omega \times]0, T[; V)$, $v(\omega, T) = 0$ ω-a.s. ;

(iv) $\int_0^T \left[<f, v> + \left(\frac{dv}{dt}, u - M \right) \right] ds + (u_0, v_0) = 0$ ω-a.s. ; for all $v \in L^2_{ad}(\Omega \times]0, T[; V]$ *such that* $dv/dt \in L^2_{ad}(\Omega \times]0, T[; V)$, $v(\omega, T) = 0$, ω-a.s., *where* $v_0(\omega) = v(\omega, 0)$.

Proof. $(i) \Rightarrow (ii)$. Let $v \in W_{ad}(\Omega \times]0, T[; V)$ be arbitrary. Hence

$$v(t) = v_0 + \int_0^T v'(s) ds, \quad \text{for all } t \in [0, T] ; \ \omega\text{-a.s.} \tag{1.14}$$

By Energy equality we have

$$(1/2)|u(t)|^2 = (1/2)|u_0|^2 + \int_0^t <f, u> ds + \int_0^t (u, dM) + 1/2 <M>_t \tag{1.15}$$

$$(1/2)|v(t)|^2 = (1/2)|v_0|^2 + \int_0^t <v', v> ds \tag{1.16}$$

$$(1/2)|v(t) + u(t)|^2 = (1/2)|v_0 + u_0|^2 + \int_0^t <f + v', v + u> ds + \int_0^t (v + u, dM) + (1/2) <M>_t \tag{1.17}$$

and substracting from Equality (1.17) Equalities (1.15) and (1.16), member by member, we infer (ii). In particular we allways have

$$(M(t), v(t)) = \int_0^t \left(\frac{dv}{dt}, M \right) ds + \int_0^t (v, dM), \tag{1.18}$$

for all $M \in \mathcal{M}^2(0, T ; H)$ and $v \in L^2_{ad}(\Omega ; C(0, T ; H))$ such that $dv/dt \in L^2_{ad}(\Omega \times]0, T[; H)$.

The implications $(ii) \Rightarrow (iii) \Rightarrow (iv)$ are obvious.

$(iv) \Rightarrow (i)$. Let $g \in L^2(0, T ; V)$ be arbitrary. We put in (iv) $v(\omega, t) = \int_0^t g(s) ds - \int_0^T g(s) ds$. If we denote $F(t) = \int_0^t f(s) ds$ we obtain by (1.18) and (iv)

$$\int_0^T <u - F - M - u_0, g> ds = 0, \ \omega\text{-a.s.} \tag{1.19}$$

Let K be a countable and dense set in $L^2(0, T ; V)$ and $\Omega_0 = \{\omega : (1.19)$ holds for all $g \in K\}$. We have $P(\Omega_0) = 1$ and (1.19) holds, by continuity, for all $g \in L^2(0, T ; V)$ and $\omega \in \Omega_0$. Hence

$$u(\omega, t) = u_0(\omega) + F(\omega, t) + M(\omega, t) \ t\text{-a.e.} ; \text{ for all } \omega \in \Omega_0, \tag{1.20}$$

and, again by continuity, from (1.20) (i) follows.

C. Let D be a bounded and open subset of $\mathbf{R}^N$ with the sufficiently smooth boundary Γ (it will be enough to assume that Γ is regular of class C^2). We set $Q =]0, T[\times D$, $\Sigma =]0, T[\times \Gamma$ and for k, r, s real numbers denote by $H^k(D)$, $H^k_0(D)$, $H^k(\Gamma)$, $H^{s,r}(Q)$, $H^{s,r}(\Sigma)$ the usual Sobolev spaces on D, Γ, Q respectively (see e.g. L i o n s & M a g e n e s [12], Chap. 4).

D. Throughout this paper we consider the conjecture $V \subset H \subset V^*$ from point *B*, this Section, with $V = H^1(D)$ and $H = L^2(D)$, and then we shall denote briefly $W(D) = W(0, T ; H^1(D)$, $\Theta = \Omega \times]0, T[\times D = \Omega \times Q$, $W_{ad}(\Theta) = W_{ad}(\Omega \times]0, T[; H^1(D))$, $L^2_{ad}(\Theta) = L^2_{ad}(\Omega \times]0, T[; L^2(D))$, $L^2_{ad}(\Omega \times \Sigma) = L^2_{ad}(\Omega \times]0, T[; L^2(\Gamma))$, $L^2_0(\Omega \times D) = L^2(\Omega, \mathcal{F}_0, P ; L^2(D))$.

§ 2. **Hypotheses.** Let D be a bounded and open subset of R^N with a sufficiently smooth boundary Γ. Let $(\Omega, \mathcal{F}, P, (\mathcal{F}_t)_{t \geq 0})$ be a probabilized stochastic basis (see Section 1-*A* 1).

We shall consider the boundary-value stochastic system

$$(2.1) \quad \begin{aligned} & du + Au dt = f(\omega, t, x) dt + dM(\omega, t, x) \ \text{ in } \Theta = \Omega \times Q \\ & \frac{\partial u}{\partial \nu} + \beta(u) \ni g(\omega, t, \sigma) \ \text{ in } \Omega \times \Sigma \\ & u(\omega, 0, x) = u_0(\omega, x) \ \text{ in } \Omega \times D, \end{aligned}$$

where $\partial u / \partial \nu$ is the outward normal derivative associated with A.

We assume that :

$(H\ 1)$ The operator A is a second order differential operator on D of the form

$$Ay = - \sum_{i, j=1}^N \frac{\partial}{\partial x_j} \left(a_{ij}(x) \frac{\partial y}{\partial x_i} \right) + a_0(x) y$$

where $a_{ij} \in C^1(\bar{D})$, $a_0 \in L^\infty(D)$, $a_{ij} = a_{ji}$ for all $i, j \in \{1, 2, ..., N\}$, $a_0(x) \geq 0$ a.e. $x \in D$, and there exists $C > 0$ such that

$$\sum_{i, j=1}^N a_{ij}(x) \xi_i \xi_j \geq c |\xi|^2 \text{ a.e. on } D$$

for all $\xi = (\xi_1, \xi_2, ..., \xi_N) \in R^N$.

$(H\ 2)$ $\quad f \in L^2_{ad}(\Theta)$, $g \in L^2_{ad}(\Omega \times \Sigma)$, $u_0 \in L^2_0(\Omega \times D)$.

$(H\ 3)$ $M \in \mathcal{M}^2(0, T ; L^2(D))$, with K covariance process and λ Doleans measure, verifies the condition

$$\sup_{t\in[0,T]} \int_{\Omega\times]0,t[} \|AS(t-s)K^{1/2}\|_2^2\, d\lambda < +\infty,$$

where $\|\cdot\|$ is the Hilbert-Schmidt norm in $L^2(D)$ and $S(t)$ is the semigroup (strongly continuous and analytic) generated by $A : H^2(D)\cap H_0^1(D)\subset L^2(D)\to \to L^2(D)$ from $(H\ 1)$.

$(H\ 4)$. β is a graph maximal monotone in $R\times R$ such that $0\in\beta(0)$, what is equivalent with $\beta=\partial\varphi$, where $\varphi : R\to]-\infty, +\infty]$ is a proper convex lower-semicontinuous function, $\varphi(x)\geq\varphi(0)$ for all $x\in R$, and $\partial\varphi$ represents the subdifferential of φ.

Definition 2.1. *A function* $u\in L^2_{ad}(\Omega\times]0, T[\,; H^1(D))\cap L^2_{ad}(\Omega\,; C(0, T\,; L^2(D)))$ *is a strong solution to* (2.1) *if there exists* $w\in L^2_{ad}(\Omega\times\Sigma)$ *such that*

$$w(\omega, t, x)\in\beta(u(\omega, t, x)) \quad a.e.\ on\ \Omega\times\Sigma \tag{2.2}$$

$$\text{a)}\quad -\int_Q u\frac{\partial v}{\partial t}\,dt dx+\int_0^T a(u, v)dt+\int_\Sigma (w-g)v dt d\sigma=\int_D u_0 v(\omega, 0, x)dx+ \tag{2.3}$$

$$+\int_Q f v dt dx+\int_0^T\int_D v(\omega, t, x)dM(\omega, t, x)dx,\quad \omega\text{-a.s.}\,;$$

for all $v\in W_{ad}(\Theta)$, $v(\omega, T, x)=0$ *a.e. on* $\Omega\times D$, *or equivalently*

$$\text{b)}\quad \int_Q (-u+M)\frac{\partial v}{\partial t}\,dt dx+\int_0^T a(u, v)dt+\int_\Sigma (w-g)v dt d\sigma=$$

$$=\int_D u_0 v(\omega, 0, x)dx+\int_Q f v dt dx,\quad \omega\text{-a.s.}\,;$$

for all $v\in L^2_{ad}(\Omega\times]0, T[\,; H^1(D))$ *such that* $\partial v/\partial t\in L^2_{ad}(\Omega\times]0, T[\,; H^1(D))$ *and* $v(\omega, T, x)=0$ *a.e. on* $\Omega\times D$.

Here $a : H^1(D)\times H^1(D)\to R$ is the bilinear functional

$$a(y, z)=\sum_{i,j=1}^N\int_D a_{ij}\frac{\partial y}{\partial x_i}\frac{\partial z}{\partial x_j}+\int_D a_0 yz dx, \tag{2.4}$$

$y, z\in H^1(D)$. According to Prop. 1.2, Conditions 2.3 can be written also in the form

$$u(t)+\int_0^t A_1 u(s)ds+\int_0^t h(u)(s)ds=u_0+\int_0^t \bar f(s)ds+M(t), \tag{2.5}$$

in $(H^1(D))^*$, for all $t\in[0, T]$; ω-a.s., where $A_1 : H^1(D)\to(H^1(D))^*$, $h(u)$, $\bar f\in L^2_{ad}(\Omega\times]0, T[\,; (H^1(D))^*)$ are given by

$$a)\quad <A_1 y, z>=a(y, z),\ y, z\in H^1(D),$$

$$b)\quad <h(u), y>=\int_\Gamma w y|_\Gamma\, d\sigma, \tag{2.6}$$

$$c)\quad <\bar f, y>=\int_D f y dx+\int_\Gamma g y|_\Gamma\, d\sigma.$$

We have denoted by $y/_\Gamma$ the trace of y at Γ, and remark that $<A_1 y, z>= =<Ay, z>+(\partial y|_\Gamma/\partial\nu, z)_{L^2(\Gamma)}$ for every $y\in H^2(D)$.

Remark 2.1. The condition $0\in\beta(0)$ is not an efective restriction since if $r_0\in\beta(s_0)$ then (2.2)+(2.3) is equivalent with the same problem for $\bar\beta$, $\bar g$, $\bar u_0$ and $\bar f$, where $\bar\beta(r)=\beta(r+s_0)-r_0$, $\bar g=g-r_0$, $\bar u_0=u_0-s_0$, $\bar f=f-a_0 s_0$ and, now, $0\in\bar\beta(0)$, $\bar u=u+s_0$.

§ 3. The main result. Theorem 3.1. *If* $\varphi(u_0)\in L^1(\Omega\times D)$ *then, under hypotheses* $(H\ 1-H4)$, *the System* (2.1) *has a unique strong solution.*

Proof. The uniqueness of the solution is obtained easily from (2.5). For proving the existence we shall divide System 2.1 in three systems

$$\begin{cases} dy_1(t)+Ay_1 dt=f(t)dt & \text{in } \Theta\\ y_1(\omega, t, x)=0 & \text{on } \Omega\times\Sigma\\ y_1(\omega, 0, x)=0 & \text{on } \Omega\times D,\end{cases} \tag{3.1}$$

$$\begin{cases} dy_2(t)+Ay_2 dt=dM(t) & \text{in } \Theta\\ y_2(\omega, t, x)=0 & \text{on } \Omega\times\Sigma\\ y_2(\omega, 0, x)=0 & \text{on } \Omega\times D,\end{cases} \tag{3.2}$$

$$\begin{cases} dz(t)+Az dt=0 & \text{in } \Theta\\ \dfrac{\partial z}{\partial\nu}+\beta(z)\ni h & \text{in } \Omega\times\Sigma\\ z(\omega, 0, x)=u_0(\omega, x), & \text{in } \Omega\times D,\end{cases} \tag{3.3}$$

and we will show in a subsequent paper that, for $h=g-\partial(y_1+y_2)/\partial\nu$, $u= =(y_1+y_2)+z$ is a strong solution of System (2.1).

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ON THE ENERGY RELEASE RATE IN THERMOELASTODYNAMIC CRACK PROPAGATION

BY

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1. Introduction. In a recent series of papers Gurtin [1—3] and Gurtin and Yatomi [4] established important results involving the energy release rate for a sharp crack in a hyperelastic body. The papers [1, 2] are concerned with the thermodynamics of fracture in the quasi-static theory. In [4], a unified treatment of the energy release rate in elastodynamic crack propagation has been given.

It is the purpose of this paper to give the thermoelastic counterpart to several results established in [4]. We restrict our attention to edge cracks in two-dimensional bodies.

2. Basic equations. Preliminaries. Throughout this paper light-face letters indicate scalars, bold-face lower case letters indicate vectors, bold-face upper case letters indicate tensors. $\mathbf{A}^T$ is the transpose of $\mathbf{A}$. $\mathbf{A}\cdot\mathbf{B} = A_{ij}B_{ij}$, div $\mathbf{S}$ is the vector field with components $\partial S_{ij}/\partial x_j$, $\nabla\mathbf{u}$ is the tensor field with components $\partial u_i/\partial x_j$, div $\mathbf{q}$ is the scalar $\partial q_i/\partial x_i$. Here we have used standard indicial notation and Cartesian coordinates. Let $L^p(\mathcal{R})$ be the class of all functions f on $\mathcal{R}$ with $|f|^p$ integrable on $\mathcal{R}$.

We consider a two-dimensional regular body B, which we identify with the regular region of R^2 it occupies in a fixed reference configuration. Let B^0 be the interior of B.

A thermodynamic process for B is specified by the following functions of the material point $x \in B$ and time $t \in [0, T]$: $\mathbf{u}$, the displacement vector; $\mathbf{S}$, the (Piola-Kirchhoff) stress tensor; ε, the internal energy density; η, the entropy density; θ, the absolute temperature; $\mathbf{q}$, the heat flux. We assume that body forces and heat supply are absent.

The equation of motion is

$$\text{div}\,\mathbf{S} = \rho\ddot{\mathbf{u}}, \tag{2.1}$$

where ρ is the density in the reference configuration. A superposed dot denotes differentiation with respect to time. We assume that ρ is constant.

The first two laws of thermodynamics lead (see Carlson [5]) to the relations

$$\dot{\varepsilon} = \mathbf{S}\cdot\nabla\dot{\mathbf{u}} + \text{div}\,\mathbf{q}, \quad \dot{\eta} \geq -\text{div}\left(\frac{1}{\theta}\mathbf{q}\right). \tag{2.2}$$