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ON THE ENERGY RELEASE RATE IN THERMOELASTODYNAMIC CRACK PROPAGATION

BY

D. IEŞAN

1. Introduction. In a recent series of papers Gurtin [1—3] and Gurtin and Yatomi [4] established important results involving the energy release rate for a sharp crack in a hyperelastic body. The papers [1, 2] are concerned with the thermodynamics of fracture in the quasi-static theory. In [4], a unified treatment of the energy release rate in elasto-dynamic crack propagation has been given.

It is the purpose of this paper to give the thermoelastic counterpart to several results established in [4]. We restrict our attention to edge cracks in two-dimensional bodies.

2. Basic equations. Preliminaries. Throughout this paper light-face letters indicate scalars, bold-face lower case letters indicate vectors, bold-face upper case letters indicate tensors. $\mathbf{A}^T$ is the transpose of $\mathbf{A}$. $\mathbf{A}\cdot\mathbf{B}=A_{ij}B_{ij}$, div $\mathbf{S}$ is the vector field with components $\partial S_{ij}/\partial x_j$, $\nabla\mathbf{u}$ is the tensor field with components $\partial u_i/\partial x_j$, div $\mathbf{q}$ is the scalar $\partial q_i/\partial x_i$. Here we have used standard indicial notation and Cartesian coordinates. Let $L^p(\mathcal{R})$ be the class of all functions f on $\mathcal{R}$ with $|f|^p$ integrable on $\mathcal{R}$.

We consider a two-dimensional regular body B, which we identify with the regular region of R^2 it occupies in a fixed reference configuration. Let B^o be the interior of B.

A thermodynamic process for B is specified by the following functions of the material point $x\in B$ and time $t\in[0, T]$: $\mathbf{u}$, the displacement vector; $\mathbf{S}$, the (Piola-Kirchhoff) stress tensor; ε, the internal energy density; η, the entropy density; θ, the absolute temperature; $\mathbf{q}$, the heat flux. We assume that body forces and heat supply are absent.

The equation of motion is

$$\text{div}\,\mathbf{S}=\rho\ddot{\mathbf{u}}, \tag{2.1}$$

where ρ is the density in the reference configuration. A superposed dot denotes differentiation with respect to time. We assume that ρ is constant.

The first two laws of thermodynamics lead (see Carlson [5]) to the relations

$$\dot{\varepsilon}=\mathbf{S}\cdot\nabla\dot{\mathbf{u}}+\text{div}\,\mathbf{q},\quad \dot{\eta}\geq-\text{div}\left(\frac{1}{\theta}\mathbf{q}\right). \tag{2.2}$$

The constitutive equations for homogeneous elastic materials are

$$\mathbf{S}=\hat{\mathbf{S}}(\mathbf{F},\theta),\quad \varepsilon=\hat{\varepsilon}(\mathbf{F},\theta),\quad \eta=\hat{\eta}(\mathbf{F},\theta),\quad \mathbf{q}=\mathbf{q}(\mathbf{F},\theta,\nabla\theta), \tag{2.3}$$

where $\mathbf{F}=\nabla\mathbf{u}$. As it is well known, compatibility with thermodynamics require that [5]

(i) the free energy

$$\psi=\hat{\psi}(\mathbf{F},\theta)=\hat{\varepsilon}(\mathbf{F},\theta)-\theta\hat{\eta}(\mathbf{F},\theta), \tag{2.4}$$

determines the stress and entropy through the classical relations

$$\mathbf{S}=\frac{\partial\hat{\psi}}{\partial\mathbf{F}},\quad \hat{\eta}=-\frac{\partial\hat{\psi}}{\partial\theta}; \tag{2.5}$$

(ii) the heat flux obey the inequality

$$\mathbf{q}(\mathbf{F},\theta,\nabla\theta)\cdot\nabla\theta\leqslant 0. \tag{2.6}$$

By (2.5),

$$\dot{\psi}=\mathbf{S}\cdot\nabla\dot{\mathbf{u}}-\eta\dot{\theta}, \tag{2.7}$$

and, in view of (2.4), the energy equation $(2.2)_1$ takes the form

$$\theta\dot{\eta}=\operatorname{div}\mathbf{q}. \tag{2.8}$$

We assume that B contains at time t a straight edge crack C_t. The tip of the crack at time t is denoted by $\mathbf{z}_t$. We assume that $\mathbf{z}_t$ is C^2 in t with velocity $\mathbf{c}(t)=d\mathbf{z}_t/dt$ and that $\mathbf{z}_t\in B^0$ for $t\in[0,T]$. We will study fields $\varphi(\mathbf{x},t)$ which are defined at $\mathbf{x}$ in $B\setminus C_t$ and each $t\in[0,T]$. Following [4], the field φ is a C^n fracture field ($n\geqslant 0$ an integer) if :

(i) the derivatives of φ of order $\leqslant n$ exist away from the crack ;

(ii) φ and its derivatives of order $\leqslant n$ are continuous away from the crak and, except at the tip, are continuous up to the crack from either side.

We write $\varphi\in L^p(B)$ if $\varphi(\cdot,t)\in L^p(B)$ at each $t\in[0,T]$. If $\varphi\in L^p(B)$, then given any one-parameter family $B_\delta(\delta>0)$ of regular subregions of B with area $(B\setminus B_\delta)\to 0$ as $\delta\to 0$,

$$\int_{B_\delta}|\varphi(x,t)|^p da\to\int_B|\varphi(x,t)|^p da,$$

as $\delta\to 0$; when this limit is uniform in $t\in[0,T]$, we say that $\varphi\in L^p(B)$ uniformly in time.

We consider $\varphi(\mathbf{x},t)$ as a function $\varphi(\mathbf{z}_t+\mathbf{r},t)$ of t and the position vector $\mathbf{r}$ from the tip. The derivative of this function with respect to t holding $\mathbf{r}$ fixed will be denoted by φ'. We have $\varphi'=\dot{\varphi}+\nabla_c\varphi$, with $\nabla_c\varphi=\nabla\varphi\cdot\mathbf{c}$.

In what follows $\Omega_\delta=\Omega_\delta(t)$ is the disc of radius δ centered at the crack tip, $\mathbf{z}_t$. We will have the occasion to use the following lemma established in [4].

Lemma 1. *Let* $\mathbf{f}$ *be a* C^1 *fracture field with* $\operatorname{div}\mathbf{f}\in L^1(B)$. *Assume further that*

$$\lim_{\delta\to 0}\int_{\Omega_\delta}\mathbf{f}\cdot\mathbf{n}\,ds=0,$$

and that, for $\mathbf{e}$ *a unit vector normal to* C_t, $\mathbf{f}(x,t)\cdot\mathbf{e}\to 0$ *as* x *approaches* $C_t\setminus\{\mathbf{z}_t\}$ *from either side. Then*

$$\int_{\partial B}\mathbf{f}\cdot\mathbf{n}\,ds=\int_B\operatorname{div}\mathbf{f}\,da.$$

In the boundary integrals the letter $\mathbf{n}$ designate the outward unit normal.

3. Energy release rate. Henceforth, in boundary integrals the fields $\mathbf{s}$ and h will denote the stress traction and the heat flux, respectively,

$$\mathbf{s}=\mathbf{S}\mathbf{n},\quad h=\mathbf{q}\cdot\mathbf{n}. \tag{3.1}$$

We assume that the thermoelastic fields satisfy the following conditions

(A_1) $\mathbf{u}$ and θ are C^2 fracture fields ; $\mathbf{S}$, $\mathbf{q}$, η and ε are C^1 fracture fields ;

(A_2) for $\mathbf{f}=\mathbf{u},\mathbf{u}',\mathbf{u}'',\mathbf{u}'',\theta,\theta'$: $\mathbf{f}$ is bounded ; $\mathbf{S}$, $\mathbf{q}$, $\nabla\mathbf{f}\in L^2(B)$ uniformly in time ; ε, ε', η, η', $\nabla\nabla f\in L^1(B)$ uniformly in time ;

(A_3) the surface traction and the heat flux vanish on the crack ;

(A_4) $\lim\limits_{\delta\to 0}\int_{\partial\Omega_\delta}h\,ds=0$, $\lim\limits_{\delta\to 0}\int_{\partial\Omega_\delta}\mathbf{s}\cdot\mathbf{v}\,ds=0$ for any bounded vector field $\mathbf{v}$ on B.

Of course, the thermoelastic fields satisfy the field equations (2.1), (2.8) and the constitutive equations (2.3) away from the crack, and the constitutive equations are consistent with the thermodynamic restrictions.

The kinetic energy per unit volume is

$$\chi=\rho\dot{\mathbf{u}}^2/2.$$

The function E on $[0,T]$ defined by

$$E=\int_{\partial B}(\mathbf{s}\cdot\mathbf{u}+h)\,ds-\frac{d}{dt}\int_B(\varepsilon+\chi)\,da, \tag{3.1}$$

is called the dynamic energy release rate.

Following [4], we can see that E is well defined and the definition of E is independent of the region $\mathcal{R}$ surrounding the tip.

Theorem 1. *At each time* t

$$E(t)=\lim_{\mathcal{R}\to\mathbf{z}_t}\int_{\partial\mathcal{R}}\left\{\left(\psi+\frac{1}{2}\rho|\nabla_c\mathbf{u}|^2-\mathbf{S}^T\nabla_c\mathbf{u}+\theta\eta\right)\right\}\cdot\mathbf{n}\,ds. \tag{3.2}$$

The relation (3.2) gives an alternative formula for E in terms of a boundary integral over a region which shrinks to the crack tip. This result is a direct consequence of Theorem 1 from [4]. We note that the relation (3.2) does not involve the heat flux. It is clear from (2.4) that (3.2) has the same form as in the isothermal case [4, 6]. Here the temperature is involved by means of ε and $\mathbf{S}$. Henceforth, if t_0 is a fixed time in $[0, T]$, we will denote the derivative with respect to t at t_0 by $(d/dt)_{t_0}$. Given a fracture field φ, we write φ_t for the function on B/C_t defined by $\varphi_t(x)=\varphi(x,t)$.

Theorem 2.

$$E(t_0)=\left(\frac{d}{dt}\right)_{t_0}\int_B\int_{t_0}^{t}\{(S_t-S_\lambda)\cdot\nabla\dot{u}_\lambda+\rho(\ddot{u}_t-\ddot{u}_\lambda)\cdot\dot{u}_\lambda+(\theta_t-\theta_\lambda)\dot{\eta}_\lambda\}d\lambda\,da. \tag{3.3}$$

Proof. If we denote

$$\gamma(t)=\int_{t_0}^{t}\int_{\partial B}(\mathbf{s}\cdot\dot{\mathbf{u}}+h)ds-\int_B(\varepsilon+\chi)da,$$

then

$$E=\dot{\gamma}. \tag{3.4}$$

In view of (1_3)–(1_4), (2.1), (2.8), and the Lemma 1 we can write

$$\int_{t_0}^{t}\int_{\partial B}(\mathbf{s}\cdot\dot{\mathbf{u}}+h)ds=\int_{t_0}^{t}\int_{\partial B}\mathbf{s}_t\cdot\dot{\mathbf{u}}_\lambda ds\,d\lambda+\int_{t_0}^{t}\int_{\partial B}(\mathbf{s}_\lambda-\mathbf{s}_t)\cdot\dot{\mathbf{u}}_\lambda\,ds\,d\lambda+\int_{t_0}^{t}\int_B\theta_t\dot{\eta}_\lambda d\lambda\,da+$$

$$+\int_{t_0}^{t}\int_B(\theta_\lambda-\theta_t)\dot{\eta}_\lambda d\lambda\,da=\int_{\partial B}\mathbf{s}_t\cdot(\mathbf{u}_t-\mathbf{u}_{t_0})ds+\int_{t_0}^{t}\int_{\partial B}(\mathbf{s}_\lambda-\mathbf{s}_t)\cdot\dot{\mathbf{u}}_\lambda ds\,d\lambda+$$

$$+\int_B\theta_t(\eta_t-\eta_{t_0})da+\int_{t_0}^{t}\int_B(\theta_\lambda-\theta_t)\dot{\eta}_\lambda d\lambda\,da=\int_B\{\mathbf{S}_t\cdot(\nabla\mathbf{u}_t-\nabla\mathbf{u}_{t_0})+\rho\ddot{\mathbf{u}}_t\cdot(\mathbf{u}_t-\mathbf{u}_{t_0})+$$

$$+\theta_t(\eta_t-\eta_{t_0})\}da+\int_{t_0}^{t}\int_{\partial B}(\mathbf{s}_\lambda-\mathbf{s}_t)\cdot\dot{\mathbf{u}}_\lambda ds\,d\lambda+\int_{t_0}^{t}\int_B(\theta_\lambda-\theta_t)\dot{\eta}_\lambda d\lambda\,da.$$

Since

$$\frac{d}{dt}\left[\int_{t_0}^{t}\int_{\partial B}(\mathbf{s}_\lambda-\mathbf{s}_t)\cdot\dot{\mathbf{u}}_\lambda ds\,d\lambda+\int_{t_0}^{t}\int_B(\theta_\lambda-\theta_t)\dot{\eta}_\lambda d\lambda\,da\right]=$$

$$=-\int_{t_0}^{t}\int_{\partial B}\dot{\mathbf{s}}_t\cdot\dot{\mathbf{u}}_\lambda\,ds\,d\lambda-\int_{t_0}^{t}\int_{\partial B}\dot{\theta}_t\dot{\eta}_\lambda d\lambda\,da.$$

we obtain

$$\left(\frac{d}{dt}\right)_{t_0}\int_{t_0}^{t}\int_B(\mathbf{s}\cdot\mathbf{u}+h)ds=\left(\frac{d}{dt}\right)_{t_0}\int_B\{\mathbf{S}_t\cdot(\nabla\mathbf{u}_t-\nabla\mathbf{u}_{t_0})+\rho\ddot{\mathbf{u}}_t\cdot(\mathbf{u}_t-\mathbf{u}_{t_0})+$$

$$+\theta_t(\eta_t-\eta_{t_0})\}da=\left(\frac{d}{dt}\right)\int_B\int_{t_0}^{t}(\mathbf{S}_t\cdot\nabla\dot{\mathbf{u}}_\lambda+\rho\ddot{\mathbf{u}}_t\cdot\dot{\mathbf{u}}_\lambda+\theta_t\dot{\eta}_\lambda)d\lambda\,da.$$

Using (2.2) and (2.8), we have

$$\frac{d}{dt}\int_B(\varepsilon+\chi)da=\frac{d}{dt}\int_B\int_{t_0}^{t}(\dot{\varepsilon}+\dot{\chi})d\lambda\,da=\frac{d}{dt}\int_B\int_{t_0}^{t}(\mathbf{S}_\lambda\cdot\nabla\dot{\mathbf{u}}_\lambda+\rho\ddot{\mathbf{u}}_\lambda\cdot\dot{\mathbf{u}}_\lambda+\theta_\lambda\dot{\eta}_\lambda)d\lambda\,da.$$

The last two results and (3.4) yield (3.3). In the absence of thermal effects the relation (3.3) reduces to those established in [4], [7].
As in [4], we write $E(t_0)=U(t_0)+\mathcal{K}(t_0)$ with

$$U(t_0)=\left(\frac{d}{dt}\right)_{t_0}\int_B\int_{t_0}^{t}[(\mathbf{S}_t-\mathbf{S})\cdot\nabla\dot{\mathbf{u}}_\lambda+(\theta_t-\theta_\lambda)\dot{\eta}_\lambda]d\lambda\,da,$$

$$\mathcal{K}(t_0)=\left(\frac{d}{dt}\right)_{t_0}\int_B\int_{t_0}^{t}\rho(\ddot{\mathbf{u}}_t-\ddot{\mathbf{u}}_\lambda)\cdot\dot{\mathbf{u}}_\lambda d\lambda\,da.$$

In what follows we consider the linear theory of thermoelasticity. In this case the constitutive equations are [5]

$$\mathbf{S}=\mathbf{C}\nabla\mathbf{u}-\mathbf{M}\vartheta,\quad \eta=\mathbf{M}\cdot\nabla\mathbf{u}+a\vartheta,\quad \mathbf{q}=\mathbf{K}\nabla\vartheta, \tag{3.5}$$

where $\mathbf{C}$, $\mathbf{M}$ are symmetric tensors, $\vartheta=\theta-\theta_0$, and θ_0 is the constant absolute temperature of natural state.

Theorem 3.

$$U(t_0)=\frac{1}{2}\left(\frac{d}{dt}\right)_{t_0}\int_B\{(\mathbf{S}_t-\mathbf{S}_{t_0})\cdot(\nabla\mathbf{u}_t-\nabla\mathbf{u}_{t_0})+(\vartheta_t-\vartheta_{t_0})(\eta_t-\eta_{t_0})\}da= \tag{3.6}$$

$$=\frac{1}{2}\left(\frac{d}{dt}\right)_{t_0}\int_B\{(\nabla\mathbf{u}_t-\nabla\mathbf{u}_{t_0})\cdot\mathbf{C}(\nabla\mathbf{u}_t-\nabla\mathbf{u}_{t_0})+a(\vartheta_t-\vartheta_{t_0})^2\}da,$$

$$E(t_0)=\frac{1}{2}\left(\frac{d}{dt}\right)_{t_0}\int_B\{(\mathbf{S}_t-\mathbf{S}_{t_0})\cdot(\nabla\mathbf{u}_t-\nabla\mathbf{u}_{t_0})+\rho(\ddot{\mathbf{u}}_t-\ddot{\mathbf{u}}_{t_0})\cdot(\mathbf{u}_t-\mathbf{u}_{t_0})+$$

$$+(\vartheta_t-\vartheta_{t_0})(\eta_t-\eta_{t_0})\}da.$$

Proof. We can write $U(t_0)$ in the form

$$U(t_0)=\frac{1}{2}\left(\frac{d}{dt}\right)_{t_0}\left\{\int_B[(\mathbf{S}_t-\mathbf{S}_{t_0})\cdot(\nabla\mathbf{u}_t-\nabla\mathbf{u}_{t_0})+(\vartheta_t-\vartheta_{t_0})(\eta_t-\eta_{t_0})]da+\right. \tag{3.7}$$

$$+\int_B\int_{t_0}^{t}[(\mathbf{S}_t+\mathbf{S}_{t_0}-2\mathbf{S}_\lambda)\cdot\nabla\dot{\mathbf{u}}_\lambda+(v_t+v_{t_0}-2v_\lambda)\dot{\eta}_\lambda]\,d\lambda\,da.$$

Next,

$$\int_{t_0}^{t}[(\mathbf{S}_t+\mathbf{S}_{t_0}-2\mathbf{S}_\lambda)\cdot\nabla\dot{\mathbf{u}}_\lambda+(v_t+v_{t_0}-2v_\lambda)\dot{\eta}_\lambda]\,d\lambda= \tag{3.8}$$

$$=(\mathbf{S}_t+\mathbf{S}_{t_0})\cdot(\nabla\mathbf{u}_t-\nabla\mathbf{u}_{t_0})+(v_t+v_{t_0})(\eta_t-\eta_{t_0})-2\int_{t_0}^{t}(\mathbf{S}\cdot\nabla\dot{\mathbf{u}}+\theta\dot{\eta})\,d\lambda.$$

By (3.5)

$$\mathbf{S}\cdot\nabla\dot{\mathbf{u}}+\theta\dot{\eta}=\nabla\dot{\mathbf{u}}\cdot\mathbf{C}\nabla\mathbf{u}+a\theta\dot{\theta}=\dot{w}/2, \tag{3.9}$$

where

$$w=\mathbf{S}\cdot\nabla\mathbf{u}+\theta\eta=\nabla\mathbf{u}\cdot\mathbf{C}\nabla\mathbf{u}+a\theta^2. \tag{3.10}$$

Thus we have

$$\int_{t_0}^{t}(\mathbf{S}\cdot\nabla\dot{\mathbf{u}}+v\dot{\eta})\,d\lambda=\mathbf{S}_t\cdot\nabla\mathbf{u}_t-\mathbf{S}_{t_0}\cdot\nabla\mathbf{u}_{t_0}+v_t\eta_t-v_{t_0}\eta_{t_0}. \tag{3.11}$$

In view of (3.5) we can write $\mathbf{S}_t\cdot\nabla\mathbf{u}_{t_0}+v_t\eta_{t_0}=\nabla\mathbf{u}_{t_0}\cdot\mathbf{C}\nabla\mathbf{u}_t+av_tv_{t_0}$.

Since $\mathbf{C}$ is symmetric, we obtain

$$\mathbf{S}_t\cdot\nabla\mathbf{u}_{t_0}+v_t\eta_{t_0}=\mathbf{S}_{t_0}\cdot\nabla\mathbf{u}_t+v_{t_0}\eta_t. \tag{3.12}$$

From (3.8), (3.11), (3.12) we arrive at $(3.6)_{1,2}$. The relation $(3.6)_3$ is a direct consequence of $(3.6)_1$ and Theorem 3 of [4].

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STRESS DISTRIBUTION IN AN INFINITE REGION BOUNDED BY A CIRCLE UNDER BIAXIAL FORCES

BY

M. LAL and R. S. SIKARWAR

1. Introduction. Complex variable approach for the solution of two-dimensional problems in elasticity has proved to be very useful. This technique was propounded by K o l o s o v [1] and M u s k h e l i s h v i l i [2]. It was later developed and adopted by A d k i n s, G r e e n et al. [3, 4] to be used in solving problems in second order elasticity. Subsequently C a r l s o n and S h i e l d [5] have used this technique to study of second and higher order effects in a class of problems in plane finite elasticity. Of late, B h a r g a v a and P a n d e [6, 7] have further medified this technique to make it suitable for the solution of more general problems in plane strain. Recently L a l and P a n d e [8, 9] have used his technique to solve several problems in the second order elasticity. A brief resume of the same is given below. Let

$$\zeta=x_1+ix_2 \text{ and } z=\xi+i\eta$$

be the coordinates of a point of an elastic body in its undeformed state $\mathcal{B}_0$ and the deformed state $\mathcal{B}$ respectively. The Airy's stress function φ and the displacement vector $D=u_1+iu_2$, being analytic, can be expanded as

$$\begin{aligned}\varphi&=\mu\varepsilon\{\varphi_0(z,\bar{z})+\varepsilon\varphi_1(z,\bar{z})+\ldots\},\\ D&=\varepsilon\{D_0(z,\bar{z})+\varepsilon D_1(z,\bar{z})+\ldots\},\end{aligned} \tag{1.1}$$

where ε is a parameter depending upon the physical condition of the problem.

Equilibrium equations are satisfied if

$$T^{11}=\overline{T}^{22}=-4\frac{\partial^2\varphi}{\partial\bar{z}^2}\,;\quad T^{12}=4\frac{\partial^2\varphi}{\partial z\partial\bar{z}}, \tag{1.2}$$

where

$$T^{11}=\overline{T}^{22}=p_{\xi\xi}-p_{\eta\eta}-2ip_{\xi\eta},\quad T^{12}=p_{\xi\xi}+p_{\eta\eta}. \tag{1.3}$$

If the resultant stress along the boundary is zero, then

$$\frac{\partial\varphi}{\partial z}=\frac{\partial\varphi}{\partial\bar{z}}=0, \tag{1.4}$$