Now a suitable application of theorem 4.1. provides an upper estimate on $|\Delta^{r_1+r_2}u(x, y)|$ in terms of known functions which is

$$|\Delta^{r_1+r_2}u(x, y)| \leqslant |\Delta^{r_1+r_2}u(x, 0)+\Delta^{r_1+r_2}u(0, y)-\Delta^{r_1+r_2}u(0, 0)| + \sum_{s=0}^{x-1}\sum_{t=0}^{y-1} A_1(s, t)v(s+1, t+1, x, y), \tag{5.5}$$

where $A_1(s, t)$, $v(s+1, t+1, x, y)$ are same as in theorem 4.1 with

$$a(x, y) = |\Delta^{r_1+r_2}u(x, 0)+\Delta^{r_1+r_2}u(0, y)-\Delta^{r_1+r_2}u(0, 0)|, \quad b(x, y)=1.$$

Similarly, corollary 4.2 provides

$$|\Delta^{r_1+r_2}u(x, y)| \leqslant |\Delta^{r_1+r_2}u(x, 0)+\Delta^{r_1+r_2}u(0, y)-\Delta^{r_1+r_2}u(0, y)| + \left[\sum_{s=0}^{x-1}\sum_{t=0}^{y-1} A_1(s, t)\right]\prod_{s=0}^{x-1}\left[1+\sum_{t=0}^{y-1} B_1(s, t)\right]. \tag{5.6}$$

Next, (5.5) or (5.6) can be used in lemma 3,5, and we find upper estimates for $|\Delta^{i+j}u(x, y)|$, $0 \leqslant i \leqslant r_1$, $0 \leqslant j \leqslant r_2$.

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GLOBAL SUFFICIENT OPTIMALITY CONDITIONS FOR A FAMILY OF NON-CONVEX OPTIMIZATION PROBLEMS

BY

TEODOR PRECUPANU

§0. In this paper we shall be concerned with the following family of optimization problems

$$(P_y) \qquad \min\{f(x)-g(Ax-y)\ ;\ x\in X\}, \quad y\in Y,$$

where X, Y are two separated linear topological spaces, $A : X\to Y$ is a linear continuous operator and $f : X\to]-\infty, \infty]$, $g : Y\to[-\infty, \infty[$ are two proper extended real-valued functions, i.e. $\operatorname{dom} f=\{x\in X\ ;\ f(x)<\infty\}\neq\varnothing$, $\operatorname{dom} g=\{y\in Y\ ;\ g(y)>-\infty\}\neq\varnothing$.

It is well known that in the convex case some optimality or duality properties are expresed in terms of the following set :

$$H=\{(Ax-y, f(x)-g(y)+r)\in Y\times R\ ;\ x\in \operatorname{dom} f,\ y\in \operatorname{dom} g,\ r\geqslant 0\}. \tag{0.1}$$

In an earlier paper [16], we have established the following two results :

Theorem 0.1. *The problems* $(P_y)_{y\in Y}$ *have solutions whenever* $\operatorname{val} P_y\neq-\infty$ *and the value function* $y\to \operatorname{val} P_y$, $y\in Y$, *is lower-semicontinuous on* Y *if and only if the set* H *is closed in* $Y\times R$.

Theorem 0.2. *If* P_y *has the finite value at least an element* $y\in Y$, *then the family* $(P_y)_{y\in Y}$ *is dually stable on* Y (i.e. $\min P_y=\sup D_y\neq-\infty$ *for all* $y\in Y$) *if and only if the set* H *is convex and closed.*

We recall that the dual by conjugacy of F e n c h e l type [1] of P_y, denoted by D_y, is as follows

$$(D_y) \qquad \max_{y^*\in Y^*}\{g^*(y^*)-f^*(A^*y^*)-(y, y^*)\}, \quad y\in Y,$$

where f^* is the (convex-) conjugate of f and g^* is the (concave-) conjugate of g. These two results can be regard as refinements of L e v i n e - P o m e r o l's result from the convex case [12]. Thus, by Theorem 0.1 we can obtain various sufficient optimality conditions, which assure the lower-semicontinuity of value function using criteria for the closedness of the set H. If in addition the set H is convex, different sufficient stability conditions are obtained. In this line, several main results derived from sufficient closedness conditions of H was given by J. P. D e d i e u [4], U. D i e t e r [5], K y F a n [7], W. K r a b s [8], [9], K. K r e t s c h m e r [10], P. L e v i n e - J. C h. P o m e r o l [11], [12], T. N a k a m u r a - M. Y a m a s a k i [13], R. N o z a w a [14], J. P o n s t e i n [15], T. P r e c u p a n u [17], C. R a f f i n [18], M. Y a m a s a k i [20], C. Z ă l i n e s c u [21]. In the linear case, T. N a k a m u r a - M. Y a m a s a k i

[13] remark that the set H can be considered in several ways as a difference of two sets and thus applying D i e u d o n n é's criterion [6] they obtain various sufficient conditions for closedness of H.

The present paper is a continuation of our work [16] its aim being to extend some results established by T. N a k a m u r a - M. Y a m a s a k i [13] in the linear case to the case of problems $(P_y)_{y \in Y}$ (generally non-convex) of F e n c h e l - R o c k a f e l l a r type [19]. This is done using D e d i e u's generalization of D i e u d o n n é's criterion [3]. In §1 we give some results derived from [3] and with their aid we obtain in §2 some sufficient conditions for the global optimality of the family $(P_y)_{y \in Y}$ and at same time, for the lower-semicontinuity of the value function. The special case of the positively homogeneous functions is considered in §3.

§1. J. D e d i e u [3] gives a sufficient condition for the closedness of the image of a closed set by a multivalued mapping replacing the usual local compactness condition from the convex case by the stronger condition of the asymptotical compactness. In the following we point out some special results concerning the closedness of the difference of two sets.

Definition 1. [4]. *For a given set K, the set K_∞ defined by*

$$K_\infty = \bigcap_{\varepsilon > 0} \overline{]0, \varepsilon] K} \tag{1.1}$$

is called the recession (asymptotic) cone of K.

Remark 1. The set K_∞ is a cone consisting from all limits of the convergent nets of the form $(\lambda_i x_i)_{i \in I}$, where $(\lambda_i)_{i \in I}$ is a net of positive numbers convergent to zero and $(x_i)_{i \in I}$ is a net of elements of K. If the set K is convex we obtain the usual concept of the recession cone introduced by G. C h o q u e t [2].

Definition 2 [3]. *A set K is called asymptotically compact if there exist $\varepsilon_0 > 0$ and a neighborhood V_0 of the origin such that $(]0, \varepsilon_0] K) \cap V_0$ is relatively compact.*

Remark 2. A closed convex set or a closed cone is asymptotically compact if and only if is locally compact. Also, if K is closed and asymptotically compact then K and K_∞ are locally compact, but the converse is false [3].

Theorem 1.1. *Let X_1, X_2 be two separated linear topological spaces and let T be a positively homogeneous operator defined on a cone of X_1 into X_2. Let $B \subset X_1$, $C \subset X_2$ be two closed sets such that* con $B \subset$ dom T *and the restrictions of T to* con B *is continuous. If*

$$B_\infty \cap T^{-1}(C_\infty) = \{0\} \tag{1.2}$$

and B is asymptotically compact, then $T(B) - C$ is closed.

Proof. Since B is asymptotically compact there exist $\varepsilon_0 > 0$ and a circled open neighborhood V_0 of the origin of X_1 such that the set $B_0 = (]0, \varepsilon_0] B) \cap V_0$ is relatively compact. Let $(Tb_i - c_i)_{i \in I}$ be a net of elements of $T(B) - C$ which converges to an element $x_0 \in X_2$. Without loss of generality we can suppose that $b_i \neq 0$ for all $i \in I$. For each $i \in I$ there exist positive numbers $\lambda > 0$ such that $\lambda b_i \in V_0$ since V_0 is absorbent. Let us $\lambda_i = \min(\{\varepsilon_0\} \cup \{\lambda ; \lambda > 0$ and $2\lambda b_i \notin V_0\})$. It is clear that $0 < \lambda_i \leq \varepsilon_0$ and $\lambda_i b_i \in V_0$ for all $i \in I$. Therefore $(\lambda_i b_i)_{i \in I} \subset B_0$. Since B_0 is relatively compact and (λ_i) is bounded, we can consider that $(\lambda_i b_i)_{i \in I}$ is convergent to an element $b_0 \in X_1$ and (λ_i) is convergent to a number $\lambda_0 \geq 0$ (taking a subnet). We claim that $\lambda_0 \neq 0$. To prove this, we observe that if $\lambda_0 = 0$, there exists $i_0 \in I$ such that $\lambda_i < \varepsilon_0$ for all $i > i_0$. Thus

$$2\lambda_i b_i \notin V_0 \text{ for all } i > i_0. \tag{1.3}$$

On the other hand, $(\lambda_i(Tb_i - c_i))_{i \in I}$ being convergent to zero and $(\lambda_i b_i)_{i \in I}$ being convergent to $b_0 \in B_\infty$, by continuity of $T/_{\overline{\text{con}\, B}}$ it follows that $(\lambda_i Tb_i)_{i \in I}$ converges to Tb_0. Therefore $(\lambda_i c_i)_{i \in I}$ is also convergent to an element of C_∞, i.e. $b_0 \in B_\infty \cap T^{-1}(C_\infty)$. By (1.2) it follows $b_0 = 0$ which contradicts the property (1.3). Thus $\lambda_0 \neq 0$ and then $(b_i)_{i \in I}$ is convergent to $1/\lambda_0\, b_0 \in B$ and $(T(b_i))_{i \in I}$ convergent to an element of TB. Also, (c_i) is convergent to an element of the closed set C. Finally, we obtain that $x_0 \in T(B) - C$, i.e. $T(B) - C$ is closed.

Corollary 1. *Let T and B be as in Theorem 1.1. If*

$$B_\infty \cap \ker T = \{0\}, \tag{1.4}$$

then $T(B)$ is closed. If in addition B is a locally compact cone then $T(B)$ is also locally compact cone (hence so much the more is closed).

Proof. The first part of statement follows taking $C = \{0\}$. For the second part of corollary we recall that a cone is locally compact if and only if it is generated by a compact set which does not contain the origin. Thus, if $B =$ con K, where K is compact and $0 \notin K$, then $TB =$ con TK, where TK is compact and $0 \notin TK$ (according to condition (1.4) for $B_\infty = B$).

Corollary 2. *Let B, C be two closed sets such that*

$$B_\infty \cap C_\infty = \{0\}. \tag{1.5}$$

If B or C is asymptotically compact then their difference is also closed.

Proof. Let us consider T as identity operator.

Remark 3. Theorem 1.1 can be obtained from the general result of J. D e d i e u announced in [3]. At the same time it is an extension of the result of T. N a k a m u r a - M. Y a m a s a k i [13] used in the proof of several sufficient conditions for duality theorems in infinite linear programming problems. Also it is easy to observe that the proof of Theorem 1.1 rest valid if $B \subset$ dom T and the following two weaker properties hold

(1.6) *if $(b_i) \subset B$, $(c_i) \subset C$, such that $b_i \to b$, $Tb_i - c_i \to y_0$ then $Tb - y_0 \in C$,*

(1.7) *if $(b_i) \subset B$, $(c_i) \subset C$, $(\lambda_i) \subset R_+$ such that $\lambda_i(Tb_i - c_i) \to 0$, $\lambda_i b_i \to b_0$ and $\lambda_i \to 0$, then $b_0 = 0$.*

It is clear that the property (1.6) is equivalent to the closedness of the graph of the multivalued function $x \to Tx - C$, $x \in B$, in $X_1 \times X_2$, while the second properties (1.7) replaces the asymptotical separation property (1.2).

§2. It is easy to see that the set H associated to the family of problems $(P_y)_{y \in Y}$ can be written as a difference of two sets in the following four ways:

$$H = M^i - N^i, \quad i = 1, 2, 3, 4, \tag{2.1}$$

where

(2.1$_1$) $M^1 = \{(Ax, f(x) + r) ;\ x \in \text{dom} f, r \geq 0\}$, $N^1 = \{(y, g(y)) ;\ y \in \text{dom}\, g\}$,

(2.1$_2$) $M^2 = \{(Ax, f(x)) ;\ x \in \text{dom} f\}$, $N^2 = \{(y, g(y) - s) ;\ y \in \text{dom}\, g, s \geq 0\}$,

(2.1$_3$) $M^3 = M^1$, $N^3 = N^2$,

(2.1$_4$) $M^4 = \{(Ax - y, f(x) - g(y)) ;\ x \in \text{dom} f, y \in \text{dom}\, g\}$, $N^4 = -\{0\} \times R_+$.

If f and g are arbitrary functions the above sets can be non-closed and non-convex (except N^4 which is a closed convex cone).

Let us consider the operators $A_f : \operatorname{dom} f \to Y \times R$, $\widetilde{A} : X \times R \to Y \times R$, defined by

(2.2) $$A_f(x) = (Ax, f(x)), \quad x \in \operatorname{dom} f,$$

(2.3) $$\widetilde{A}(x, \alpha) = (Ax, \alpha), \quad (x, \alpha) \in X \times R.$$

Then we have

(2.4) $$M^1 = \widetilde{A}(\operatorname{epi} f) = A_f(\operatorname{dom} f) + \{0\} \times R_+,$$

(2.5) $$M^2 = \widetilde{A}(\operatorname{graph} f) = A_f(\operatorname{dom} f).$$

Since the set N^4 is a locally compact cone (hence is closed and asymptotically compact) combining Corollary 2 with Theorems 0.1 and 0.2 we obtain

Theorem 2.1. *Suppose that f and g have the following property:*

(c_4) $$\text{if } (x_i)_{i \in I} \subset \operatorname{dom} f, (y_i)_{i \in I} \subset \operatorname{dom} g, \lambda_i > 0, \lambda_i \to 0,$$
$$\lambda_i(Ax_i - y_i) \to 0 \text{ and } \lambda_i(f(x_i) - g(y_i)) \to a_0 \leq 0 \text{ then } a_0 = 0.$$

If M^4 is closed then (P_y) have solutions whenever val $P_y \neq -\infty$. If M_4 is also convex and (P_0) has a finite value, then the family $(P_y)_{y \in Y}$ is dually stable.

Indeed, it is sufficient to observe that $M^4_\infty \cap (\{0\} \times R_-) = \{(0, 0)\}$ if and only if (c_4) holds.

Now, we can give sufficient conditions in order that M^4 be closed applying again Corollary 2 because $M^4 = M^2 - N^1$. To use Corollary 2 for the decompositions (2.1_{1-3}) we observe that the condition (1.5) may be expressed with the aid of the *recession functions* (of convex and concave type respectively) defined as

(2.6) $$f_\infty(x) = \sup_{\varepsilon > 0} \liminf_{u \to x} \inf_{0 < \lambda \leq \varepsilon} \lambda f(u/\lambda), \quad x \in X,$$

(2.7) $$g_\infty(y) = \inf_{\varepsilon > 0} \limsup_{v \to y} \sup_{0 < \mu \leq \varepsilon} \mu g(v/\mu), \quad y \in Y.$$

Both functions are positively homogeneous. Also, f_∞ is lower-semicontinuous and g_∞ is upper-semicontinuous. The following equalities are well known

(2.8) $$(\operatorname{epi} f)_\infty = \operatorname{epi} f_\infty, \quad (\operatorname{hypo} g)_\infty = \operatorname{hypo} g_\infty.$$

According to Remark 1, we obtain that the condition $M^i_\infty \cap N^i_\infty = \{0\}$ is equivalent to the condition (c_i), $i = 1, 2, 3$, where

(c_1) *if* $Au_i \to y_0$, $y_i \to y_0$ *and* $\pm\infty \neq \lim \lambda_i[f(u_i/\lambda_i) + r_i] = \lim \mu_i g(y_i/\mu_i)$, *where* $\lambda_i, \mu_i > 0$, $r_i \geq 0$, $\lambda_i, \mu_i \to 0$ *then* $y_0 = 0$ *and* $\lim \mu_i g(y_i/\mu_i) = 0$,

(c_2) *if* $Au_i \to y_0 \in \operatorname{dom} g_\infty$, $\lambda_i > 0$, $\lambda_i \to 0$ *and* $-\infty \neq \lim \lambda_i f(u_i/\lambda_i) \leq g_\infty(y_0)$, *then* $y_0 = 0$ *and* $\lim \lambda_i f(u_i/\lambda_i) = 0$,

(c_3) *if* $Au_i \to y_0 \in \operatorname{dom} g_\infty$, $\lambda_i > 0$, $\lambda_i \to 0$ *and* $-\infty \neq \lim \lambda_i[f(u_i/\lambda_i) + r_i] \leq g_\infty(y_0)$ *where* $r_i \geq 0$, *then* $y_0 = 0$ *and* $\lim \lambda_i[f(u_i/\lambda_i) + r_i] = 0$.

If A is the identity operator, the above three conditions are equivalent and can be expressed as

(c) *if* $f_\infty(x) \leq g_\infty(x)$ *with* $x \in \operatorname{dom} f_\infty \cap \operatorname{dom} g_\infty$, *then* $x = 0$.

Theorem 2.2. *The problems $(P_y)_{y \in Y}$ have solutions whenever val $P_y \neq -\infty$ and the value function is lower-semicontinuous if one of the following conditions holds:*

(i) M^1 and graph g are closed, the condition (c_1) is fulfilled and M^1 or graph g is asymptotically compact;

(ii) g is upper-semicontinuous, hypo g is asymptotically compact and

(1) M^2 is closed and (c_2) is fulfilled, or

(2) M^1 is closed and (c_3) is fulfilled,

(iii) g is upper-semicontinuous and

(1) M^2 is closed and asymptotically compact and (c_2) is fulfilled, or

(2) M^1 is closed asymptotically compact and (c_3) is fulfilled.

Remark 4. If a function is bounded from below (above) and its graph is closed, then it is lower (upper)-semicontinuous. On the other hand, if hypograph or epigraph of a function is asymptotically compact then its graph is also asymptotically compact. In particular, if N^1 is asymptotically compact it follows N^2 is asymptotically compact. Also, if dom g is asymptotically compact, then graph g and hypo g are asymptotically compact.

§3. In the special case of positively homogeneous functions all sets M^i, N^i are cones (with vertex at origin). Because of $K_\infty = K$, if K is a closed cone, and $f_\infty = f$ ($g_\infty = g$), if f(g) is lower (upper)-semicontinuous and positively homogeneous, the results of §2 become more simple.

Let us suppose in the following that

(A) *f and $-g$ are positively homogeneous lower-semicontinuous functions.*

In this case we have $f_\infty = f$ and $g_\infty = g$. Thus, if the sets M^i and N^i are closed then the coercivity conditions (c_i), $i = 1, 2, 3, 4$, and (c) may be replaced by:

(c_1^0) *if* $f(x) \leq g(Ax)$, *then* $Ax = 0$,

(c_2^0) = (c_3^0) *if* $f(x) \leq g(Ax)$, *then* $Ax = 0$ *and* $f(x) = 0$,

(c_4^0) *if* $f(x) \leq g(Ax)$, *then* $f(x) = g(Ax)$,

(c^0) *if* $f(x) \leq g(x)$, *then* $x = 0$,

or equivalently

(c_1^0) $f(x) > g(Ax)$ *if* $x \notin \ker A$,

(c_2^0) = (c_3^0) $f(x) > g(Ax)$ *for all* $x \notin \ker A \cap \ker f$,

(c_4^0) $f(x) \geq g(Ax)$ *for all* $x \in X$,

(c^0) $f(x) > g(x)$ *for all* $x \in X \setminus \{0\}$.

According to relations 2.1_{1-4} several sufficient conditions for closedness of H can be derived from Corollary 2.

Theorem 3.1. *The set H is closed if one of the following three conditions is fulfilled:*

(i) (c_1^0) and one of the cones M^1 or N^1 is locally compact and the other is closed,

(ii) (c_2^0) and

(1) hypo *g is locally compact and M^1 (or M^2) is closed, or*

(2) M^1 or M^2 is locally compact,

(iii) (c_4^0) and M^4 is closed.

Since the operators A_f and $\widetilde{A}$ are positively homogeneous using Corollary 1 we can obtain sufficient conditions in order that M^i be closed or locally compact. For this we consider the following stronger coercivity condition

$(\tilde{c}_1^0)$ $f(x) \leqslant g(Ax)$ *implies* $x=0$.

It is clear that $(\tilde{c}_1^0)$ is just (c_0) if g is replaced by $g \circ A$.

Theorem 3.2. *If the condition* $(\tilde{c}_1^0)$ *holds, then each of the following conditions is sufficient for closedness of* H :

(i) epi f (*or* hypo $g \circ A$) *is locally compact,*

(ii) graph f (*or* graph $g \circ A$) *is locally compact,*

(iii) dom f (*or* dom $g \circ A$) *is locally compact.*

Proof. (i) Consider $T=\tilde{A}$ and $E=$epi f in Corollary 1. It is clear that (c_2^0) and epi $f \cap \ker \tilde{A}=\{(0, 0)\}$ is equivalent to $(\tilde{c}_1^0)$. Since graph $\tilde{A}$ is closed and epi f is locally compact it follows that M^1 is locally compact. By Theorem 3.1 (ii_2) we obtain that H is closed. To prove the sufficiency of conditions (ii) and (iii) we observe that (iii)→(ii) and (ii)→(i). Indeed, epi f=graph $f+\{0\} \times R_+$ and graph $f=I_f$ (dom f), where I_f is the operator associated by (2.2) to the identity operator. Now, we apply Corollaries 2 and 1. The symmetrical form of Theorem follows replacing f, g by $-g \circ A$, $-f$ respectively.

Remark 5. According to Remark 3 the result can be extended if A is a positively homogeneous operator with closed graph.

Remark 6. If in addition the set H is convex, from Theorems 2.1, 2.2, 3.1 one can obtain sufficient conditions for dual stability. In particular, if

$$f(x)=\langle x, x_0^*\rangle+I_P(x), \quad g(y)=-I_Q(y), \tag{3.1}$$

where P and Q are two closed convex cones and I_P is the indicator function of the set P, we obtain the sufficient conditions for stability in infinite linear programming problems. Several results of this type are given by T. Nakamura-M. Yamasaky [13]. In fact, the proofs of the results of our paper are based on an idea of this authors via the result of J. Dedieu for non-convex case. In the special case above mentioned the conditions (c_1^0) become the conditions (F_i) from [13]. Other results of stability for linear case with the aid of the locally compactity were established by Ky Fan [7], U. Dieter [5], W. Krabs [8], K. Kretschmer [10].

Theorem 3.1 may be extended to the non-homogeneous case under analogous conditions for the recession functions taking into account Remark 2. Also, the conditions of local compactness (or asymptotical compactness) may be assured by the following dual interiority conditions: *the polar has a non-void interior with respect to Mackey topology.*

In the next we consider the following homogeneous program

(P_y') min $a(x)$
subject to $x \in P$, $Ax \in y+Q$,

where $a: X \to]-\infty, +\infty[$ is a positively homogeneous lower-semicontinuous function, $A: X \to Y$ is a linear continuous operator, y_0 is a fixed element of Y and $P \subset X$, $Q \subset Y$ are closed cones. This minimization problem is of P_y type for

$$f=a+I_P, \quad g=-I_Q$$

and so, as dual problem we have

(D_y') max $\langle y, y^*\rangle$
subject to $y^* \in Q^0$, $A^* y^* \in \partial a_P(0)$,

where $\partial a_P(0)$ is the subdifferential of a at origin with respect to P, i.e.

$$\partial a_P(0)=\{x^* \in X^* ;\ x^*(x) \leqslant a(x), \text{ for all } x \in P\}.$$

If $a \in X^*$ it is clear that $\partial a_P(0)=a+P^0$,
where P^0 is the polar of the cone P.

Also, we have

$$H=\{(Ax-y,\ a(x)+r) :\ x \in P \cap \operatorname{dom} a,\ y \in Q,\ r \geqslant 0\}. \tag{3.2}$$

The coercivity conditions (c_i), $i=1,4$, become

(H_1) *if* $x \in P \cap A^{-1}(Q)$ *and* $(a(x) \leqslant 0$, *then* $Ax=0$,
(H_2) = (H_3) *if* $x \in P \cap A^{-1}(Q)$ *and* $a(x) \leqslant 0$, *then* $Ax=0$ *and* $a(x)=0$,
(H_4) *if* $x \in P \cap A^{-1}(Q)$ *and* $a(x) \leqslant 0$, *then* $a(x)=0$.

From Theorem 3.1 we obtain the following result.

Theorem 3.3. *The homogeneous programs* $(P_y')_{y \in Y}$ *have solutions for every* $y \in Y$ *such that val* $P_y' \neq -\infty$ *and the value function* $y \to$ *val* P_y', $y \in Y$, *is lower-semicontinuous on* Y *if one of the following conditions holds :*

(i) Q *is locally compact and*

(1) (H_1) *is fulfilled and* $\tilde{A}$ (epi $a/_P$) *is closed or*

(2) (H_2) *is fulfilled and* $A_a(P \cap$ dom $a)$ *is closed.*

(ii) (H_1) *is fulfilled and* $\tilde{A}$ (*epi* $a/_P$) *is locally compact.*

(iii) (H_2) *is fulfilled and* A ($P \cap$dom a) *is locally compact.*

(iv) (H_4) *is fulfilled and* A_a ($P \cap$dom $a)-Q \times R$ *is locally compact.*

Proof. It is sufficient to observe that

$M^1=M^3=\{(Ax, a(x)+r) ;\ x \in P \cap \operatorname{dom} a,\ r \geqslant 0\}=\tilde{A}(\text{epi } a/_P)$, where $a/_P$ is the restriction of a to the cone P.

$M^2=\{(Ax, a(x)) ;\ x \in P \cap \operatorname{dom} a\}=\tilde{A}(\text{graph } a/_P)=A_a(P \cap \operatorname{dom} a)$, where A_a is defined by (2.2).

$N^1=Q \times \{0\}$, $N^2=N^3=Q \times R_-$, $N^4=\{0\} \times R_-$, $M^4=M^2-N^2$.

The local compactness which appear in conditions (ii—iv) can be derived from Corollary 1. In this may, the coercivity conditions (H_1—H_4) are replaced by the following stronger conditions

(H_0) *if* $x \in P \cap A^{-1}(Q)$ *and* $a(x) \leqslant 0$ *then* $x=0$,

which is analogous to the condition $(\tilde{c}_1^0)$.

Thus, if the condition (H_0) is fulfilled we also obtain the following three sufficient conditions.

(v) $P \cap$ dom a *is locally compact and graph* $a/_P$ *is closed in* $P \cap$ dom a,

(vi) graph $a/_P$ *is locally compact.*

(vii) epi $a/_P$ *is locally compact.*

If $a \in X^*$ the conditions of this type are given in [13]. In fact if a is a convex positively homogeneous function the above results can be also derived from [13] since the problem (P_y') is equivalent to the following problem on $X \times R$:

$\min \lambda$

Subject to $(x, \lambda) \in \operatorname{epi} a,\ x \in P,\ Ax \in y + Q.$

But this problem is linear and its constraints $(\operatorname{epi} a) \cap (P \times R)$ and Q are obviously convex closed cones in $X \times R$ and Y respectively.

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COHOMOLOGIE À VALEURS DANS UN FAISCEAU CELLULAIRE SIMPLE

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Les faisceaux cellulaires simples apparaissent dans la calcul des K_G-groupes de certains G-espaces ; ils ont été introduits dans [2], mais la notion générale a été ensuite dégagée dans [4].

Le but de cette note est de donner la cohomologie d'un simplexe euclidien quelconque, à valeurs dans un faisceau cellulaire simple.

1. Nous rappelons d'abord quelques définitions et résultats de [3]. On note ci-après par $X = [x_0, x_1, \ldots, x_n]$ le simplexe euclidien de dimension n (engendré par $x_0, x_1, \ldots, x_n$) et par $\mathcal{F}$ un faisceau de base X, à valeurs dans la catégorie Ab.

Définition 1.1. *Le faisceau $\mathcal{F}$ est dit cellulaire simple si, pour toute face ouverte Y de X, le faisceau induit par $\mathcal{F}$ dans Y est simple (i.e.: $\mathcal{F}_{|Y} = G_Y \in$ $\in \mathrm{Ob}\,|\,Ab$).*

Dans ce travail, on designe par $G_{i_0, \ldots, i_q}$ le faisceau induit par $\mathcal{F}$ dans la q-ème face ouverte $X_{i_0 \ldots i_q} = [x_{i_0}, \ldots, x_{i_q}]$, pour toutes les suites strictement croissantes $i_0 < \ldots < i_q$ d'entiers $\geqslant 0$ et $\leqslant n$.

Proposition 2.1. *Soient $\mathcal{F}$ un faisceau cellulaire simple de base X et Y, Y', Y'' trois faces ouvertes de X.*

a) *Si $Y \cap \overline{Y'} \neq \emptyset$ alors il existe un homomorphisme $f^{Y}_{Y'} : G_Y \to G_{Y'}$.*

b) *Si $Y' \cap \overline{Y''} \neq \emptyset$ et $Y \cap \overline{Y'} \cap \overline{Y''} \neq \emptyset$ alors on a $f^{Y'}_{Y''} \circ f^{Y}_{Y'} = f^{Y}_{Y''}$.*

Il est clair que la condition $Y \cap \overline{Y'} \neq \emptyset$ se traduit par le fait que Y est une face de Y'. Alors, pour $q < r$, on convient de noter les homomorphismes de la proposition 1.2 par :

$$f^{i_0 \ldots i_q}_{i_0 \ldots i_r} : G_{i_0 \ldots i_q} \to G_{i_0 \ldots i_r}$$

où $0 \leqslant i_0 < \ldots < i_q < \ldots < i_r < \ldots \leqslant n$.

2. Soit $\dot{X}^q$ la réunion des q-èmes faces fermées et $\mathring{X}^q$ la réunion des q-èmes faces ouvertes de X ; alors on a : $\dot{X}^0 = \{x_0, \ldots, x_n\}$, $\dot{X}^{n-1} = \mathrm{Fr}\,X$ (la frontière de X) et $\dot{X}^q - \dot{X}^{q-1} = \mathring{X}^q$ pour tout $q = 1, 2, \ldots, n-1$.

En utilisant la formule :

$$H^n_c(\mathbf{R}^m; G) = \begin{cases} G & \text{si } m = n \\ 0 & \text{si } m \neq n \end{cases} \tag{1}$$