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ON DISCRETE INEQUALITIES INVOLVING HIGHER ORDER PARTIAL DIFFERENCES

BY

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1. **Introduction.** Recently in [1—3, 8—10, 12] discrete analogs of several integral and differential inequalities known in the continuous case e.g. see [4,11 and references therein] have been studied. In [5, 6] several new inequalities involving higher order partial derivatives of two independent variables have been obtained. In the present paper we shall discuss discrete analogs of these inequalities.

2. **Definitions and Notations.** Let N be a set of points $\{0, 1, 2, \ldots\}$. The expression $\sum_{s=0}^{t-1} b(s)$ represents the solution of the linear difference equation $\Delta u(t)=b(t)$ for all $t\in N$ with the initial condition $u(0)=0$, where Δ is the forward difference operator defined by $\Delta u(t)=u(t+1)-u(t)$. It is supposed that $\sum_{s=0}^{-1} b(s)=0$. The expression $\prod_{s=0}^{t-1} c(s)$ represents a solution of the linear difference equation $u(t+1)=c(t)u(t)$ for all $t\in N$ with the initial condition $u(0)=1$. It is assumed that $\prod_{s=0}^{-1} c(s)=1$. For all $x,y\in N\times N$, we define $\Delta_x u(x, y) = u(x+1, y) - u(x, y)$, $\Delta_y u(x, y) = u(x, y+1) - u(x, y)$, $\Delta^2_{xy}u(x, y)=u(x+1, y+1)-u(x, y+1)-u(x+1, y)+u(x, y)$ and so on. $(m)^{(n)}$ is the usual factorial notation i.e. $(m)^{(n)}=m(m-1)(m-2)\ldots(m-n+1)$. We shall follow the convention $(0)^{(0)}=1$.

3. **Some Lemmas. Lemma 3.1.** [8]. *Let $g(x, y)$ be defined on $N\times N$. Then, the problem*

(3.1) $\Delta^2_{st}v(s, t, x, y)=g(s, t)v(s+1, t+1, x, y)$, (3.2) $v(x, t, x, y)=v(s, y, x, y)=1$.

has a unique solution $v(s, t, x, y)$. Further, if $g(x, y)$ is non-negative then $v(s, t, x, y)\geq 1$.

Lemma 3.2. [8]. *Let $g(x, y)$ and $h(x, y)$ be defined and non-negative on $N\times N$ and the following inequality be satisfied*

$$\Delta^2_{xy}u(x, y)\leq g(x, y)u(x, y)+h(x, y), \tag{3.3}$$ $$u(x, 0)=u(0, y)=0. \tag{3.4}$$

Then

$$u(x, y)\leq \sum_{s=0}^{x-1}\sum_{t=0}^{y-1} h(s, t)v(s+1, t+1, x, y), \tag{3.5}$$

where $v(s, t, x, y)$ is the solution of (3.1), (3.2).

Lemma 3.3. [8]. *Let $a(x, y)$, $b(x, y)$ and $u(x, y)$ be defined and non-negative on $N\times N$. Further, let $a(x, y)$ be non-decreasing and the following equality be satisfied*

$$u(x, y)\leqslant a(x, y)+\sum_{s=0}^{x-1}\sum_{t=0}^{y-1} b(s, t)u(s, t). \tag{3.6}$$

Then,

$$u(x, y)\leqslant a(x, y)\prod_{p=0}^{x-1}\left[1+\sum_{q=0}^{y-1} b(p, q)\right]. \tag{3.7}$$

Lemma 3.4. [7]. *Let $u(x)$ be some function defined on N. Then, for $0\leqslant i\leqslant r_1-1 (r_1\geqslant 1)$*

$$\Delta^i u(x)=\sum_{p=i}^{r_1-1}\frac{(x)^{(p-i)}}{(p-i)!}\Delta^p u(0)+\frac{1}{(r_1-i-1)!}\sum_{s=0}^{x-r_1+i}(x-s-1)^{(r_1-i-1)}\Delta^{r_1}u(s)$$

where $0\leqslant x\in N$.

Lemma 3.5. *Let $u(x, y)$ be some function defined on $N\times N$. Then, for $0\leqslant i\leqslant r_1-1$, $0\leqslant j\leqslant r_2-1 (r_1, r_2\geqslant 1)$*

$$\Delta_{xy}^{i+j}u(x, y)=\Phi_{ij}(x, y)+\frac{1}{(r_1-i-1)!\,(r_2-j-1)!}\sum_{s=0}^{x-r_1+i}\sum_{t=0}^{y-r_2+j}(x-s-1)^{(r_1-i-1)}\times$$
$$\times(y-t-1)^{(r_2-j-1)}\,\Delta_{st}^{r_1+r_2}u(s, t) \tag{3.8}$$

where

$$\Phi_{ij}(x, y)=\sum_{p=i}^{r_1-1}\frac{(x)^{(p-i)}}{(p-i)!}\Delta_{xy}^{p+j}u(0, y)+\sum_{q=j}^{r_2-1}\frac{(y)^{(q-j)}}{(q-j)!}\Delta_{xy}^{i+q}u(x, 0)-$$
$$-\sum_{p=i}^{r_1-1}\sum_{q=j}^{r_2-1}\frac{(x)^{(p-i)}}{(p-i)!}\frac{(y)^{(q-j)}}{(q-j)!}\Delta_{xy}^{p+q}u(0, 0)$$

and $0\leqslant x$, $0\leqslant y\in N$.

Proof. From lemma 3.4, we find

$$\Phi_{ij}(x, y)=\sum_{q=j}^{r_2-1}\frac{(y)^{(q-j)}}{(q-j)!}\Delta_{xy}^{i+q}u(x, 0)+\sum_{p=i}^{r_1-1}\frac{(x)^{(p-i)}}{(p-i)!}\times$$
$$\times\frac{1}{(r_2-j-1)!}\sum_{t=0}^{y-r_2+j}(y-t-1)^{(r_2-j-1)}\,\Delta_{xt}^{p+r_2}u(0, t).$$

Thus, the right side of (3.8) is same as

$$\sum_{q=j}^{r_2-1}\frac{(y)^{(q-j)}}{(q-j)!}\Delta_{xy}^{i+q}u(x, 0)+\frac{1}{(r_2-j-1)!}\sum_{t=0}^{y-r_2+j}(y-t-1)^{(r_2-j-1)}\times$$
$$\times\left\{\sum_{p=i}^{r_1-1}\frac{(x)^{(p-i)}}{(p-i)!}\Delta_{xt}^{p+r_2}u(0, t)+\frac{1}{(r_1-i-1)!}\sum_{s=0}^{x-r_1+i}(x-s-1)^{(r_1-i-1)}\Delta_{st}^{r_1+r_2}u(s, t)\right\},$$

which is, on applying lemma 3.4 successively

$$\sum_{q=j}^{r_2-1}\frac{(y)^{(q-j)}}{(q-j)!}\Delta_{xy}^{i+q}u(x, 0)+\frac{1}{(r_2-j-1)!}\sum_{t=0}^{x-r_1+j}(y-t-1)^{(r_2-j-1)}\Delta^{i+r_2}u(x, t)=$$
$$=\Delta_{xy}^{i+j}u(x, y).$$

4. Linear Inequalities. Theorem 4.1. *Let $a(x, y)$, $b(x, y)$, $h_{ij}(x, y)$ and $\Delta_{xy}^{i+j}u(x, y)$, $0\leqslant i\leqslant r_1$, $0\leqslant j\leqslant r_2$ be defined and non-negative on $N\times N$. Let $v(s, t, x, y)$ be the unique solution of the problem*

$$\Delta_{st}^2 v(s, t, x, y)-B_1(s, t)v(s+1, t+1, x, y)=0, \tag{4.1}$$
$$v(x, t, x, y)=v(s, y, x, y)=1. \tag{4.2}$$

Let for all $(x, y)\in N\times N$ the following inequality be satisfied

$$\Delta_{xy}^{r_1+r_2}u(x, y)\leqslant a(x, y)+b(x, y)\sum_{i=0}^{r_1}\sum_{j=0}^{r_2}\sum_{s=0}^{x-1}\sum_{t=0}^{y-1}h_{ij}(s, t)\Delta_{st}^{i+j}u(s, t). \tag{4.3}$$

Then,

$$\Delta_{xy}^{r_1+r_2}u(x, y)\leqslant a(x, y)+b(x, y)\sum_{s=0}^{x-1}\sum_{t=0}^{y-1}A_1(s, t)v(s+1, t+1, x, y) \tag{4.4}$$

where

$$A_1(x, y)=a(x, y)h_{r_1r_2}(x, y)+\sum_{i=0}^{\alpha}\sum_{j=0}^{\beta}h_{ij}(x, y)\left\{\varphi_{ij}(x, y)+\right.$$
$$\left.+\frac{1}{(r_1-i-1)!\,(r_2-j-1)!}\sum_{s=0}^{x-r_1+i}\sum_{t=0}^{y-r_2+j}(x-s-1)^{(r_1-i-1)}(y-t-1)^{(r_2-j-1)}a(s, t)\right\} \tag{4.5}$$

and

$$B_1(x, y)=b(x, y)h_{r_1r_2}(x, y)+\sum_{i=0}^{\alpha}\sum_{j=0}^{\beta}h_{ij}(x, y)\times$$
$$\frac{1}{(r_1-i-1)!\,(r_2-j-1)!}\sum_{s=0}^{x-r_1+i}\sum_{t=0}^{y-r_2+j}(x-s-1)^{(r_1-i-1)}(y-t-1)^{(r_2-j-1)}b(s, t) \tag{4.6}$$

$\alpha=r_1-1$, $\beta=r_2$ or $\alpha=r_1$, $\beta=r_2-1$. *For $i=\alpha=r_1$ or $j=\beta=r_2$ the sums in* (4.5) *and* (4.6) *are interpreted in the usual way.*

Proof. For $(x, y)\in N\times N$, we define

$$R(x, y)=\sum_{i=0}^{r_1}\sum_{j=0}^{r_2}\sum_{s=0}^{x-1}\sum_{t=0}^{y-1}h_{ij}(s, t)\Delta_{st}^{i+j}u(s, t), \tag{4.7}$$

then, it follows that

$$\Delta_{xy}^2 R(x, y)=\sum_{i=0}^{r_1}\sum_{j=0}^{r_2}h_{ij}(x, y)\Delta_{xy}^{i+j}u(x, y) \tag{4.8}$$

$R(0, y)=R(x, 0)=0$.

Using lemma 3.5 in (4.8), we get

$$\Delta_{xy}^2 R(x, y)=h_{r_1r_2}(x, y)\Delta_{xy}^{r_1r_2}u(x, y)+\sum_{i=0}^{\alpha}\sum_{j=0}^{\beta}h_{ij}(x, y)\left\{\varphi_{ij}(x, y)+\right. \tag{4.9}$$
$$\left.+\frac{1}{(r_1-i-1)!\,(r_2-j-1)!}\sum_{s=0}^{x-r_1+i}\sum_{t=0}^{y-r_2+j}(x-s-1)^{(r_1-i-1)}(y-t-1)^{(r_2-j-1)}\Delta_{st}^{r_1+r_2}u(s, t)\right\}.$$

From (4.3) and (4.7), we have

$$(4.10)\qquad \Delta_{xy}^{r_1+r_2}u(x, y)\leqslant a(x, y)+b(x, y)R(x, y).$$

Using (4.10) in (4.9) and nondecreasing nature of $R(x, y)$, we obtain

$$(4.11)\qquad \Delta_{xy}^{2}R(x, y)\leqslant A_1(x, y)+B_1(x, y)R(x, y).$$

Now from lemma 3.2, we find

$$(4.12)\qquad R(x, y)\leqslant \sum_{s=0}^{x-1}\sum_{t=0}^{y-1} A_1(s, t)v(s+1, t+1, x, y).$$

Substituting (4.12) in (4.10), the result (4.4) follows.

Remark. From inequality (4.4) and lemma 3.5, we can find $\Delta_{xy}^{i+j}u(x, y)$ for all $0\leqslant i\leqslant r_1-1$, $0\leqslant j\leqslant r_2-1$

$$\Delta_{xy}^{i+j}u(x, y)\leqslant \varphi_{ij}(x, y)+\frac{1}{(r_1-i-1)!\,(r_2-j-1)!}\sum_{s=0}^{x-r_1+i}\sum_{t=0}^{y-r_2+j}(x-s-1)^{(r_1-i-1)}\times$$
$$\times(y-t-1)^{(r_2-j-1)}\left[a(s, t)+b(s, t)\sum_{p=0}^{s-1}\sum_{q=0}^{t-1}A_1(p, q)v(p+1, q+1, s, t)\right].$$

Corollary 4.2. *In theorem* 4.1, *inequality* (4.4) *can be replaced by*

$$(4.13)\qquad \Delta_{xy}^{r_1+r_2}u(x, y)\leqslant a(x, y)+b(x, y)\left[\sum_{s=0}^{x-1}\sum_{t=0}^{y-1}A_1(s, t)\right]\prod_{s=0}^{x-1}\left[1+\sum_{t=0}^{y-1}B_1(s, t)\right].$$

Proof. As in theorem 4.1, from (4.11) we find

$$(4.14)\qquad R(x, y)\leqslant \sum_{s=0}^{x-1}\sum_{t=0}^{y-1}A_1(s, t)+\sum_{s=0}^{x-1}\sum_{t=0}^{y-1}B_1(s, t)R(s, t).$$

Using lemma 3.3, in (4.14), we get

$$(4.15)\qquad R(x, y)\leqslant\left[\sum_{s=0}^{x-1}\sum_{t=0}^{y-1}A_1(s, t)\right]\prod_{s=0}^{x-1}\left[1+\sum_{t=0}^{y-1}B_1(s, t)\right].$$

Substituting (4.15) in (4.10) the inequality (4.13) follows.

Theorem 4.3. *Let* $a(x, y)$, $b(x, y)$, $h(x, y)$ *and* $\Delta_{xy}^{2i}u(x, y)$, $0\leqslant i\leqslant r$ *be defined and non-negative on* $N\times N$ *and* $b(x, y)\geqslant 1$. *Let* $v_i(s, t, x, y)$ *be the unique solutions of the problems*

$$\Delta_{st}^{2}v_1(s, t, x, y)-[h(s, t)b(s, t)+h(s, t)+rb(s, t)+(r-1)]v_1(s+1, t+1, x, y)=0,$$
$$\Delta_{st}^{2}v_i(s, t, x, y)-[h(s, t)b(s, t)+h(s, t)+(r-i+1)b(s, t)+(r-i-1)]v_i(s+$$
$$+1, t+1, x, y)=0,\quad 2\leqslant i\leqslant r,$$
$$\Delta_{st}^{2}v_{r+1}(s, t, x, y)-[h(s, t)(b(s, t)-1)]v_{r+1}(s+1, t+1, x, y)=0,$$
$$v_j(x, t, x, y)=v_j(s, y, x, y)=1\ ;\quad 1\leqslant j\leqslant r+1.$$

Let for all $(x, y)\in N\times N$ *the following inequality be satisfied*

$$(4.16)\qquad \Delta_{xy}^{2r}u(x, y)\leqslant a(x, y)+b(x, y)\sum_{i=0}^{r}\sum_{s=0}^{x-1}\sum_{t=0}^{y-1}h(s, t)\Delta_{st}^{2i}u(s, t).$$

Then,

$$(4.17)\qquad \Delta_{xy}^{2r}u(x, y)\leqslant a(x, y)+b(x, y)\sum_{s=0}^{x-1}\sum_{t=0}^{y-1}[H(s, t)+h(s, t)B_r(s, t)]\times$$
$$\times v_{r+1}(s+1, t+1, x, y)$$

where $B_1(x, y)=\sum_{s=0}^{x-1}\sum_{t=0}^{y-1}H(s, t)v_1(s+1, t+1, x, y)$,

$$B_i(x, y)=\sum_{s=0}^{x-1}\sum_{t=0}^{y-1}[H(s, t)+B_{i-1}(x, y)]v_i(s+1, t+1, x, y),\quad 2\leqslant i\leqslant r$$

and

$$H(x, y)=h(x, y)\left\{a(x, y)+\sum_{i=0}^{r-1}\left[\varphi_{ii}(x, y)+\frac{1}{((r-i-1)!)^2}\times\right.\right.$$
$$\left.\left.\times\sum_{s=0}^{x-r+i}\sum_{t=0}^{y-r+i}(x-s-1)^{(r-i-1)}(y-t-1)^{(r-i-1)}a(s, t)\right]\right\}.$$

Proof. We define

$$(4.18)\qquad R_1(x, y)=\sum_{i=0}^{r}\sum_{s=0}^{x-1}\sum_{t=0}^{y-1}h(s, t)\Delta_{st}^{2i}u(s, t).$$

Then, it follows that

$$(4.19)\qquad \Delta_{xy}^{2}R_1(x, y)=h(x, y)\left[\Delta_{xy}^{2r}u(x, y)+\sum_{i=0}^{r-1}\Delta_{xy}^{2i}u(x, y)\right].$$

From (4.16) and (4.18), we have

$$(4.20)\qquad \Delta_{xy}^{2r}u(x, y)\leqslant a(x, y)+b(x, y)R_1(x, y)$$

and hence from lemma 3.5, we find

$$(4.21)\qquad \Delta_{xy}^{2i}u(x, y)\leqslant\varphi_{ii}(x, y)+\frac{1}{((r-i-1)!)^2}\sum_{s=0}^{x-r+i}\sum_{t=0}^{y-r+i}(x-s-1)^{(r-i-1)}\times$$
$$\times(y-t-1)^{(r-i-1)}[a(s, t)+b(s, t)R_1(s, t)].$$

Using (4.20) and (4.21) in (4.19), we obtain

$$(4.22)\qquad \Delta_{xy}^{2}R_1(x, y)+h(x, y)R_1(x, y)\leqslant H(x, y)+h(x, y)b(x, y)R_1(x, y)+$$
$$+h(x, y)R_2(x, y).$$

where

$$R_2(x, y)=R_1(x, y)+\sum_{i=0}^{r-1}\frac{1}{((r-i-1)!)^2}\sum_{s=0}^{x-r+i}\sum_{t=0}^{y-r+i}(x-s-1)^{(r-i-1)}\times$$
$$\times(y-t-1)^{(r-i-1)}b(x, t)R_1(s, t).$$

Since $R_1(x, y) \leqslant R_2(x, y)$, it follows from (4.22) that

$$\Delta^2_{xy} R_2(x, y) + R_2(x, y) \leqslant H(x, y) + [h(x, y)b(x, y) + h(x, y) + b(x, y)]R_2(x, y) + R_3(x, y)$$

where

$$R_3(x, y) = R_2(x, y) + \sum_{i=0}^{r-2} \frac{1}{((r-i-2)\,!)^2} \sum_{s=0}^{x-r+i+1} \sum_{t=0}^{y-r+i+1} (x-s-1)^{(r-i-2)} \times (y-t-1)^{(r-i-2)} b(s, t) R_2(s, t).$$

Once again on using $R_2(x, y) \leqslant R_3(x, y)$, we find

$$\Delta^2_{xy} R_3(x, y) + R_3(x, y) \leqslant H(x, y) + [h(x, y)b(x, y) + h(x, y) + 2b(x, y) + 1]R_3(x, y) + R_4(x, y),$$

where

$$R_4(x, y) = R_3(x, y) + \sum_{i=0}^{r-3} \frac{1}{((r-i-3)\,!)^2} \sum_{s=0}^{x-r+i+2} \sum_{t=0}^{y-r+i+2} (x-s-1)^{(r-i-3)} \times (y-t-1)^{(r-i-3)} b(s, t) R_3(s, t).$$

Continuing in this way, we get

$$\Delta^2_{xy} R_r(x, y) + R_r(x, y) \leqslant H(x, y) + [h(x, y)b(x, y) + h(x, y) + (r-1)b(x, y) + (r-2)]R_r(x, y) + R_{r+1}(x, y), \tag{4.23}$$

where $R_{r+1}(x, y) = R_r(x, y) + \sum_{s=0}^{x-1} \sum_{t=0}^{y-1} b(s, t) R_r(s, t)$ and hence on using $R_r(x, y) \leqslant R_{r+1}(x, y)$ in (4. 23), we obtain

$$\Delta^2_{xy} R_{r+1}(x, y) \leqslant H(x, y) + [h(x, y)b(x, y) + h(x, y) + rb(x, y) + (r-1)]R_{r+1}(x, y).$$

Since $R_j(x, 0) = R_j(0, y) = 0$, $1 \leqslant j \leqslant r+1$, an application of lemma 3.2, provides

$$R_{r+1}(x, y) \leqslant \sum_{s=0}^{x-1} \sum_{t=0}^{y-1} H(s, t) v_1(s+1, t+1, x, y) = B_1(x, y). \tag{4.24}$$

Using (4.24) in (4.23) and applying lemma 3.2, we get $R_r(x, y) \leqslant B_2(x, y)$. Continuing in this way, we find

$$R_2(x, y) \leqslant B_r(x, y). \tag{4.25}$$

Substituting (4.25) in (4.22), we obtain $\Delta^2_{xy} R_1(x, y) \leqslant H(x, y) + h(x, y)B_r(x, y) + [h(x, y)(b(x, y) - 1)]R_1(x, y)$ and lemma 3.2 provides

$$R_1(x, y) \leqslant \sum_{s=0}^{x-1} \sum_{t=0}^{y-1} [H(s, t) + h(s, t)B_r(s, t)] v_{r+1}(s+1, t+1, x, y) \tag{4.26}$$

and the result (4.17) follows from (4.20) and (4.26).

Remark. For $b(x, y)$ not necessarily greater than 1, the conclusion of theorem 4.3 remains valid if v_r, v_{r+1} and B_r are replaced by v^*_r, v^*_{r+1} and B^*_r, defined by

$$\Delta^2_{st} v^*_r(s, t, x, y) - [h(s, t)b(s, t) + h(s, t) + b(s, t)] v^*_r(s+1, t+1, x, y) = 0,$$

$$\Delta^2_{st} v^*_{r+1}(s, t, x, y) - h(s, t)b(s, t) v^*_{r+1}(s+1, t+1, x, y) = 0,$$

$$v^*_r(x, t, x, y) = v^*_r(s, y, x, y) = 1, \quad v^*_{r+1}(x, t, x, y) = v^*_{r+1}(s, y, x, y) = 1,$$

and $B^*_r(x, y) = \sum_{s=0}^{x-1} \sum_{t=0}^{y-1} [H(s, t) + B_{r-1}(x, y)] v^*_r(s+1, t+1, x, y)$.

Remark. The result which can be deduced from theorem 4.1 for the inequality (4.16) does not seem to be comparable with the one obtained in theorem 4.3.

Theorem 4.4. *Let $h(x, y)$ and $\Delta^{2i}_{xy} u(x, y)$, $0 \leqslant i \leqslant r$ be defined and non-negative on $N \times N$. Let for all $(x, y) \in N \times N$ the following inequality be satisfied*

$$\Delta^{2r}_{xy} u(x, y) \leqslant a(x) + b(y) + \sum_{i=0}^{r} \sum_{s=0}^{x-1} \sum_{t=0}^{y-1} h(s, t) \Delta^{2i}_{st} u(s, t) \tag{4.27}$$

where $a(x)$, $b(y)$, $\Delta a(x)$, $\Delta b(y)$ are defined and non-negative on N. Then,

$$\Delta^{2r}_{xy} u(x, y) \leqslant B_{r+1}(x, y) \tag{4.28}$$

where $B_1(x, y) = [a(x) + b(y)] \prod_{s=0}^{x-1} \left[1 + \sum_{t=0}^{y-1} [h(s, t) + r] \right]$ *and*

$$B_i(x, y) = [a(x) + b(y)] + \sum_{s=0}^{x-1} \sum_{t=0}^{y-1} (h(s, t) + (r-i+1)) B_{i-1}(s, t), \quad 2 \leqslant i \leqslant r+1.$$

Proof. We define $R_1(x, y)$ to be the right side of (4.27). Then, as in theorem 4.3, we find

$$\Delta^2_{xy} R_1(x, y) \leqslant h(x, y) R_2(x, y) \tag{4.29}$$

where

$$R_2(x, y) = R_1(x, y) + \sum_{i=0}^{r-1} \frac{1}{((r-i-1)\,!)^2} \sum_{s=0}^{x-r+i} \sum_{t=0}^{y-r+i} (x-s-1)^{(r-i-1)} \times (y-t-1)^{(r-i-1)} R_1(s, t).$$

On using $R_1(x, y) \leqslant R_2(x, y)$, we get

$$\Delta^2_{xy} R_2(x, y) \leqslant h(x, y) R_2(x, y) + R_3(x, y), \tag{4.30}$$

where

$$R_3(x, y) = R_2(x, y) + \sum_{i=0}^{r-2} \frac{1}{((r-i-2)\,!)^2} \sum_{s=0}^{x-r+i+1} \sum_{t=0}^{y-r+i+1} (x-s-1)^{(r-i-2)} \times (y-t-1)^{(r-i-2)} R_2(s, t).$$

Again on using $R_2(x, y) \leqslant R_3(x, y)$, we obtain

$$\Delta^2_{xy} R_3(x, y) \leqslant [h(x, y) + 1] R_3(x, y) + R_4(x, y). \tag{4.31}$$

Continuing in this way, it follows that

$$\Delta^2_{xy} R_r(x, y) \leqslant [h(x, y) + (r-2)] R_r(x, y) + R_{r+1}(x, y), \tag{4.32}$$

where $R_{r+1}(x, y)=R_r(x, y)+\sum_{s=0}^{x-1}\sum_{t=0}^{y-1} R_r(s, t)$. On using $R_r(x, y)\leqslant R_{r+1}(x, y)$, we find

$$\Delta^2_{xy}R_{r+1}(x, y)\leqslant [h(x, y)+(r-2)]R_{r+1}(x, y)+R_{r+1}(x, y)+R_{r+1}(x, y)= \tag{4.33}$$
$$=[h(x, y)+r]R_{r+1}(x, y).$$

From the definition of $R_i(x, y)$ it follows that $R_i(x, y)\leqslant R_{i+1}(x, y)$, $1\leqslant i\leqslant r$ and $R_j(x, 0)=a(x)+b(0)$, $R_j(0, y)=a(0)+b(y)$, $1\leqslant j\leqslant r+1$.

Summing both sides of (4.33), we get $R_{r+1}(x, y)\leqslant a(x)+b(y)+ + \sum_{s=0}^{x-1}\sum_{t=0}^{y-1}[h(s, t)+r]$ and an application of lemma 3.3 provides

$$R_{r+1}(x, y)\leqslant B_1(x, y). \tag{4.34}$$

Using (4.34) and $R_r(x, y)\leqslant R_{r+1}(x, y)$ in (4.32), we obtain $\Delta^2_{xy}R_r(x, y)\leqslant \leqslant[h(x, y)+(r-1)]B_1(x, y)$, and hence $R_r(x, y)\leqslant B_2(x, y)$. Continuing in this way the result (4.28) follows.

Our next result is connected with the following inequality

$$\Delta^{r_1+r_2}_{xy}u(x, y)\leqslant a(x, y)+b(x, y)\left[\sum_{k=1}^{n-1} E_k\left(x, y, \sum_{i=0}^{r_1}\sum_{j=0}^{r_2}\Delta^{i+j}_{**}u(*, *)\right)+\right. \tag{4.35}$$
$$\left. +E_n(x, y, \Delta^{r_1+r_2}_{**}u(*, *))\right]$$

where for all $1\leqslant k\leqslant n$ in E_k, $(*, *)$ is (s_k, t_k) and

$$E_k(x, y, \odot)=\sum_{s_1=0}^{x-1}\sum_{t_1=0}^{y-1} f_{k1}(s_1, t_1)\sum_{s_2=0}^{s_1-1}\sum_{t_2=0}^{t_1-1} f_{k2}(s_2, t_2)\ldots\sum_{s_k=0}^{s_{k-1}-1}\sum_{t_k=0}^{t_{k-1}-1} f_{kk}(s_k, t_k)\odot.$$

Theorem 4.5. *Let all the functions which appear in (4.35) be defined and non-negative on $N\times N$. Let $v(s, t, x, y)$ be the unique solution of the problem*

$$\Delta^2_{st}v(s, t, x, y)-\left[\sum_{k=1}^{n-1}\Delta^2_{st}E_k(s, t, B_2(*, *))+\Delta^2_{st}E_n(S, t, b(*, *))\right]\times$$
$$\times v(s+1, t+1, x, y)=0, \qquad v(x, t, x, y)=v(s, y, x, y)=1.$$

Let for all $(x, y)\in N\times N$ the inequality (4.35) be satisfied. Then,

$$\Delta^{r_1+r_2}_{xy}u(x, y)\leqslant a(x, y)+b(x, y)\sum_{s=0}^{x-1}\sum_{t=0}^{y-1}\left[\sum_{k=0}^{n-1}\Delta^2_{st}E_k(s, t, A_2(*, *))+\right. \tag{4.36}$$
$$\left.+\Delta^2_{st}E_n(s, t, a(*, *))\right]v(s+1, t+1, x, y)$$

where A_2 and B_2 are same as A_1 and B_1 with $h_{ij}(x, y)\equiv 1$ for all $1\leqslant i\leqslant r_1$, $1\leqslant j\leqslant r_2$ on $N\times N$.

Proof. Let $R(x, y)$ be the part of the right side appearing in the bracket of (4.35). Then, we have as in theorem 4.1.

$$\Delta^2_{xy}R(x, y)=\sum_{k=1}^{n-1}\Delta^2_{xy}E_k\left(x, y, \sum_{i=0}^{r_1}\sum_{j=0}^{r_2}\Delta^{i+j}_{**}u(*, *)\right)+\Delta^2_{xy}E_n(x, y, \Delta^{r_1+r_2}_{**}u(*, *))=$$
$$=\sum_{k=1}^{n-1}\Delta^2_{xy}E_k\left(x, y, \sum_{i=0}^{\alpha}\sum_{j=0}^{\beta}\Delta^{i+j}_{**}u(*, *)\right)+\sum_{k=1}^{n}\Delta^2_{xy}E_k(x, y, \Delta^{r_1+r_2}_{**}u(*, *))\leqslant$$
$$\leqslant\sum_{k=0}^{n-1}\Delta^2_{xy}E_k\left(x, y, \sum_{i=0}^{\alpha}\sum_{j=0}^{\beta}\left\{\varphi_{ij}(*, *)+\frac{1}{(r_1-i-1)!\,(r_2-j-1)!}\times\right.\right.$$
$$\left.\left.\times\sum_{s=0}^{*=r_1+i}\sum_{t=0}^{*=r_2+j}(*-s-1)^{(r_1-i-1)}(*-t-1)^{(r_2-j-1)}[a(*, *)+b(*, *)R(*, *)]\right\}\right)+$$
$$+\sum_{k=0}^{n}\Delta^2_{xy}E_k(x, y, a(*, *)+b(*, *)R(*, *))\leqslant$$
$$\leqslant\left[\sum_{k=1}^{n-1}\Delta^2_{xy}E_k(x, y, A_2(*, *))+\Delta^2_{xy}E_n(x, y, a(*, *))\right]+$$
$$+\left[\sum_{k=1}^{n-1}\Delta^2_{xy}E_k(x, y, B_2(*, *))+\Delta^2_{xy}E_n(x, y, b(*, *))\right]R(x, y).$$

The result (4.36) now follows from lemma 3.2 and $\Delta^{r_1+r_2}_{xy}u(x, y)\leqslant a(x, y)+ +b(x, y)R(x, y)$.

5. Some Applications. The inequalities obtained in section 4 will have as many applications as their continuous analogs [5, 6]. To show some of these, we consider (r_1+1+r_2+1)-th order partial difference equation

$$\Delta^{r_1+1+r_2+1}_{xy}u(x, y)=f(x, y, u(x, y), \Delta_x u(x, y),\ldots,\Delta^{r_1+r_2}_{xy}u(x, y)) \tag{5.1}$$

where

$$|f(x, y, u_{00}, u_{10},\ldots,u_{r_1r_2})|\leqslant\sum_{i=0}^{r_1}\sum_{j=0}^{r_2}h_{ij}(x, y)|u_{ij}|. \tag{5.2}$$

and $h_{ij}(x, y)$, $0\leqslant i\leqslant r_1$, $0\leqslant j\leqslant r_2$ are nonnegative functions defined on $N\times N$.

Let $u(x, y)$ be any solution of (5.1), then this also satisfies

$$\Delta^{r_1+r_2}u(x, y)=\Delta^{r_1+r_2}u(x, 0)+\Delta^{r_1+r_2}u(0, y)-\Delta^{r_1+r_2}u(0, 0)+ \tag{5.3}$$
$$+\sum_{s=0}^{x-1}\sum_{t=0}^{y-1}f(s, t, u(s, t), \Delta_s u(s, t),\ldots,\Delta^{r_1+r_2}_{st}u(s, t)).$$

Using (5.2) in (5.3) we find

$$|\Delta^{r_1+r_2}u(x, y)|\leqslant|\Delta^{r_1+r_2}u(x, 0)+\Delta^{r_1+r_2}u(0, y)-\Delta^{r_1+r_2}u(0, 0)|+ \tag{5.4}$$
$$+\sum_{s=0}^{x-1}\sum_{t=0}^{y-1}\sum_{i=0}^{r_1}\sum_{j=0}^{r_2}h_{ij}(s, t)|\Delta^{i+j}_{st}u(s, t)|.$$

Now a suitable application of theorem 4.1. provides an upper estimate on $|\Delta^{r_1+r_2}u(x, y)|$ in terms of known functions which is

$$
\begin{aligned}
&|\Delta^{r_1+r_2}u(x, y)| \leqslant |\Delta^{r_1+r_2}u(x, 0)+\Delta^{r_1+r_2}u(0, y)-\Delta^{r_1+r_2}u(0, 0)| + \\
&+\sum_{s=0}^{x-1}\sum_{t=0}^{y-1} A_1(s, t)v(s+1, t+1, x, y),
\end{aligned} \tag{5.5}
$$

where $A_1(s, t)$, $v(s+1, t+1, x, y)$ are same as in theorem 4.1 with

$$a(x, y)=|\Delta^{r_1+r_2}u(x, 0)+\Delta^{r_1+r_2}u(0, y)-\Delta^{r_1+r_2}u(0, 0)|, \quad b(x, y)=1.$$

Similarly, corollary 4.2 provides

$$
\begin{aligned}
&|\Delta^{r_1+r_2}u(x, y)| \leqslant |\Delta^{r_1+r_2}u(x, 0)+\Delta^{r_1+r_2}u(0, y)-\Delta^{r_1+r_2}u(0, y)| + \\
&+\left[\sum_{s=0}^{x-1}\sum_{t=0}^{y-1} A_1(s, t)\right]\prod_{s=0}^{x-1}\left[1+\sum_{t=0}^{y-1} B_1(s, t)\right].
\end{aligned} \tag{5.6}
$$

Next, (5.5) or (5.6) can be used in lemma 3,5, and we find upper estimates for $|\Delta^{i+j}u(x, y)|$, $0 \leqslant i \leqslant r_1$, $0 \leqslant j \leqslant r_2$.

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GLOBAL SUFFICIENT OPTIMALITY CONDITIONS FOR A FAMILY OF NON-CONVEX OPTIMIZATION PROBLEMS

BY

TEODOR PRECUPANU

§0. In this paper we shall be concerned with the following family of optimization problems

$$(P_y) \qquad \min\{f(x)-g(Ax-y)\,;\ x\in X\},\quad y\in Y,$$

where X, Y are two separated linear topological spaces, $A : X \to Y$ is a linear continuous operator and $f : X \to]-\infty, \infty]$, $g : Y \to [-\infty, \infty[$ are two proper extended real-valued functions, i.e. $\mathrm{dom}\, f=\{x\in X\,;\ f(x)<\infty\}\neq\emptyset$, $\mathrm{dom}\, g=\{y\in Y\,;\ g(y)>-\infty\}\neq\emptyset$.

It is well known that in the convex case some optimality or duality properties are expresed in terms of the following set:

$$H=\{(Ax-y, f(x)-g(y)+r)\in Y\times R\,;\ x\in \mathrm{dom}\, f,\ y\in \mathrm{dom}\, g,\ r\geqslant 0\}. \tag{0.1}$$

In an earlier paper [16], we have established the following two results:

Theorem 0.1. *The problems $(P_y)_{y\in Y}$ have solutions whenever* val $P_y \neq -\infty$ *and the value function $y\to$* val P_y, $y\in Y$, *is lower-semicontinuous on Y if and only if the set H is closed in $Y\times R$.*

Theorem 0.2. *If P_y has the finite value at least an element $y\in Y$, then the family $(P_y)_{y\in Y}$ is dually stable on Y* (i.e. min $P_y=$ sup $D_y\neq-\infty$ *for all* $y\in Y$) *if and only if the set H is convex and closed.*

We recall that the dual by conjugacy of Fenchel type [1] of P_y, denoted by D_y, is as follows

$$(D_y) \qquad \max_{y^*\in Y^*}\{g^*(y^*)-f^*(A^*y^*)-(y, y^*)\},\quad y\in Y,$$

where f^* is the (convex-) conjugate of f and g^* is the (concave-) conjugate of g. These two results can be regard as refinements of Levine-Pomerol's result from the convex case [12]. Thus, by Theorem 0.1 we can obtain various sufficient optimality conditions, which assure the lower-semicontinuity of value function using criteria for the closedness of the set H. If in addition the set H is convex, different sufficient stability conditions are obtained. In this line, several main results derived from sufficient closedness conditions of H was given by J. P. Dedieu [4], U. Dieter [5], Ky Fan [7], W. Krabs [8], [9], K. Kretschmer [10], P. Levine-J. Ch. Pomerol [11], [12], T. Nakamura-M. Yamasaki [13], R. Nozawa [14], J. Ponstein [15], T. Precupanu [17], C. Raffin [18], M. Yamasaki [20], C. Zălinescu [21]. In the linear case, T. Nakamura-M. Yamasaki