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# METRIC BASES

BY

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1. Dimension theory of topological spaces (we refer to the Menger-Urysohn, Brouwer-Čech and Čech-Lebesgue dimensions — see [1]) is an inconvenient tool for an investigation of the geometric properties of metric spaces (in the Menger's statement — see [2, Part 3]). For example, ind $(l^p(2))=\text{Ind}\ (l^p(2))=\dim\ (l^p(2))=2$ for every $p\geq 1$, but, in the case $p=1$, there exists an infinity of metric lines which contain the points (0, 0) and (1,1).

The purpose of this paper is to introduce a new dimension for metric spaces; this dimension is an isometric invariant, but, generally, it is not a homeomorphic invariant. The method used was inspired by the technique of introduction of the dimension for vector spaces. So, in 2 we define the metric bases and we prove (see the theorems 1 and 2) that, in certain conditions, there exist metric bases and every two metric bases are equivalent. In 3 we establish some sufficient conditions in order that the metric topology be generated by a metric base. We give also, some properties of the finite dimensional metric spaces and several examples.

2. Let $(M, \rho)$ be a metric space; for a finite set $A\subseteq M$ and $x\in M$, we note $A(x)=\{y:\rho(t,x)=\rho(t,y)$ for every $t\in A\}$. The set $H(A)=\{x: A(x)=\{x\}\}$ is called *the metric hull* of $A$. For an infinite set $A$, *the metric hull* of $A$ is:

$$H(A)=\{x:\ \text{there exists } B\subseteq A \text{ with } B \text{ finite and } x\in H(B)\}.$$

Assume that $H(\emptyset)=\emptyset$. Hence $H$ is an operator assigning to every set $A\subseteq M$ a set $H(A)\subseteq M$; $x\in H(A)$ iff there exists $B\subseteq A$, $B$ finite, such that for every $y\neq x$ there exists $t\in B$ with $\rho(t, x)\neq\rho(t, y)$.

We omit the proof of the following easily established,

**Lemma 1.** (i) *$A\subseteq H(A)$ for every $A\subseteq M$; $H(M)=M$,*
(ii) *if $A\subseteq B$ then $H(A)\subseteq H(B)$,*
(iii) *$H(A)=H(\bar{A})$ for every $A\subseteq M$, where $\bar{A}$ is the closure of $A$,*
(iv) *if $A$ is dense in $M$ then $H(A)=M$,*
(v) *$\overline{H(A)}\subseteq H(H(A))$.*

*Example.* Let $l^p(2)$ be the plane $R^2$ with the norm $\|x\|_p=(|x_1|^p+|x_2|^p)^{1/p}$ for every $x=(x_1, x_2)\in R^2$, where $p\geq 1$ and let $x_0=(0,0)$, $y_0=(1,0)$ and $z_0=(0,1)$ be three points in $R^2$. In $l^1(2)$ we have $H(\{x_0, y_0\})=[x_0, y_0]=\{(\alpha, 0):0\leq\alpha\leq 1\}$, $H(\{x_0, z_0\})=[x_0, z_0]=\{(0, \alpha):0\leq\alpha\leq 1\}$.

$H(\{y_0, z_0\})=\{y_0, z_0\}$, $H(\{x_0, y_0, z_0\})=\{(\alpha, \beta) : 0\le\alpha\le 1 \text{ or } 0\le\beta\le 1\}$ and $H(H(\{x_0, y_0, z_0\}))=R^2\ne H(\{x_0, y_0, z_0\})=\overline{H(\{x_0, y_0, z_0\})}$. In $l^2$ (2) we have $H(\{x_0, y_0\})=\{(\alpha, 0) : \alpha\in R\}$, $H(\{x_0, z_0\})=\{(0, \alpha) : \alpha\in R\}$ and $H(\{x_0, y_0, z_0\})=R^2$; for every $u, v\in R^2$, $H(\{u, v\})$ is a straight line $G(u, v)$ such that $u, v\in G(u, v)$.

A finite set $A\subseteq M$ is called a *metric independent set* if for every $t\in A$ $t\notin H(A-\{t\})$. It is obvious that $A=\{t\}$ is a metric independent set, for every $t\in M$. An infinite set $A$ is *metric independent* if for every finite set $B\subseteq A$, $B$ is a metric independent set.

A metric independent set $A$ with $H(A)=M$ is called a *metric base*.

**Lemma 2.** *Let $(M, \rho)$ be a metric space satisfying the condition:*

$(\ast)$ *For every $A\subseteq M$, $x, y\in M$, $x\in H(A\cup\{y\})-H(A)$ if and only if $y\in H(A\cup\{x\})-H(A)$.*

*Then:*

(1) *For every $A\subseteq M$, $x, y\notin H(A)$ we have $x\in H(A\cup\{y\})$ if and only if $y\in H(A\cup\{x\})$.*

(2) *$x\in H(A)$ if and only if $H(A)=H(A\cup\{x\})$.*

*Let $A$ be a metric independent finite set; then:*

(3) *For every $\{t_k : k=1, 2, \dots, n\}\subseteq A$ we have* $\bigcap_{k=1}^{n} H(A-\{t_k\})= H(A-\{t_1, t_2, \dots, t_n\})$.

(4) *For every $x\in M$ there exists $t\in A$ such that $x\notin H(A-\{t\})$.*

(5) *$x\notin H(A)$ if and only if $A\cup\{x\}$ is a metric independent set.*

*Proof.* (1), (2) and (5) follow directly from $(\ast)$. Let $A$ be a metric independent finite set. (3). Let $t, s\in A$ and $x\in(H(A-\{t\})\cup H(A-\{s\}))-H(A-\{t, s\})$; because $A$ is metric independent $s\notin H(A-\{s\})\supseteq H(A-\{t, s\})$ and $t\notin H(A-\{t\})\supseteq H(A-\{t, s\})$. But $x\in H(A-\{t\})=H((A-\{t, s\})\cup\{s\})$, hence, from (1), $s\in H((A-\{t, s\})\cup\{x\})$ and similarly, $t\in H((A-\{t, s\})\cup\{x\})$. Now, from (2), $H((A-\{t, s\})\cup\{x\})=H(((A-\{t, s\})\cup\{x\})\cup\{s\})=H((A-\{t\})\cup\{x\})=H(A-\{t\})$, hence $t\in H(A-\{t\})$: but this is a contradiction. Let $\{t_1, \dots, t_n\}\subseteq A$; $H(A-\{t_1\})\cap H(A-\{t_2\})\cap H(A-\{t_3\})=(H(A-\{t_1\})\cap H(A-\{t_2\}))\cap(H(A-\{t_2\})\cap H(A-\{t_3\}))=H(A-\{t_1, t_2\})\cap H(A-\{t_2, t_3\})=H((A-\{t_2\})-\{t_1\})\cap H((A-\{t_2\})-\{t_3\})=H((A-\{t_2\})-\{t_1, t_3\})=H(A-\{t_1, t_2, t_3\})$. Now, (3) follows by induction. (4) follows from (3).

*Remarks.* (1) If $A=\emptyset$ then from $(\ast)$ we have $x\in H(\{y\})$ iff $y\in H(\{x\})$. If $|C(x, r)|\ne 1$ for every $x\in M$, $r>0$ then $H(\{x\})=\{x\}$, where $C(x, r)=\{y : \rho(x, y)=r\}$.

(2) If $A=\{x\}$ and $y, z\notin H(\{x\})$, then from lemma 2 (1) we have $y\in H(\{x, z\})$ iff $z\in H(\{x, y\})$; in $l^2(2)$ this condition is equivalent with: $y$ belongs to the straight line $G(x, z)$ iff $z$ belongs to the straight line $G(x, y)$. Hence, the conditions $(\ast)$, (1)—(5) from lemma 2 have a geometric character.

**Theorem 1.** *Let $(M, \rho)$ be a metric space satisfying the condition (1) from lemma 2. Then, for every metric independent set $A$ there exists a metric base $A_0$ such that $A\subseteq A_0$*



*Proof.* Let $\mathcal{A}=\{A : A$ is a metric independent set in $M\}$; $\mathcal{A}$ is a non-empty family ordered by the relation $\subseteq$. For each linearly ordered subset $\mathcal{L}\subseteq\mathcal{A}$ there exists $A_0=\cup\mathcal{L}$ such that $A\subseteq A_0$ for every $A\in\mathcal{L}$. From the Zorn lemma, for every $A\in\mathcal{A}$ there exists a maximal element $A_0\in\mathcal{A}$ such that $A\subseteq A_0$. Let $A_0$ be a maximal element in $\mathcal{A}$ and suppose that there exists $x\in M-H(A_0)$; let $A_1=A_0\cup\{x\}$. We shall show that $A_1$ is a metric independent set. Let $B$ be a finite subset of $A_1$; if $B\subseteq A_0$ then $B$ is a metric independent set. We suppose now that $x\in B$; $x\notin H(A_0)\supseteq H(B-\{x\})$ hence $x\notin H(B-\{x\})$. Now, let $t\in B-\{x\}$; because $B-\{x\}$ is a metric independent set we have that $t\notin H(B-\{t, x\})$ and it is obvious that $x\notin H(B-\{t, x\})$. It follows from $x\notin H((B-\{t, x\})\cup\{t\})$ and from the condition (1) (lemma 2) that $t\notin H((B-\{t, x\})\cup\{x\})=H(B-\{t\})$. Hence $B$ is a metric independent set; therefore $A_1$ is a metric independent set and $A_0\subset A_1$. But $A_0$ is a maximal element; this is a contradiction. Hence $H(A_0)=M$: so, $A_0$ is a metric base.

**Theorem 2.** *Let $(M, \rho)$ be a metric space satisfying the condition $(\ast)$ from lemma 2. Then, every two metric bases are cardinal equivalent.*

*Proof.* I. Let $A_0$, $B_0$ be two metric bases with $|A_0|=n<|B_0|$, where $|A|$ is the cardinal number of $A$. From lemma 2 (4), for every $x_1\in B_0$ there exists $t_1\in A_0$ such that $x_1\notin H(A_0-\{t_1\})$. Let $A_1=(A_0-\{t_1\})\cup\{x_1\}$; from lemma 2 (5), $A_1$ is a metric independent set. But, $x_1, t_1\notin H(A_0-\{t_1\})$ and $x_1\in M=H(A_0)=H((A_0-\{t_1\})\cup\{t_1\})$, hence, from lemma 2 (1), $t_1\in H(A_1)$. From lemma 2 (2), $H(A_1)=H(A_1\cup\{t_1\})=H(A_0\cup\{x_1\})=H(A_0)=M$; hence $A_1$ is a metric base. Let now $x_2\in B_0-\{x_1\}$; because $B_0$ is a metric independent set, $x_2\notin H(\{x_1\})=H(\cap\{A_1-\{t\} : t\ne x_1\})=$(from lemma 2 (3))$=\cap\{H(A_1-\{t\}) : t\in A-\{t_1\}\}$. Hence there exists $t_2\in A-\{t_1\}$ such that $x_2\notin H(A_1-\{t_2\})$; let $A_2=(A_1-\{t_2\})\cup\{x_2\}$. Similarly, it follows that $A_2$ is a metric base. After $n$ steps we obtain a metric base $A_n=\{x_1, x_2, \dots, x_n\}\subset B_0$. Let $x\in B_0-A_n$; because $B_0$ is metric independent, it follows that $A_n\cup\{x\}$ is a metric independent set. Hence $x\notin H(A_n)=M$; but this is a contradiction. It follows that $B_0$ is a finite set and $|B_0|\le|A_0|$. Symmetrical, if suppose that $|B_0|<|A_0|$ we obtain a contradiction. Therefore $|A_0|=|B_0|$.

II. Let $A_0$, $B_0$ be two infinite metric bases; for every $x\in A_0\subseteq M=H(B_0)$ there exists $B_x\subseteq B_0$ with $B_x$ finite and $x\in H(B_x)$. Let $B'=\cup\{B_x : x\in A\}$ and suppose that there exists $y\in B_0-B'$. Then, from $y\in M=H(A_0)$, there exists $A_y\subseteq A$ with $A_y$ finite and $y\in H(A_y)$. For every $x\in A_y$ $x\in H(B_x)\subseteq H(B')$; from lemma 2 (2), it follows that $H(B')=H(B'\cup\{x\})$. Because $A_y$ is a finite set, it follows that $H(B')=H(B'\cup A_y)\supseteq H(A_y)$, hence $y\in H(B')$. Then there exists a finite set $B''\subseteq B'$ such that $y\in H(B'')$. It follows that $B''$ is not a metric independent set. But $B''\subseteq B'\subseteq B_0$: this is a contradiction. Hence $B_0=B'$; it follows that $|B_0|\le|A_0|\cdot\aleph_0=|A_0|$. Similarly, we obtain $|A_0|\le|B_0|$.

Let $(M, \rho)$ be a metric space which satisfies the condition $(\ast)$ and let $A_0$ be a metric base; the cardinal number $|A_0|$ is called *the metric dimension* of $(M, \rho)$ and we note $\dim(M, \rho)=|A_0|$.

**3.** Let $(M, \rho)$ be a metric space and $A \subseteq M$; let $\tau_A$ be the topology generated by the family of mappings $\{\rho(t,.) : t \in A\}$ ($\tau_A$ is the coarsest topology in the class of all topologies on $M$ that make all $\rho(t,.)$'s continuous). For every $t \in A$ let $d_t : M \times M \to R_+$ defined by $d_t(x, y) = |\rho(t, x) - \rho(t, y)|$, $x, y \in M$. Then $\{d_t : t \in A\}$ is a family of pseudometrics on $M$: the uniformity generated by this family, $\mathcal{U}_A$, induce the topology $\tau_A$. If $(A, \rho_A)$ is a separable metric subspace of $(M, \rho)$, then $\mathcal{U}_A$ is pseudometrizable.

We get easily the following:

**Lemma 3.** *For every $A \subseteq M$, $\mathcal{U}_A = \mathcal{U}_{\bar{A}}$.*

If $\tau_A = \tau_\rho$ we say that $A$ *generates* the metric topology of $(M, \rho)$.

**Corollary 1.** *In every separable metric space $(M, \rho)$ there exists $A \subseteq M$ such that $|A| \leq \aleph_0$ and $A$ generates the metric topology.*

**Theorem 3.** *Let $(M, \rho)$ be a finite compact metric space (every closed and bounded subset of $M$ is compact). If $A \subseteq M$ and $H(A) = M$ then $A$ generates the topology of $(M, \rho)$.*

*Proof.* For every $x \in M$, $r > 0$ let $S(x, r) = \{y : \rho(x, y) < r\}$ and $\bar{S}(x, r) = \{y : \rho(x, y) \leq r\}$. Because $x \in H(A)$ there exists $B = \{t_1, t_2, \ldots, t_n\} \subseteq A$ such that $x \in H(B)$. Let $V_p = \bigcap_{k=1}^{n} [S(t_k, \rho(t_k, x) + 1/p) - \bar{S}(t_k, \rho(t_k, x) - 1/p)]$, $p \in N$. For every $p \in N$, $V_p$ is a neighbourhood of $x$ in $\tau_A$. We suppose that for every $p \in N$ $V_p \nsubseteq S(x, r)$; hence there exists $(x_p)_{p \in N} \subseteq M - S(x, r)$ such that $\rho(t_k, x) - 1/p < \rho(t_k, x_p) < \rho(t_k, x) + 1/p$ for every $p \in N$ and $k \in \{1, 2, \ldots, n\}$. $(x_p)_{p \in N}$ is a bounded set in the finite compact metric space $(M, \rho)$. Hence there exist $x_0 \in M$ and $(x_{p_m})_{m \in N}$ a subsequence of $(x_p)_{p \in N}$ such that $x_{p_m} \xrightarrow{\tau_\rho} x_0$ and $\rho(t_k, x) - 1/p_m < \rho(t_k, x_{p_m}) < \rho(t_k, x) + 1/p_m$ for every $m \in N$. If $m \to +\infty$ we have $\rho(t_k, x) = \rho(t_k, x_0)$ for every $k = 1, 2, \ldots, n$. But $x_0 \notin S(x, r)$, hence $x \neq x_0$. This is a contradiction ($x \in H(B)$). Hence, there exists $p \in N$ such that $V_p \subseteq S(x, r)$. Therefore $\tau_A = \tau_\rho$.

**Corollary 2.** *In a finite compact metric space every metric base generates the metric topology.*

Let $A \subseteq M$ and $d_A : M \times M \to R_+$ defined by $d_A(x, y) = \sup \{d_t(x, y) : t \in A\}$ for every $x, y \in M$. Then $d_A$ is a pseudometric on $M$ and $\mathcal{U}_A \subseteq \mathcal{U}_{d_A} \subseteq \mathcal{U}_\rho$, where $\mathcal{U}_{d_A}$ and $\mathcal{U}_\rho$ are the uniformities induced by $d_A$ and $\rho$. We say that $(M, \rho)$ is A-*complete* if $(M, d_A)$ is a complete pseudometric space.

**Theorem 4.** *Let $A$ be a finite set which generates the metric topology of $(M, \rho)$. Then $(M, \rho)$ is a finite compact metric space if and only if $(M, \rho)$ is $A$-complete.*

*Proof.* Suppose that $(M, \rho)$ is a finite compact metric space and let $(x_n)_{n \in N}$ be a $d_A$ — Cauchy sequence; then $(x_n)_{n \in N}$ is a bounded sequence, hence there exist $x_0 \in M$ and a subsequence $(x_{n_k})_{k \in N}$ of $(x_n)_{n \in N}$ such that $x_{n_k} \xrightarrow{\tau_\rho} x_0$. Now, $x_n \xrightarrow{\rho} x_0$ and because $d_A \leq \rho$, it follows that $x_n \xrightarrow{d_A} x_0$.

Conversely, if $A = \{t_1, t_2, \ldots, t_n\}$ and $(M, \rho)$ is $A$-complete then the mapping $\varphi : M \to R^n$ defined by $\varphi(x) = (\rho(t_1, x), \ldots, \rho(t_n, x))$ for every $x \in M$, is an uniformly continuous one-to-one mapping ($d_A(x, y) \leq \|\varphi(x) - \varphi(y)\| \leq$



$\sqrt{n} \cdot d_A(x, y)$ for every $x, y \in M$, where the norm $\|\cdot\|$ on $R^n$ is the usual one). Because $(M, \rho)$ is *A-complete*, $\varphi(M)$ is complete in $R^n$, hence $\varphi(M) = \varphi(M)$. Let $K$ be a $\rho$ — bounded set in $M$; then $K$ is $d_A$ — bounded. It follows that $\overline{\varphi(K)} \subseteq \varphi(M)$ is compact in $R^n$. Hence $\bar{K}$ is compact in $(M, d_A)$. But $\tau_A = \tau_{d_A} = \tau_\rho$, hence $\bar{K}$ is compact in $(M, \rho)$. Therefore $(M, \rho)$ is finite compact.

*Examples.* (1) Let $(M, \rho) = (R^n, \|\cdot\|_p) = l^p(n)$, where $\|x\|_p = (\sum_{k=1}^{n} |x_k|^p)^{1/p}$ for every $x = (x_1, x_2, \ldots, x_n)$. Then, for $1 \leq p < +\infty$ every set $A_0$ of $n+1$ points which do not belong to the same hyperplane is a metric base; because $(M, \rho)$ is a finite compact metric space, $A_0$ generates the metric topology of $R^n$.

(2) In $l^1(n)$ and $l^\infty(n)$ no bounded set generates the metric topology hence every metric base is unbounded and, because these spaces are separable, it follows that $\mathrm{dm}\,(l^1(n)) = \mathrm{dm}\,(l^\infty(n)) = \aleph_0$.

(3) If $\rho$ is the discrete metric on a finite set $M$ then a set $A$ generates the metric topology iff $|M - A| \leq 1$: every set $A \subseteq M$ with $|M - A| = 1$ is a metric base, hence $\mathrm{dm}\,(M, \rho) = |M| - 1$.

## REFERENCES


1. Engelking, R. — *General Topology*, PWN — Polish Scientific Publishers MM 60, Warszawa, 1977
2. Blumenthal, L.M., Menger K. — *Studies in Geometry*, W.H. Freeman and Comp., San Francisco, 1970


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