Proof. Indeed, φ gives rise to a H-map between the saturated spaces Y_1, $X_1 \cdot \varphi^{-1}(G)$ makes sense and is a finely open set in Y_1.

Lemma 7.11. *There exists a base β for the natural topology on X, such that $B^{X \setminus U}$ and $B^{Y \setminus \varphi^{-1}(U)}$ are balayages, $\forall U \in \beta$.*

Proof. The reasoning is entirely analogous to the proof of th. 3 from [7].

Corollary 7.12. *Let S, T be standard H-cones of functions on X, Y. If $\varphi : Y \to X$ is a naturally continuous H-map and satisfies*

$$(^{**}) \qquad \varphi(B^{X \setminus U} s) = B^{Y \setminus \varphi^{-1}(U)}(\varphi(s)),$$

$\forall s \in S$, $U \in \beta$, then $\varphi|_{\varphi^{-1}(U)}$ is a H-map from $\mathcal{S}(U)$ to $\mathcal{T}(\varphi^{-1}(U))$ $\forall U \in \beta$,

Theorem 7.13. *Let S, T be standard H-cones of functions on X, Y. Let $\varphi : Y \to X$ be a naturally continuous H-map which satisfies $(^{**})$. If $\mathcal{T}$* (resp. *$\mathcal{T}'$) is a natural sheaf, then φ_U is a H-map from $\mathcal{S}(U)$ to $\mathcal{T}(\varphi^{-1}(U))$ (resp. $\mathcal{T}'(\varphi^{-1}(U))$) for any naturally open U.*

Proof. Let (U_α) be covering for U, with $U_\alpha \in \beta$. Let $s \in \mathcal{S}(U)$. Hence $s|_{U_\alpha} \in \mathcal{S}(U_\alpha)$. From cor. 7.12 we have $s \circ \varphi|_{\varphi^{-1}(U_\alpha)} \in \mathcal{T}(\varphi^{-1}(U_\alpha))$. $\mathcal{T}$ being a sheaf. it follows that $s \circ \varphi \in \mathcal{T}(\varphi^{-1}(U))$.

Remark. Since $(B^{X \setminus G} s) \circ \varphi \leqslant B^{Y \setminus \varphi^{-1}(G)}(s \circ \varphi)$ for any finely open G, it suffices that $(^{**})$ holds only for a generator $p \in S$.

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EXTENSION OF VECTOR AND GROUP-VALUED SUBMEASURES

BY

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The results of this paper are related to those from L. D r e w n o w s k i's article [4] and W. H e r e r's work [5]. In [4] the author obtains an extension procedure for a scalar submeasure; in [5] H e r e r gives an extension method for a group-valued measure.

In this article we define the concepts of vector submeasure and group-valued submeasure, the last being obviously more general than the previous one. Using Drewnowski's procedure, we give an extension procedure for a vector submeasure (Theorem 1.2), and next, taking into account Herer's method we obtain an extension method for a group-valued submeasure (Theorem 2.3). In the last section we compare the two extension methods described in Theorem 1.2 and Theorem 2.3 and we show that, under certain conditions, the two theorems mentioned above lead to the same extending σ-ring $\mathcal{R}_0$ and the same extending vector submeasure η_0 (see Theorem 3.2).

In the sequel we shall, generally, use the notations and terminology from [4].

§1. Let $(\mathcal{R}, \Delta, \cap)$ be a ring of subsets of a fixed set R and X a linear space ordered by the relation „$\leqslant$" induced by the convex cone P.

Definition 1.1. *A set function $\eta : \mathcal{R} \to X$ is subadditive if*

(1v) $\eta(\varnothing) = 0$;

(2v) $\eta(A \cup B) \leqslant \eta(A) + \eta(B)$ *for all A, $B \in \mathcal{R}$ with $A \cap B = \varnothing$.*

Definition 1.2. *We call vector submeasure a subadditive set function η defined on $\mathcal{R}$ with values in P which satisfies the condition:*

(3v) $\eta(A) \leqslant \eta(B)$ *for every A, $B \in \mathcal{R}$ with $A \subseteq B$.*

Example 1.1. Let $m : \mathcal{R} \to \mathbf{R}_+$ be a finite scalar submeasure and X a linear ordered space with the positive cone P. For every $p \in P$ we define the set function $\eta_p : \mathcal{R} \to P$ by $\eta_p(A) = m(A).p \in P$, for every $A \in \mathcal{R}$. It is obvious that η_p verifies the conditions (1v), (2v), (3v) so that η_p is a vector submeasure on $\mathcal{R}$.

Theorem 1.1. *If η is a vector submeasure on $\mathcal{R}$ then the family of classes $\mathcal{U}_p = \{A \in \mathcal{R} / \eta(A) \leqslant p\}$, $p \in P$, $p \neq 0$ is a normal base of neighbourhoods (ns.) of $\varnothing$ for a FN-topology $\Gamma(\eta)$.* (We refer to $\Gamma(\eta)$ as the Fréchet-Nikodym topology generated by η on $\mathcal{R}$).

Proof. Easily follows from the definition of the normal base of ns. at $\varnothing$.

Let X be a normed ordered space such that the positive cone P is closed and normal, i.e., there exist $\delta\in\mathbf{R}$, $\delta>0$ such that $\|x\|\leqslant\delta\|y\|$, for all $x, y\in P$ with $x\leqslant y$.

Definition 1.3. *A vector submeasure η is σ-subadditive if for every sequence of mutually disjoint sets $\{E_n\}_{n\in N}\subset\mathcal{R}$ with $\bigcup\limits_{n=1}^{\infty}E_n\in\mathcal{R}$, such that the sequence $\{\sum\limits_{k=1}^{n}\eta(E_k)\}_{n\in N}$ is convergent in P, one has: $\eta(\bigcup\limits_{n=1}^{\infty}E_n)\leqslant\sum\limits_{n=1}^{\infty}\eta(E_n)$.*

To prove the following theorem we shall use L. Drewnowski's procedure from [4]; taking into account the new situation, the reasoning from [4] implies important changes.

Theorem 1.2. *Consider X a normed space which is a complete vector lattice with the normal cone P, $\mathcal{R}$ a ring of subsets of a fixed set R and $\eta:\mathcal{R}\to\to P$ an exhaustive (exh.) and order-continuous (o.c.) vector submeasure on $\mathcal{R}$. Then there is a σ-ring $\mathcal{R}_0$ of subsets of R and an o.c. vector submeasure $\tilde{\eta}:\mathcal{R}_0\to P$ so that the following conditions hold: a) $\mathcal{R}\subset\mathcal{R}_0$ and $\mathcal{R}_0=\overline{\mathcal{R}}^{\Gamma(\tilde{\eta})}$, where $\overline{\mathcal{R}}^{\Gamma(\tilde{\eta})}$ denotes the closure of $\mathcal{R}$ in topology $\Gamma(\tilde{\eta})$, b) $\tilde{\eta}/\mathcal{R}=\eta$, c) The σ-ring $\mathcal{R}_0$ is null-complete with respect to $\tilde{\eta}$ d) $\tilde{\eta}$ is the unique o.c. vector submeasure on $\mathcal{R}_0$ which verifies the condition b), e) The σ-ring $\mathcal{R}_0$ is the null-completion of the σ-ring $\mathcal{S}$ generated by $\mathcal{R}$.*

Proof. Let us consider the class

$$\mathcal{R}_\sigma=\left\{A/A=\bigcup_{n=1}^{\infty}A_n,\ A_n\in\mathcal{R},\ n=1,2,\ldots\right\}. \tag{1}$$

$\mathcal{R}_\sigma$ has the following properties: 1) $\mathcal{R}_\sigma$ is closed with respect to countable unions and finite intersections. 2) $E\setminus F\in\mathcal{R}_\sigma$ for every $E\in\mathcal{R}_\sigma$ and $F\in\mathcal{R}$. 3) It is easy to see that every element from $\mathcal{R}_\sigma$ is generated by an increasing sequence of sets of $\mathcal{R}$.

We define the set function $\hat{\eta}:\mathcal{R}_\sigma\to P$ by

$$\hat{\eta}(E)=\sup\{\eta(F)/F\subseteq E,\ F\in\mathcal{R}\}\ \text{for}\ E\in\mathcal{R}_\sigma. \tag{2}$$

If $\{F_n\}_{n\in N}$ is a sequence of sets of $\mathcal{R}$ such that $F_n\searrow E$, we have:

$$\hat{\eta}(E)=\lim_{n\to\infty}\eta(F_n). \tag{3}$$

This relation (3) is a consequence of the o-continuity of the vector submeasure η. One can show that the set function $\hat{\eta}$ has the following properties: 4) $\hat{\eta}/\mathcal{R}=\eta$; 5) $\hat{\eta}$ satisfies the conditions (1v), (2v), (3v); 6) $\hat{\eta}$ is σ-subadditive in the sense of Definition 1.3; 7) If $E_n\nearrow E$ then $\eta(E\setminus E_n)\searrow 0$ for $n\to\infty$; 8) $\hat{\eta}$ is exh. on $\mathcal{R}_\sigma$. Indeed, let $\{F_n\}_{n\in N}\subset\mathcal{R}_\sigma$ be a sequence of disjoint sets. For each $n\in\mathbf{N}$, we have $F_n=\bigcup\limits_{k=1}^{\infty}E_n^k$, $E_n^k\in\mathcal{R}$ $E_n^k\subseteq E_n^{k+1}$, $\forall n\in N$. By 7) $\hat{\eta}(F_n\setminus E_n^k)\searrow 0$ for $k\to\infty$ and besides $\hat{\eta}(F_n)\leqslant\hat{\eta}(F_n\setminus E_n^k)+\eta(E_n^k)$ also

holds. But η is exh. and P is normal; thus for each $\varepsilon>0$ there exists $n_\varepsilon\in\mathbf{N}$ such that for every $n\geqslant n_\varepsilon$, $\|\hat{\eta}(F_n)\|\leqslant\delta\|\hat{\eta}(F_n\setminus E_n^k)\|+\varepsilon$, $\delta\in\mathbf{R}$, $\delta>0$.

We obtain (when $k\to\infty$) $\|\hat{\eta}(F_n)\|\leqslant\varepsilon$, for every $n\geqslant n_\varepsilon$, that is $\hat{\eta}(F_n)\to 0$ for $n\to\infty$. 9) $\hat{\eta}$ is o.c. on $\mathcal{R}_\sigma$.

Now we define the class $\mathcal{R}_0$ and the set function $\tilde{\eta}$ and we prove that $\mathcal{R}_0$ and $\tilde{\eta}$ satisfy the conditions of the theorem. Let us take

$$\tilde{\mathcal{R}}=\{E/E\subseteq R\ \text{and}\ E\subseteq F\ \text{for some}\ F\in\mathcal{R}_\sigma\}\ \text{and} \tag{4}$$

$$\tilde{\eta}(E)=\inf\{\hat{\eta}(F)/E\subseteq F\in\mathcal{R}_\sigma\}\ \text{for each}\ E\in\tilde{\mathcal{R}}. \tag{5}$$

Clearly, we have: 10) $\tilde{\mathcal{R}}$ is a σ-ring of sets; 11) $\tilde{\eta}/\mathcal{R}_\sigma=\hat{\eta}$; 12) $\tilde{\eta}$ is a vector submeasure on $\tilde{\mathcal{R}}$. Next we denote:

$$\mathcal{R}_0=\overline{\mathcal{R}}_\sigma^{\Gamma(\tilde{\eta})},\ \text{where}\ \overline{\mathcal{R}}_\sigma^{\Gamma(\tilde{\eta})}\ \text{is the closure of}\ \mathcal{R}_\sigma\ \text{in}\ \Gamma(\tilde{\eta})\ \text{topology}. \tag{6}$$

Since $\mathcal{R}_\sigma\subset\tilde{\mathcal{R}}$, we get $\mathcal{R}_0\subseteq\tilde{\mathcal{R}}$ and then $\tilde{\eta}$ is so much the more a vector submeasure on $\mathcal{R}_0$.

Now let us verify the conditions a)—e). It is obvious (for a)) that $\overline{\mathcal{R}}^{\Gamma(\tilde{\eta})}\subset\mathcal{R}_0$. To prove the converse inclusion consider $E\in\mathcal{R}_0$; then there is $\{E_n\}_{n\in N}\subset\mathcal{R}_\sigma$ such that $\tilde{\eta}(E\triangle E_n)\to 0$ for $n\to\infty$; moreover then for each $n\in\mathbf{N}$ there is a sequence $\{E_n^k\}_{k\in N}\subset\mathcal{R}$, $E_n^k\nearrow E_n$ for $k\to\infty$ and therefore, by 7) $\hat{\eta}(E_n\setminus E_n^k)\searrow 0$ for $k\to\infty$. Obviously there exists $\delta>0$ with $\|\tilde{\eta}(E\Delta E_n^k)\|\leqslant\leqslant\delta\|\hat{\eta}(E\Delta E_n)\|+\delta\|\hat{\eta}(E_n\Delta E_n^k)\|$. But, for each $\varepsilon>0$ (for e.g. $\varepsilon=1/k\cdot\delta$) there is $n_\varepsilon\in\mathbf{N}$ such that, $\|\hat{\eta}(E\Delta E_n)\|<1/k\cdot\delta$ for $n\geqslant n_\varepsilon$. Thus we obtain:

$\|\tilde{\eta}(E\Delta E_n^k)\|<1/k+\delta\|\hat{\eta}(E_n\Delta E_n^k)\|$ for each $n\geqslant n_\varepsilon$. Hence, for $k\to\infty$ it follows $\tilde{\eta}(E\Delta E_n^k)\to 0$, i.e. $E\in\overline{\mathcal{R}}^{\Gamma(\tilde{\eta})}$.

Next, we observe that: 13) $\mathcal{R}_0$ is a ring of sets.

In the sequel we shall use the following property:

(7) (If $E\in\mathcal{R}_0$ then there exists $\{H_n\}_{n\in N}\subset\mathcal{R}_\sigma$ such that $E\subseteq H_n$ for all $n\in\mathbf{N}$ and $\tilde{\eta}(H_n\setminus E)\to 0$ as $n\to\infty$. We can even choose a decreasing $\{H_n\}_{n\in N}$).

14) From (7) it follows that: For $E_n\in\mathcal{R}_0$ there are $H_n\in\mathcal{R}_\sigma$ such that $E_n\subseteq H_n$ and $\tilde{\eta}(H_n\setminus E_n)\to 0$ as $n\to\infty$.

15) $\tilde{\eta}$ is o.c. To prove this, let $\{E_n\}_{n\in N}$ be a sequence from $\mathcal{R}_0$, $E_n\searrow\varnothing$ for $n\to\infty$. If $E_n\in\mathcal{R}_0$, $\forall n\in\mathbf{N}$, from 14) we get that there is $H_n\in\mathcal{R}_\sigma$ such that $E_n\subseteq H_n$ and $\tilde{\eta}(H_n\setminus E_n)\to 0$ for $n\to\infty$. Hence, for each $\varepsilon>0$ there exists $n_\varepsilon\in\mathbf{N}$ such that for every $n\geqslant n_\varepsilon$ we can find $H_n\in\mathcal{R}_\sigma$ for which

(8) $\|\tilde{\eta}(H_n\setminus E_n)\|<\varepsilon/(2^n\cdot\delta)$, with δ defined below. Let $H=\bigcap\limits_{n=1}^{\infty}H_n$. Since $E_n\searrow\varnothing$ we have succesively

$$H\subseteq\bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty}(H_k\setminus E_k)\subseteq\left(\bigcup_{k=1}^{p}(H_k\setminus E_k)\right)\cup\bigcup_{k=p+1}^{\infty}H_k$$

Therefore, we infer the existence of $\delta > 0$ with

$$\|\tilde{\eta}(H)\| \leq \delta \sum_{n=1}^{\infty} \|\tilde{\eta}(H_n \setminus E_n)\| + \delta \left\| \hat{\gamma}\left(\bigcup_{n=p+1}^{\infty} H_n \right) \right\| < \varepsilon + \delta \left\| \hat{\gamma}\left(\bigcup_{n=p+1}^{\infty} H_n \right) \right\|$$

according to (8). But $\bigcup_{n=p+1}^{\infty} H_n \searrow \emptyset$ when $p \to \infty$ and as $\hat{\gamma}$ is o.c. we obtain that $\hat{\gamma}\left(\bigcup_{n=p+1}^{\infty} H_n \right) \to 0$, for $p \to \infty$, hence $\tilde{\eta}(H) = 0$.

Then, because $\tilde{\eta}(H) = \inf \{\hat{\gamma}(F) / H \subseteq F,\ F \in \mathcal{R}_\sigma\}$, for each $p \in P$, $p \neq 0$ there is $F \in \mathcal{R}_\sigma$ verifying $H \subseteq F$ and $\hat{\gamma}(F) < p$. Consequently $\|\hat{\gamma}(F)\| < \delta \|p\| < \varepsilon/\delta^2$, for $\varepsilon > 0$. Since $F \in \mathcal{R}_\sigma$, there exists $\{F_n\}_{n \in \mathbf{N}} \subset \mathcal{R}$ with $F_n \nearrow F$ for $n \to \infty$. From $F_n \subseteq F$ it results that $\hat{\gamma}(F_n) \leq \hat{\gamma}(F)$ and thus

$$\|\hat{\gamma}(F_n)\| \leq \delta \|\hat{\gamma}(F)\| < \delta\ \varepsilon/\delta^2 = \varepsilon/\delta. \tag{9}$$

Because $H_n \in \mathcal{R}_\sigma$, $\forall n \in \mathbf{N}$ there is a sequence $\{E_n^k\}_{k \in \mathbf{N}} \subset \mathcal{R}$ with $E_n^k \nearrow H_n$ for $k \to \infty$; according to 7) $\hat{\gamma}(H_n \setminus E_n^k) \searrow 0$ for $k \to \infty$, i.e., for each $\varepsilon > 0$ there is $k_\varepsilon \in \mathbf{N}$ such that $\|\hat{\gamma}(H_n \setminus E_n^k)\| < \varepsilon/2^k \delta^2$ for every $k \geq k_\varepsilon$.

Next, let us remark that:

$$\left\|\tilde{\gamma}(H_n \setminus E_n^1)\right\| \leq \delta \sum_{k=1}^{\infty} \left\|\hat{\gamma}(H_n \setminus E_n^k)\right\| < \varepsilon/\delta. \tag{10}$$

Denote $G_n = E_n^1 \setminus F_n$ for each $n \in \mathbf{N}$. Obviously $G_n \searrow \emptyset$ as $n \to \infty$. Since γ is o.c. and $G_n \in \mathcal{R}$ for $n \in \mathbf{N}$, it follows that

$$\gamma(G_n) \to 0 \quad \text{when } n \to \infty. \tag{11}$$

We have $H_n \subseteq (H_n \setminus E_n^1) \cup G_n \cup F_n$; hence

$$\|\tilde{\eta}(H_n)\| \leq \delta \|\tilde{\eta}(H_n \setminus E_n^1)\| + \delta \|\gamma(G_n)\| + \delta \|\hat{\gamma}(F_n)\| < 3\varepsilon.$$

for all $n \geq n_\varepsilon$ (according to (9), (10), (11)) and thus $\lim_{n \to \infty} \tilde{\gamma}(H_n) = 0$.

But $E_n \subseteq H_n$ for all $n \in \mathbf{N}$, so $\tilde{\eta}(E_n) \leq \tilde{\eta}(H_n)$ and we conclude that $\lim_{n \to \infty} \tilde{\eta}(E_n) = 0$ q.e.d.

Finally, note that 16) $\mathcal{R}_0$ is a σ-ring. Now, clearly $\tilde{\gamma}/\mathcal{R} = \gamma$ and b) is valid.

If $F \subseteq E$, $E \in \mathcal{R}_0$ and $\tilde{\eta}(E) = 0$, then we have $F \in \mathcal{R}_0$ and $\tilde{\eta}(F) = 0$ as in c).

For d), consider γ be an o.c. vector submeasure on $\mathcal{R}_0$ such that $\gamma/\mathcal{R} = \gamma$. One can immediately see that $\gamma/\mathcal{R}_\sigma = \hat{\gamma}$. Let now E belong to $\mathcal{R}_0$. By virtue of (7) there exists a decreasing sequence $\{H_n\}_{n \in \mathbf{N}} \subset \mathcal{R}_\sigma$ such that $E \subseteq H_n$ and $\tilde{\eta}(H_n \setminus E) \to 0$ for $n \to \infty$. Therefore, the sequence $\{\hat{\eta}(H_n)\}_{n \in \mathbf{N}}$ is decreasing and $\lim_{n \to \infty} \hat{\gamma}(H_n) = \inf_{n \in \mathbf{N}} \hat{\gamma}(H_n) = \tilde{\eta}(E)$. Moreover, since $\gamma/\mathcal{R}_\sigma = \hat{\gamma}$, we have $\tilde{\gamma}(E) = \lim_{n \to \infty} \gamma(H_n)$.

To prove the uniqueness of γ it is enough to show that $\gamma(E) = \lim_{n \to \infty} \gamma(H_n)$. Indeed, because $E \subseteq H_n$ for each $n \in \mathbf{N}$ $\gamma(E) \leq \lim_{n \to \infty} \gamma(H_n)$. From $\tilde{\gamma}(H_n \setminus E) = \inf \{\hat{\gamma}(H);\ H_n \setminus E \subseteq H,\ H \in \mathcal{R}_\sigma\}$, it follows that for each $p \in P$, $p \neq 0$ there is $G_n \in \mathcal{R}_\sigma$ with $H_n \setminus E \subseteq G_n$ and $\hat{\gamma}(G_n) \leq \tilde{\gamma}(H_n \setminus E) + (1/n)p$; so $\lim_{n \to \infty} \hat{\gamma}(G_n) = 0$

In view of inequalities:

$\gamma(H_n) \leq \tilde{\gamma}(H_n \setminus E) + \gamma(E) \leq \hat{\gamma}(G_n) + \gamma(E)$ one gets $\lim_{n \to \infty} \gamma(H_n) \leq \gamma(E)$.

The null-completion of the σ-ring $\mathcal{S}$, generated by $\mathcal{R}$, noted $\mathcal{S}^*$ and defined by

$$\mathcal{S}^* = \{A \subseteq R / A = B \cup T,\ B \in \mathcal{S},\ T \in \tilde{\mathcal{R}};\ \tilde{\eta}(T) = 0\},$$

coincides with $\mathcal{R}_0$ and therefore e) also holds.

§2. In this section we introduce the notion of group-valued submeasure and next we give an extension procedure for such a kind of set function.

A subset E of an ordered additive group G is called a full set if $x_1, x_2 \in E$ implies $[x_1, x_2] \subseteq E$ $([x_1, x_2] = \{x \in G / x_1 \leq x \leq x_2\})$.

A topology on an additive ordered group G, compatible with the group structure, is called locally full if the zero element has a base of full ns. (see [3]).

In what follows, by an ordered topological group, we shall mean an ordered group endowed with a locally full topology.

Theorem 2.1. *Let $(G, +, \leq, \tau)$ be an ordered commutative topological group with G_+ the closed cone of positive elements. Then, there exists a base of full and closed ns. of the zero element in G_+ endowed with the induced topology.*

Proof. The proof of this theorem will be immediate if one takes into account that G_+ is closed and the closure (with respect to the topology induced on G_+) of a full n. of 0 in G_+ is a full set.

Definition 2.1. *Let $(\mathcal{R}, \Delta, \cap)$ be a ring of subsets of a fixed set R and G an ordered commutative group. A set function $\eta : \mathcal{R} \to G_+$ is called a group-valued submeasure if it satisfies the conditions (1v), (2v), (3v) with their meaning in G.*

Remark 2.1. From now on G will be an ordered commutative topological group and G_+ will be considered closed.

Definition 2.2. *We say that a group-valued submeasure $\eta : \mathcal{R} \to G_+$ satisfies the condition* (C) *if*:

(C) *For every decreasing sequence $\{A_n\}_{n \in \mathbf{N}} \subset \mathcal{R}$ with $\bigcap_{n \in \mathbf{N}} A_n = \emptyset$ one has* $\lim_{n \to \infty} \eta(A_n) = 0$.

Further, $\mathcal{R}_\sigma$ is the class of sets defined by (1), $\tilde{\mathcal{R}}$ the class defined by (4) and

$$\mathcal{R}_\delta = \{Z / Z = \bigcap_{n=1}^{\infty} Z_n,\ Z_n \in \mathcal{R},\ n = 1, 2, \ldots\}. \tag{12}$$

In the following theorem we shall introduce a suitable topology on the class $\tilde{\mathcal{R}}$ structured as an commutative additive group with respect to the symmetric difference of sets as addition; for $\mathcal{U}, \mathcal{V} = \tilde{\mathcal{R}}$

$$\mathcal{U}\overset{\circ}{\Delta}\mathcal{V}=\{A\Delta B\,/\,A\in\mathcal{U} \text{ and } B\in\mathcal{V}\}.$$

Let $\eta:\mathcal{R}\to G_+$ be a group-valued submeasure which satisfies the condition (C) and Ω a base of full and closed ns. for $0\in G$, considered in G_+ with the induced topology (see Remark 2.1 and Theorem 2.1).

Definition 2.3. *For every $U\in\Omega$ the class $\mathcal{U}^*\subset\tilde{\mathcal{R}}$ is defined as follows:* (13) *$(T\in\mathcal{U}^*$ if and only if there is an $E\in\mathcal{R}_\sigma$ such that $T\subseteq E$ and for $A\in\mathcal{R}$, $A\subseteq E$ one has $\eta(A)\in U)$.*

We shall use the notation:

$$\Omega^*=\{\mathcal{U}^*\,|\,U\in\Omega\}. \tag{14}$$

Theorem 2.2. *Ω^* is a base of ns. of $\varnothing$ for a group topology on $\tilde{\mathcal{R}}$.*

Proof. According to [2] we have only to show that:

1° $\forall\mathcal{U}^*\in\Omega^*$, $\varnothing\in U^*$;

2° $\forall\mathcal{U}^*,\mathcal{V}^*\in\Omega^*$ there exists $\mathcal{W}^*\in\Omega^*$ such that $\mathcal{W}^*\subset\mathcal{U}^*\overset{\circ}{\cap}\mathcal{V}^*$;

3° $\forall\mathcal{U}^*\in\Omega^*$ there exists $\mathcal{V}^*\in\Omega^*$ such that $\mathcal{V}^*\Delta\mathcal{V}^*\subset\mathcal{U}^*$.

1° and 2° are immediate. We shall prove 3°.

Let $\mathcal{V}^*$ be such that for the corresponding $V\in\Omega$ we have $V+V\subseteq U$. Consider now $T,Z\in\mathcal{V}^*$; then there exist $E,Q\in\mathcal{R}_\sigma$ with $T\subseteq E$, $Z\subseteq Q$ and $\eta(A)\in V$ if either $A\subseteq E$ or $A\subseteq Q$, $A\in\mathcal{R}$.

We have $T\Delta Z\subseteq E\cup Q$ and $E\cup Q\in\mathcal{R}_\sigma$. Take $A\in\mathcal{R}$, $A\subseteq E\cup Q$. Since $E,\,Q\in\mathcal{R}_\sigma$ it follows that $E=\bigcup\limits_{n=1}^{\infty}A_n$, $Q=\bigcup\limits_{n=1}^{\infty}B_n$, $(\{A_n\}_{n\in N}$, $\{B_n\}_{n\in N}$ are increasing sequences from $\mathcal{R})$. Thus, we obtain $\eta(A)=\eta\left(A\cap\left(\bigcup\limits_{n=1}^{\infty}(A_n\cup B_n)\right)\right)=$

$$=\eta\left(\bigcup_{n=1}^{\infty}(A\cap(A_n\cup B_n))\right).$$

We note that for a group-valued submeasure η which satisfies the condition (C), the continuity on decreasing (increasing) sequences holds, i.e., for any decreasing (increasing) sequence $\{D_n\}_{n\in N}\subset\mathcal{R}$ with $D=\bigcap\limits_{n=1}^{\infty}D_n\in$ $\in\mathcal{R}\left(D=\bigcup\limits_{n=1}^{\infty}D_n\in\mathcal{R}\right)$, one has $\lim\limits_{n\to\infty}\eta(D_n)=\eta(D)$.

Because $A\cap A_n\subseteq E$, $A\cap B_n\subseteq Q$ and V is a closed n. of 0,

$$\eta(A)\leqslant\lim_{n\to\infty}\eta(A\cap A_n)+\lim_{n\to\infty}\eta(A\cap B_n)\in V+V.$$

Finally, $V+V\subseteq U$ implies that $\eta(A)\in U$ (We recall that U is a full set).

Remark 2.2. The group topology on $\tilde{\mathcal{R}}$ determined by Ω^* does not depend on the choice of the basis Ω (ns. of 0 in G_+ with the induced topology). This topology will be called the η-topology on $\tilde{\mathcal{R}}$ (shortly η-top $\tilde{\mathcal{R}}$).

Proposition 2.1. *The operations of union, difference and intersection for the sets from $\tilde{\mathcal{R}}$ are uniformly continuous with respect to η-top $\tilde{\mathcal{R}}$.*

Remark 2.3. The trace of η-top $\mathcal{R}$ on $\tilde{\mathcal{R}}$ determines on $\mathcal{R}$ a structure of topological ring of sets (TRS).

Theorem 2.3. *Suppose that $\mathcal{R}$ is a ring of subsets of a fixed set R and G an ordered, commutative, complete and separable T_0 topological group with the closed positive cone G_+. Consider η a group-valued submeasure defined on $\mathcal{R}$ with values in G_+ satisfying condition* (C), *exh. and uniformly continuous in η-top $\tilde{\mathcal{R}}$. Then there is only one uniformly continuous group-valued submeasure $\bar{\eta}$, extending η to the σ-ring $\bar{\mathcal{R}}$* (closure of $\mathcal{R}$ in η-top $\tilde{\mathcal{R}}$).

Proof. By Proposition 2.1 it follows immediately that $\bar{\mathcal{R}}$ is a ring of sets. To show that $\bar{\mathcal{R}}$ is a σ-ring, it is sufficient to prove that for any increasing sequence $\{M_n\}_{n\in N}\subset\bar{\mathcal{R}}$, $\eta\text{-}\lim\limits_{n\to\infty}M_n=\bigcup\limits_{n=1}^{\infty}M_n$ (η-lim denotes the limit in η-top $\tilde{\mathcal{R}}$). This can be obtained as in the similar situation from [5].

Since the uniform space G_+ is complete and T_0 with respect to the induced topology and η is uniformly continuous, the set function η can be uniquely extended to an uniformly continuous function, $\bar{\eta}$ on $\bar{\mathcal{R}}$ according to [2], chapter 2. It is not difficult to verify that $\bar{\eta}$ is a group-valued submeasure on $\bar{\mathcal{R}}$.

§3. In the sequel, remarking that any vector submeasure is at the same time a group-valued submeasure, we shall draw a parallel between the two extension methods described in Theorems 1.2 and 2.3.

The main result of this paragraph is Theorem 3.1.

Let X be a normed space which is a complete vector lattice with the positive cone P considered to be normal. This last assumption and Proposition 1, §1.4, Chapter VI from [3] assure that the norm topology is a locally full topology.

$\tilde{\mathcal{R}}$ is a ring of subsets of R with respect to symmetric difference and intersection of sets.

As these operations are uniformly continuous in η-top $\tilde{\mathcal{R}}$ (Proposition 2.1), η-top $\tilde{\mathcal{R}}$ is a r-topology on $\tilde{\mathcal{R}}$ (see [4], Part. I, §1).

In fact, according to 10) from the proof of Theorem 1.2, $\tilde{\mathcal{R}}$ is even a σ-ring. Let $\eta:\mathcal{R}\to P$ be an exh. vector submeasure and assume the hypothesis (C) (Obviously η will be o.c.). By means of the extension $\tilde{\eta}$ of η obtained in Theorem 1.2 we can define on $\tilde{\mathcal{R}}$ the family of classes

$$\tilde{\Omega}=\{\tilde{\mathcal{U}}_p\}_{\substack{p\in P\\ p\neq 0}} \tag{15}$$

where $\tilde{\mathcal{U}}_p=\{A\in\tilde{\mathcal{R}}\,/\,\tilde{\eta}(A)\leqslant p\}$ for every $p\in P$, $p\neq 0$. This family (Theorem 1.1) constitutes a normal base of ns. of $\varnothing$ for a FN-topology denoted $\Gamma(\tilde{\eta})$.

Theorem 3.1. *The families of classes Ω^* (see* (14)) *and $\tilde{\Omega}$ (see* (15)) *are normal bases of ns. of $\varnothing$ for the same FN-topology on $\tilde{\mathcal{R}}$.*

Proof. Each $\mathcal{U}^*\in\Omega^*$ is a normal subclass, that is: if $B\in\tilde{\mathcal{R}}$ and there exists $A\in\mathcal{U}^*$ with $B\subseteq A$, then $B\in\mathcal{U}^*$. Indeed, because $A\in\mathcal{U}^*$, there is (Definition 2.3) $E\in\mathcal{R}_\sigma$ such that $A\subseteq E$ and for every $M\in\mathcal{R}$, $M\subseteq E$, $\eta(M\in)U$. From $B\subseteq A$, it follows that $B\subseteq E$ and consequently $B\in\mathcal{U}^*$.

η-top $\tilde{\mathcal{R}}$ is a FN-topology. To prove this, let $\mathcal{U}^*\in\Omega^*$. $\mathcal{U}^*$ corresponds to a n. of Φ, $U\in\Omega$. For this U there exists $V\in\Omega$ such that $V\subseteq U$. Consider

$\mathcal{V}^* \in \Omega^*$, the n. of $\emptyset$ corresponding to V from above and take $B \in \tilde{\mathcal{R}}$, $B \subseteq \subseteq A \in \mathcal{V}^*$. Since every element of Ω^* is a normal subclass and $\mathcal{V}^* \subseteq \mathcal{U}^*$ we get $B \in \mathcal{U}^*$.

Further we shaw that $\tau_1 = \eta$-top $\tilde{\mathcal{R}}$ coincides with $\tau_2 = \Gamma(\tilde{\eta})$.

$\tau_1 < \tau_2$: Let $\mathcal{U}^* \in \Omega^*$ and $V \in \Omega$ with $V + V \subseteq U$, where $U \in \Omega$ is just that n. with the help of which $\mathcal{U}^*$ has been constructed. When there is $p \in \in P$, $p \neq 0$, $p \in V$, if we take $\tilde{\mathcal{U}}_p = \{A / \tilde{\eta}(A) \leqslant p\}$ then $\tilde{\mathcal{U}}_p \subset \mathcal{U}^*$. Indeed, let $A \in \tilde{\mathcal{U}}_p$; for every $q \in P$, $q \neq 0$ there is $F \in \mathcal{R}_\sigma$ with $A \subseteq F$ such that $\hat{\eta}(F) < < \tilde{\eta}(A) + q$ (due to the definition of $\tilde{\eta}$, (5)). But $\eta(H) \leqslant \hat{\eta}(F)$, $\forall H \in \mathcal{R}$, $H \subseteq F$ (according to the definition of $\hat{\eta}$, (2)) hence $\eta(H) < \tilde{\eta}(A) + q$, inequality which is valid also for p choosen above, $p \in V$. But $\tilde{\eta}(A) \leqslant p$, so $\eta(H) < p + + p \in V + V \subseteq U$. U being full it follows $\eta(H) \in U$. Then $A \in \mathcal{U}^*$ because there exists $F \in \mathcal{R}_\sigma$ such that $A \subseteq F$ and for any $H \in \mathcal{R}$, $H \subseteq F$, $\eta(H)$ belongs to U.

If $V = \{0\}$, one has $\{A / \tilde{\eta}(A) = 0\} \subset \tilde{\mathcal{U}}_p$ for every $p \in P$, $p \neq 0$. Let $A \in \in \tilde{\mathcal{U}}_p$, for $p \in P$, $p \neq 0$ arbitrary and fixed. There is $F \in \mathcal{R}_\sigma$ with $A \subseteq F$ such that $\hat{\eta}(F) < \tilde{\eta}(A) + p$. But $\eta(H) \leqslant \hat{\eta}(F)$ for any $H \in \mathcal{R}$, $H \subseteq F$ so that $\eta(H) \leqslant \leqslant p + p$, for all $p \in P$, $p \neq 0$, hence $\eta(H) = 0 \in U$. Thus, obviously $A \in \mathcal{U}^*$.

$\tau_2 < \tau_1$: Let $\tilde{\mathcal{U}}_p \in \tilde{\Omega}$, $U_1 = \{u / 0 \leqslant u \leqslant p\} = P \cap (p - P)$ is a n. of 0 in P endowed with the induced topology and $p \in U_1$. Since Ω is a base of ns. of 0 in P for the induced topology, there is $U \in \Omega$, $U \subseteq U_1$.

We have $\mathcal{U}^* \subset \mathcal{A}$ ($\mathcal{U}^*$ corresponding to $U \in \Omega$) where $\mathcal{A} = \{T \subseteq R /$ there exists $E \in \mathcal{R}_\sigma$ such that $T \subseteq E$ and for every $A \in \mathcal{R}$, $A \subseteq E$, $\eta(A) \in U_1\}$. We shall prove that the family $\mathcal{A}$ coincides with the family of sets: $\mathcal{B} = = \{T \subseteq R /$ there exists $B \in \mathcal{R}_\sigma$, $B \in \tilde{\mathcal{U}}_p$, $T \subseteq B\}$ $\mathcal{B}$ is nonvoid because $\emptyset \in \mathcal{B}$. Clearly $\mathcal{B} \subseteq \mathcal{A}$. Consider now $T \in \mathcal{A}$. This means that there is $E \in \mathcal{R}_\sigma$ such that $T \subseteq E$ and for every $A \in \mathcal{R}$, $A \subseteq E$, $\eta(A) \in U_1$. Since $E \in \mathcal{R}_\sigma$, there is an increasing sequence $\{E_n\}_{n \in \mathbf{N}} \subset \mathcal{R}$ with $E = \bigcup_{n=1}^{\infty} E_n$. We have $E \Delta E_n \subseteq \bigcup_{k=n+1}^{\infty} E_k$; $E \Delta E_n \in \mathcal{R}_\sigma$ (according to the point 2) from the proof of Theorem 1.2) and the sequence $\left\{\bigcup_{k=n+1}^{\infty} E_k\right\}_{n \in \mathbf{N}}$ is from $\mathcal{R}_\sigma$, decreasing to $\emptyset$, hence $E_n \xrightarrow{o} E$ and then $\tilde{\eta}(E_n) \to \eta(E)$ for $n \to \infty$.

But $E_n \subseteq E$, $E_n \in \mathcal{R}$ for each $n \in \mathbf{N}$; therefore $\tilde{\eta}(E_n) = \eta(E_n) \in U_1$, $\forall n \in \mathbf{N}$, i.e., $0 \leqslant \tilde{\eta}(E_n) \leqslant p$. Consequently $\tilde{\eta}(E) \leqslant p$, that is $E \in \tilde{\mathcal{U}}_p$. Moreover $E \in \mathcal{R}_\sigma$ and $T \subseteq E$, thus $T \in \mathcal{B}$. We conclude that $\mathcal{A} = \mathcal{B}$. Hence, we have $\mathcal{U}^* \subset \mathcal{A} = \mathcal{B} \subset \tilde{\mathcal{U}}_p$ and thus for any $\tilde{\mathcal{U}}_p \in \tilde{\Omega}$ there is an $\mathcal{U}^* \in \Omega^*$ such that $\mathcal{U}^* \subset \tilde{\mathcal{U}}_p$. $\tau_1 = \tau_2$ implies that the traces of these topologies on $\mathcal{R}$ coincide and thus

$$\overline{\mathcal{R}}^{\eta\text{-top}\,\tilde{\mathcal{R}}} = \overline{\mathcal{R}}^{\Gamma(\tilde{\eta})}$$

Theorem 3.2. *Suppose X is a complete vector lattice which is at the same time a complete normed space, with the normal cone P. Let be $\mathcal{R}$ a ring of subsets of a fixed set R, and $\eta: \mathcal{R} \to P$ an exh., uniformly continuous vector submeasure satisfying condition* (C). *Then:*

$$\mathcal{R}_0 = \overline{\mathcal{R}}^{\Gamma(\tilde{\eta})} = \overline{\mathcal{R}}^{\eta\text{-top}\,\tilde{\mathcal{R}}} \text{ and } \tilde{\eta} = \overline{\eta}^{\eta\text{-top}\,\tilde{\mathcal{R}}}.$$

We shall denote this vector submeasure by η_0. In addition η_0 satisfies the conditions a)—e) from Theorem 1.2.

($\overline{\mathcal{R}}^{\Gamma(\tilde{\eta})}$ and $\tilde{\eta}$ denotes the σ-ring, respectively the vector submeasure from Theorem 1.2, and $\overline{\mathcal{R}}^{\eta\text{-top}\,\tilde{\mathcal{R}}}$ and $\overline{\eta}^{\eta\text{-top}\,\tilde{\mathcal{R}}}$ the σ-ring and the vector submeasure from Theorem 2.3 respectively).

Proof. The hypotheses of the Theorem 1.2 are verified, since η satisfies the condition (C), hence it is o.c. The assumptions of the Theorem 2.3. also hold.

The cone P being normal there exists an X a locally full topology. Taking into account that X is a vector lattice which is at the same time a complete normed space we conclude that X is a complete and T_0 ordered commutative topological group.

By Theorem 3.1, $\overline{\mathcal{R}}^{\Gamma(\tilde{\eta})} = \overline{\mathcal{R}}^{\eta\text{-top}\,\tilde{\mathcal{R}}} = \mathcal{R}_0$ Since $\overline{\eta}^{\eta\text{-top}\,\tilde{\mathcal{R}}}$ is a uniformly continuous function with respect to the trace on $\mathcal{R}_0$ of the η-top $\tilde{\mathcal{R}}$, $\overline{\eta}^{\eta\text{-top}\,\tilde{\mathcal{R}}}$ is also uniformly continuous with respect to the topology induced by $\Gamma(\tilde{\eta})$ on $\mathcal{R}_0$. Uniform continuity of $\overline{\eta}^{\eta\text{-top}\,\tilde{\mathcal{R}}}$ implies the o-continuity of $\overline{\eta}^{\eta\text{-top}\,\tilde{\mathcal{R}}}$ on $\mathcal{R}_0$. Thus $\overline{\eta}^{\eta\text{-top}\,\tilde{\mathcal{R}}}$ is an o.c. vector submeasure on $\mathcal{R}_0$ and according to d) from Theorem 1.2, one must have $\overline{\eta}^{\eta\text{-top}\,\tilde{\mathcal{R}}} = \tilde{\eta} = \eta_0$. Clearly, η_0 satisfies the conditions a)—e) of the Theorem 1.2.

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