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SOME APPLICATIONS OF THE REGULAR MULTIMEASURES

BY

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In an earlier paper [11] we have studied some continuity properties of the multimeasures and we have introduced the notion of regular multimeasure.

Since on every uniform semigroup, there exists a family of triangle functionals (see [9]) induced by a family of special pseudometrics which generates the uniformity of semigroup, we can associate to every multimeasure a variation with respect to every triangle functional.

In the present paper, we point out some regularity properties of multimeasures connected with their variations. As applications we obtain certain theorems of Nikodym-type for regular multimeasures and some results concerning the locally measurable sets with respect to a multimeasure.

§1. **Basic notions, terminology and notations.** Let Ω be a set and let $\mathcal{P}$ be the family of all subsets of Ω organized as a Boolean ring for the symmetric difference and intersection of sets, where the sum $\oplus$ in $\mathcal{P}$ is given by the symmetric difference and the product $\odot$ by the intersection. In this ring the empty set, $\emptyset$, is the null element and the set Ω is the unity. We shall also use the following notations:

$$A'=\Omega\setminus A,\ [0, A]=\{B\in\mathcal{P}\ ;\ B\subseteq A\},\ [A, B]=\{E\in\mathcal{P}\ ;\ A\subseteq E\subseteq B\},$$

$$A\oplus[0, B]=\{A\oplus C\ ;\ C\subseteq B\}=[A\setminus B,\ A\cup B]\ \text{for every}\ A,\ B\in\mathcal{P}.$$

Let $\mathcal{F}$ be the family of finite sets in Ω and $\mathcal{F}'=\{A\subset\Omega\ ;\ A'\in\mathcal{F}\}$, which is directed by inclusion. If Ω is infinite, then $\mathcal{F}'$ is a filter on Ω.

For every $A\in\mathcal{P}$, the family

$$\mathcal{B}(A)=\{A\oplus[0, F]\ ;\ F\in\mathcal{F}'\}$$

is a base of neighbourhoods of A for a topology $\mathcal{O}$ on $\mathcal{P}$ called the *order topology on* $\mathcal{P}$.

From [7] it follows:

(1.1) The family $\mathcal{W}(F)=\{(A, B)\ ;\ A\oplus B\subseteq F,\ F\in\mathcal{F}'\}$ is a base of vicinities in $\mathcal{P}$ for a uniformity $\mathcal{W}$ on Ω.

(1.2) The operations $(A, B)\to A\cup B$, $(A, B)\to A\setminus B$, $(A, B)\to A\cap B$, $(A, B)\to A\Delta B$ are uniformily continuous (The topology $\mathcal{O}$ agrees with the topology induced by the uniformity $\mathcal{W}$).

Thus $(\mathcal{P}, \oplus, \odot, \mathcal{O})$ is a topological ring of sets.

(1.3) $(\mathcal{P}, \oplus, \odot, O)$ is a totally disconnected compact Hausdorff topological ring.

(1.4) The family $\mathcal{F}$ is dense in $\mathcal{P}$.

(1.5) If $\mathcal{R}$ is a subring of $\mathcal{P}$ and $\mathcal{S}$ is the closure of $\mathcal{R}$ in O-topology, then $(\mathcal{S}, \oplus, \odot)$ is the smallest order complete Boolean subring.

Let X be a Hausdorff locally convex topological vector space, τ its topology and $\mathcal{V}$ a base of absolutely convex closed neighborhoods of the origin in X. Using notations as in [2] $\mathcal{A}(X)$ is the family of all nonempty subsets of X, and $\mathcal{C}(X)$, the family of closed sets of X, respectively. $\bar{A}$ denotes the closure of A in X. If we define the equivalence relation „ρ" on $\mathcal{A}(X)$ by

$$A\rho B \Rightarrow \bar{A}=\bar{B}, \tag{1}$$

then the factorization of the semigroup $(\mathcal{A}(X), +)$ by ρ, where $+$ is the usual addition, may be identified with $(\mathcal{C}(X), \dot{+})$, where the addition $\dot{+}$ is defined by

$$A\dot{+}B=\overline{A+B}. \tag{2}$$

In this way, Σ represents a finite sum or a series with respect to $\dot{+}$.

If $\mathcal{U}$ is the invariant uniformity on X compatible with $\tilde{\tau}$ then the uniformity $\tilde{\mathcal{U}}$ on $\mathcal{A}(X)$ defined by the following base of vicinities:

$$\mathcal{W}(U)=\{(A, B)\in\mathcal{A}(X)\times\mathcal{A}(X)\,;\ A\subset B\dot{+}U,\ B\subset A\dot{+}U\},\ U\in\mathcal{U}$$

is said to be *the exponential extension of* $\mathcal{U}$ to $\mathcal{A}(X)$.

The topology on $\mathcal{A}(X)$ induced by $\tilde{\mathcal{U}}$ will be denoted by $\tilde{\tau}$.

Using the equivalence relation (1) we may identify $(\mathcal{C}(X), \tilde{\tau})$ with $(\mathcal{A}(X), \tilde{\tau})/\rho$ and $(\mathcal{C}(X), \mathcal{U})$ with the separated uniform space associated with $(\mathcal{A}(X), \tilde{\mathcal{U}})$.

If X is metrizable, then $(\mathcal{A}(X), \tilde{\mathcal{U}})$ becomes a semi-metrizable space for the Hausdorff distance between sets. If in addition X is complete then so is $\mathcal{C}(X)$. If $(X, \mathcal{U})$ is a complete uniform space, then so is $(\mathcal{C}(X), \tilde{\mathcal{U}})$.

If (E_n) is a sequence of members of $\mathcal{A}(X)$ we say that the series $\sum_{n=1}^{\infty} E_n$ $\tilde{\tau}$-converges if the sequence $S_n=\sum_{k=1}^{n} E_k$, $\tilde{\tau}$-converges in $\mathcal{A}(X)$ and we write $(\tilde{\tau})\sum_{n=1}^{\infty} E_n$.

In what follows we shall consider correspondences μ from a ring $\mathcal{R}$ of subsets of Ω to X, that is set functions from $\mathcal{R}$ to $\mathcal{A}(X)$ or $\mathcal{C}(X)$. In fact a correspondence μ may be regarded as a function from $\mathcal{R}$ into either the semigroup $(\mathcal{A}(X), +)$ or, with the above convention about „ρ" which permits to identify two sets with the same closure, the semigroup $(\mathcal{C}(X), \dot{+})$.

Definition 1.6. *A correspondence μ from $\mathcal{R}$ to X is said to be:*

I) *additive* (respectively *subadditive*) *iff*

$$\mu(A\cup B)=(\text{resp.}\subset)\mu(A)\dot{+}\mu(B) \tag{3}$$

for every $A, B\in\mathcal{R}$ such that $A\cap B=\emptyset$.

II) *σ-additive* (resp. *σ-subadditive*) *iff*:

$$\mu\Big(\bigcup_{n=1}^{\infty} A_n\Big)=(\text{resp.}\subset)(\tilde{\tau})\sum_{n=1}^{\infty}\mu(A_n) \tag{4}$$

for every disjoint sequence $(A_n)_{n\in N}$ in $\mathcal{R}$ with $\bigcup_{n=1}^{\infty} A_n\in\mathcal{R}$.

Definition 1.7. *The correspondence μ from $\mathcal{R}$ to X is said to be exhaustive (locally exhaustive) iff $\mu(A_n)\to 0$ for every disjoint sequence $(A_n)\subset\mathcal{R}$ (for every $T\in\mathcal{R}$ and every disjoint sequence $(A_n)\subset\mathcal{R}$ such that $A_n\subset T$).*

Definition 1.8. *A correspondence from $\mathcal{R}$ to X is said to be a multimeasure iff*

a) $\mu(\emptyset)=0$, *where o denotes the set $\{0\}$ in $\mathcal{C}(X)$;*

b) *μ is additive.*

Definition 1.9. *The multimeasure μ is said to be o-continuous at $A\in\mathcal{R}$ (on $\mathcal{R}$) iff the function $\mu:(\mathcal{R}, O)\to(\mathcal{C}(X), \tilde{\tau})$ is continuous at A (on $\mathcal{R}$).*

§ 2. **Variation of an additive set function.** It is easy to remark that $(\mathcal{C}(X), \dot{+}\,;\tilde{\mathcal{U}})$ is a commutative uniform semi-group with the neutral element $\{0\}$. According to [9] the topology $\tilde{\tau}$ is generated by a family of pseudometrics. By virtue of theorem 1, § 1.5, [9] this family of pseudometrics generates a family $\mathcal{F}=\{f;\ f:\mathcal{C}(X)\to R\}$ of triangle functionals that is a family of functions f which satisfy the following three properties:

(F₁) $f(A\dot{+}B)\leqslant f(A)+(B)$, for every $A, B\in\mathcal{C}(X)$;

(F₂) $f(A\dot{+}B)\geqslant f(A)-f(B)$, for every $A, B\in\mathcal{C}(X)$;

(F₃) $f(0)=0$.

Remark. From the above conditions we easily obtain that f is non-negative.

Following E. P a p [10] we define the variation and the semi-variation of a multimeasure with respect to a triangle functional f.

Definition 2.1. *Let $\mu:\mathcal{R}\to\mathcal{C}(X)$ be a multimeasure and $f\in\mathcal{F}$. The variation of μ is the non-negative set function ν defined on $\mathcal{R}$ by*

$$\nu(E)=\sup_{\pi}\sum_{E_i\in\pi} f(\mu(E_i)),$$

where the supremum is extended over all partitions $\pi=\{E_i\}$ of E into a finite number of pairwise disjoint members of $\mathcal{R}$.

It is easy to prove the following properties of the variation:

$$\nu(\emptyset)=0\,; \tag{5}$$

$$f(\mu(E))\leqslant\nu(E) \text{ for every } E\in\mathcal{R}; \tag{6}$$

$$\nu(A)\leqslant\nu(B) \text{ for } A\subset B \text{ and } A, B\in\mathcal{R}\,; \tag{7}$$

(8) ν is an additive set function on $\mathcal{R}$ that is $\nu(A\cup B)=\nu(A)+\nu(B)$ for every $A,\ B\in\mathcal{R}$ with $A\cap B=\varnothing$.

(9) ν is a superadditive set function on $\mathcal{R}$ that is $\nu(\bigcup_{i\in I}E_i)\geq\sum_{i\in I}\nu(E_i)$ for every at most countable pairwise disjoint family

$$(E_i)_{i\in I}\subset\mathcal{R}\ \text{with}\ \bigcup_{i\in I}E_i\in\mathcal{R}.$$

Definition 2.2. *The multimeasure $\mu:\mathcal{R}\to\mathcal{C}(X)$ is said to be*
I) *of bounded variation iff $\nu(A)<\infty$ for every $A\in\mathcal{R}$;*
II) *complete iff $\nu(A)=0$ and $B\subset A\Rightarrow B\in\mathcal{R}$.*

Definition 2.3. *If μ is a multimeasure from $\mathcal{R}$ to $\mathcal{C}(X)$, the set function $\nu^*:\mathcal{P}\to\bar{R}_+$ ($\mathcal{P}$ — the family of all subsets of Ω) defined by*

(10) $$\nu^*(A)=\inf\{\nu(E)\ ;\ A\subset E\in\mathcal{R}\},\ \forall A\in\mathcal{P},$$

is said to be the outer measure induced by ν.

Definition 2.4. *If μ is a multimeasure from $\mathcal{R}$ to $\mathcal{C}(X)$ and $f\in\mathcal{F}$, the non-negative set function $|\cdot|$ defined on $\mathcal{R}$ by*

(11) $$|\mu|(A)=\sup\{f(\mu(T))\ ;\ T\subset A,\ T\in\mathcal{R}\},\ \forall A\in\mathcal{R},$$

is said to be the semi-variation of μ on $\mathcal{R}$.

Definition 2.5. *A correspondence μ from $\mathcal{R}$ to X is said to be f-superaditive iff*

$$f(\mu(\bigcup_{i\in I}E_i))\geq\sum_{i\in I}f(\mu(E_i))$$

for every at most countable pairwise disjoint family

$$(E_i)_{i\in I}\subset\mathcal{R}\ \text{with}\ \bigcup_{i\in I}E_i\in\mathcal{R}.$$

§ 3. **Regular multimeasures.** Let $(\Omega,\mathcal{T})$ be a Hausdorff locally compact space, $\mathcal{K}$, $\mathcal{K}_0$ respectively the family of compact, compact G_δ subsets of Ω, $\delta(\mathcal{K}_0)$ the δ-ring generated by $\mathcal{K}_0$ and $\mu:\mathcal{R}\to\mathcal{C}(X)$ a multimeasure.

Definition 3.1. I) *A set $A\in\mathcal{R}$ is said to be outer regular with respect to multimeasure μ iff for every vicinity $W\in\mathcal{U}$, there exists an open set $D\supset A$ such that $(\mu(E)\dot{+}T,\ T)\in W$ for every $E\in\mathcal{R}$ with $E\subset D\setminus A$, and every $T\in\mathcal{C}(X)$.*

II) *A set $A\in\mathcal{R}$ is said to be inner regular with respect to multimeasure μ iff for every vicinity $W\in\mathcal{U}$, there exists a compact set $K\subset A$ such that $(\mu(E)\dot{+}T,\ T)\in W$ for every $E\in\mathcal{R}$ with $E\subset A\setminus K$ and every $T\in\mathcal{C}(X)$.*

III) *A set $A\in\mathcal{R}$ is said to be regular with respect to multimeasure μ iff for every vicinity $W\in\mathcal{U}$ there exists an open set $D\supset A$ and a compact set $K\subset A$ such that $(\mu(E)\dot{+}T,\ T)\in W$ for every $E\in\mathcal{R}$ with $E\subset D\setminus K$ and every $T\in\mathcal{C}(X)$.*

Definition 3.2. *A multimeasure $\mu:\mathcal{R}\to\mathcal{C}(X)$ is said to be regular (outer regular, inner regular) iff every $A\in\mathcal{R}$ is a regular set (respectively outer regular, inner regular set).*

Example. If μ is a σ-additive and locally exhaustive multimeasure from $\delta(\mathcal{K}_0)$ to $\mathcal{C}(X)$, then by virtue of [6] (lemma 3.2) it follows that μ is a regular multimeasure.

Proposition 3.3. *If $A\in\mathcal{R}$ is an outer regular set with respect to multimeasure μ then for every vicinity $W\in\mathcal{U}$ there exists an open set $D\supset A$ such that $(\mu(S),\ \mu(A))\in W$ for every $S\in\mathcal{R}$ with $A\subset S\subset D$.*

Proof. Since $A\in\mathcal{R}$ is outer regular, for every $W\in\mathcal{U}$ there exists an open set $D\supset A$ such that

(12) $$(\mu(E)\dot{+}T,\ T)\in W$$

for every $E\in\mathcal{R}$ with $E\subset D\setminus A$ and every $T\in\mathcal{C}(X)$. Take $S\in\mathcal{R}$ such that $A\subset S\subset D$. Because $S\setminus A\subset D\setminus A$ and $S\setminus A\in\mathcal{R}$, from (12) we have

(13) $$(\mu(S\setminus A)\dot{+}T,\ T)\in W$$

for every $T\in\mathcal{C}(X)$. On the other hand, $S=A\cup(S\setminus A)$, where $A\cap(S\setminus A)=\varnothing$, which yields $\mu(S)=\mu(A)\dot{+}\mu(S\setminus A)$. Now writting (13) for $T=\mu(A)$ we have $(\mu(S),\ \mu(A))=(\mu(A)\dot{+}\mu(S\setminus A),\ \mu(A))\in W$ as claimed.

In the same way we can prove

Proposition 3.4. *If $A\in\mathcal{R}$ is an inner regular set with respect to multimeasure $\mu:\mathcal{R}\to\mathcal{C}(X)$, then for every vicinity $W\in\mathcal{U}$ there exists a compact set $K\subset A$ such that $(\mu(A),\ \mu(S))\in W$ for every $S\in\mathcal{R}$ with $K\subset S\subset A$.*

Example. If $\mathcal{P}$ is the family of all subsets of the Hausdorff locally compact space Ω, $\mathcal{F}$ the family of all finite sets in Ω and μ an o-continuous multimeasure from $\mathcal{P}$ to $\mathcal{C}(X)$ then μ satisfies the property mentioned in the proposition 3.4. Indeed, let $A\in\mathcal{P}$. Since μ is o-continuous at A it follows that for every $W\in\mathcal{U}$ there exists an O-neighborhood of A, that is an interval $[A\cap K,\ A\cup K']$, where $K\in\mathcal{F}$, such that $(\mu(S),\ \mu(A))\in W$ for every $S\in\mathcal{P}$ with $S\in[A\cap K,\ A\cup K']$.

But $K_0=A\cap K$ is again a compact set and $K_0\subset A\subset A\cup K'$.

Now it is easy to observe that $(\mu(S),\ \mu(A))\in W$ whenever $S\in\mathcal{R}$, $S\in[K_0,\ A]$.

Proposition 3.5. *If $A\in\mathcal{R}$ is a regular set with respect to the multimeasure $\mu:\mathcal{R}\to\mathcal{C}(X)$, then for every vicinity $W\in\mathcal{U}$ there exist an open set $D\supset A$ and a compact set $K\subset A$ such that $(\mu(S),\ \mu(A))\in W$ for every $S\in\mathcal{R}$ with $K\subset S\subset D$.*

Proof. If $W\in\mathcal{U}$ there exists a symmetric vicinity $W_1\in\mathcal{U}$ such that $W_1\circ W_1\subset W$.

From the regularity of $A\in\mathcal{R}$, for W_1 there exist $D\in\mathcal{T}$ and $K\in\mathcal{K}$ such that $K\subset A\subset D$ and

(14) $$(\mu(E)\dot{+}T,\ T)\in W_1,$$

for every $E\in\mathcal{R}$ with $E\subset D\setminus K$ and every $T\in\mathcal{C}(X)$.

Take $S\in\mathcal{R}$ with $K\subset S\subset D$. We see that $A\setminus S\subset D\setminus K$, $S\setminus A\subset D\setminus K$ and $S=(A\cap S)\cup(S\setminus A)$, where $A\setminus S,\ S\setminus A,\ A\cap S\in\mathcal{R}$.

From (14) we obtain $(\mu(A\setminus S)\dot{+}T, T)\in W_1$ and $(\mu(S\setminus A)\dot{+}T, T)\in W_1$ for every $T\in\mathcal{C}(X)$, so that

$$(15)\qquad (\mu(A\setminus S)\dot{+}T, \mu(S\setminus A)\dot{+}T)\in W_1 \circ W_1\subset W, \text{ for every } T\in\mathcal{C}(X).$$

If $T=\mu(A\cap S)$, from (15) we have

$$(\mu(A\setminus S)\dot{+}\mu(A\cap S), \mu(S\setminus A)\dot{+}\mu(S\cap A))=(\mu(A), \mu(S))\in W$$

as claimed.

Remark. It is easy to see that if $A\in\mathcal{R}$ is regular with respect to multimeasure μ, then the function μ defined on $\mathcal{R}$ with values in $(\mathcal{C}(X), \tilde{\mathcal{U}})$ is continuous at $A\in\mathcal{R}$ for the topology Γ on $\mathcal{R}$ which has as a base the family $\mathcal{B}=\{[K, D]\ ;\ K\in\mathcal{K},\ D\in\mathcal{T}\}$.

Theorem 3.6. *A set $A\in\mathcal{R}$ is regular with respect to multimeasure μ if and only if A is simultaneously outer regular and inner regular with respect to μ.*

Proof. „If part" is trivial.

To prove the *„only if part"* assume $A\in\mathcal{R}$ simultaneously outer and inner regular.

Let $W\in\tilde{\mathcal{U}}$ and $W_1\in\tilde{\mathcal{U}}$ such that $W_1 \circ W_1\subset W$. From the outer regularity of $A\in\mathcal{R}$ there exists an open set $D\supset A$ such that

$$(16)\qquad (\mu(E)\dot{+}T, T)\in W,$$

for every $E\subset D\setminus A$ and every $T\in\mathcal{C}(X)$.

By inner regularity of $A\in\mathcal{R}$ we obtain a compact set $K\subset A$ such that $(\mu(E)\dot{+}T, T)\in W_1$, for every $E\subset A\setminus K$ and every $T\in\mathcal{C}(X)$. Now let us take $S\in\mathcal{R}$ with $S\subset D\setminus K$. We observe that $S\setminus A\subset D\setminus A$, $S\cap A\subset A\setminus K$ and $S=(S\cap A)\cup(S\setminus A)$, where $(S\cap A)\cap(S\setminus A)=\emptyset$ and $S\setminus A$, $S\cap A\in\mathcal{R}$. From (15) and (16) we have

$$(15')\qquad \mu((S\setminus A)\dot{+}T, T)\in W_1, \text{ for every } T\in\mathcal{C}(X) \text{ and}$$

$$(16')\qquad \mu((S\cap A)\dot{+}T, T)\in W_1, \text{ for every } T\in\mathcal{C}(X).$$

Taking $T=\mu(S\cap A)$ in (15') we obtain

$$(17)\qquad (\mu(S), \mu(S\cap A))=(\mu S\setminus A)\dot{+}\mu(S\cap A), \mu(S\cap A))\in W_1,$$

which implies, by virtue of the invariance of $\tilde{\mathcal{U}}$,

$$(18)\qquad (\mu(S)\dot{+}T, \mu(S\cap A)\dot{+}T)\in W_1, \text{ for every } T\in\mathcal{C}(X).$$

Now from (16) and (18) it follows

$$(\mu(S)\dot{+}T, T)\in W_1 \circ W_1\subset W, \text{ for every } T\in\mathcal{C}(X),$$

which completes the proof.

Let $A\in\mathcal{R}$. We remark that the families $\{\mu(K)\ ;\ K\subset A, K\in\mathcal{R}\cap\mathcal{K}\}$ and $\{\mu(D)\ ;\ D\supset A, D\in\mathcal{T}\cap\mathcal{R}\}$ are nets with respect to inclusion.

If, moreover, the set $A\in\mathcal{R}$ satisfies the condition (*) „for every compact set $K\subset A$ there exists a compact set $K_0\in\mathcal{R}$ such that $K\subset K_0\subset A$", then we have.

Proposition 3.7. *If $A\in\mathcal{R}$ is inner regular with respect to the multimeasure μ and satisfies the property (*) then*

$$(19)\qquad \mu(A)=\lim\{\mu(K)\ ;\ K\subset A,\ K\in\mathcal{R}\cap\mathcal{K}\}.$$

Proof. If $A\in\mathcal{R}$ is inner regular, according to the proposition 3.4 for every $W\in\tilde{\mathcal{U}}$ there exists a compact set $K_0\subset A$ such that

$$(20)\qquad (\mu(A),\ \mu(S))\in W, \text{ for every } S\in\mathcal{R} \text{ with } K_0\subset S\subset A.$$

In particular (20) holds for every $K\in\mathcal{R}\cap\mathcal{K}$ such that $K_0\subset K\subset A$. Hence for every $W\in\tilde{\mathcal{U}}$ there is a $K_0\in\mathcal{K}$ such that $(\mu(A),\ \mu(K))\in W$ for every $K\in\mathcal{R}\cap\mathcal{K}$ with $K_0\subset K\subset A$, that is the inequality (19) holds.

Similarly if $A\in\mathcal{R}$ satisfies the property:

(**) „for every open set $D\supset A$ there exists an open set $D_0\in\mathcal{R}$ such that $A\subset D\subset D_0$"

we can prove

Proposition 3.8. *If $A\in\mathcal{R}$ is outer regular with respect to multimeasure μ and satisfies (**), then*

$$(21)\qquad \mu(A)=\lim\{\mu(D)\ ;\ D\supset A,\ D\in\mathcal{R}\cap\mathcal{T}\}.$$

Also the following result is true

Proposition 3.9. *If $A\in\mathcal{R}$ satisfies the properties (*) and (**) and it is regular with respect to the multimeasure μ, then*

$$(22)\qquad \mu(A)=\lim\{\mu(D)\ ;\ D\supset A,\ D\in R\cap T\}=\lim\{\mu(K)\ ;\ K\subset A,\ K\in K\cap R\}.$$

Remark. If $\mathcal{R}$ is the ring generated by the family $\mathcal{K}$ then every $A\in\mathcal{R}$ satisfies simultaneously (*) and (**) and hence every regular multimeasure μ on $\mathcal{R}$ satisfies (22).

Special case. If X is a real normed space and $\mathcal{C}_b(X)$ is the familly of all bounded closed sets of $\mathcal{A}(X)$ endowed with the Hausdorff metric D we define

$$\|A\|=D(A, \{0\})=D(A, 0),$$

for every $A\in\mathcal{C}_b(X)$.

In this case, taking into account the special form of the vicinities of $\tilde{\mathcal{U}}$, we observe that if $A\in\mathcal{R}$ is a regular set (respectively outer regular, inner regular) with respect to a multimeasure μ then the definition 3.1 can be restated as

Definition 3.10. *$A\in\mathcal{R}$ is said to be regular (outer regular, inner regular) with respect to the multimeasure μ if for every $\varepsilon>0$ there exist a compact set $K\subset A$ and an open set $D\supset A$ (respectively an open set $D\supset A$, a compact set $K\subset A$) such that $D(\mu(S)\dot{+}T, T)<\varepsilon$ for every $S\in\mathcal{R}$ with $S\subset D\setminus K$ (resp. $S\subset D\setminus A$, $S\subset A\setminus K$) and every $T\in\mathcal{C}_b(X)$.*

In this case the variation of μ is defined by

$$\nu(A)=\sup_{\pi}\{\sum_{E_i\in\pi}\|\mu(E_i)\|,\ \forall A\in\mathcal{R}\},\tag{23}$$

where the supremum is extended over all partitions $\pi=\{E_i\}$ of A into a finite number of pairwise disjoint members of $\mathcal{R}$.

It is clear that

$$\|\mu(A)\|\leqslant\nu(A),\ \text{for every } A\in\mathcal{R}.\tag{24}$$

Now assuming μ with bounded variation and taking into account (23) we obtain

Proposition 3.11. *If the variation ν of the multimeasure μ is regular (resp. outer regular, inner regular) then μ is regular (resp. outer regular, inner regular).*

Generally the converse does not hold except for the inner regularity. Indeed it is true the following

Theorem 3.12. *The multimeasure μ is inner regular if and only if the total variation ν is inner regular on $\mathcal{R}$.*

Proof „Only if part" is trivial by proposition 3.11

To prove „if part", consider $A\in\mathcal{R}$ inner regular. By the definition of ν we find for every $\varepsilon>0$ a finite partition $(A_i)_{i=1}^{n}\subset\mathcal{R}$ of A for which

$$\nu(A)<\sum_{i=1}^{n}\|\mu(A_i)\|+\varepsilon/2.\tag{25}$$

But μ is inner regular on every A_i and then according to proposition 3.4 there exists a compact $K_i\subset A_i$, $i=1,2,\dots n$, such that

$$D(\mu(A_i),\ \mu(S))<\varepsilon/2n,\tag{26}$$

for every $S\in\mathcal{R}$ with $K_i\subset S\subset A_i$, $i=1,2,\dots,n$.

Let us take $K=\bigcup_{i=1}^{n}K_i$. It is clear that K is compact and $K\subset A$.

If $E\in\mathcal{R}$ and $K\subset E\subset A$, let us denote $E_i=E\cap A_i$, $i=1,\dots n$. Obviously $E_i\in\mathcal{R}$, $K_i\subset E_i\subset A_i$ and $E_i\cap E_j=\varnothing$ for $i\neq j$.

Now using (25) and (26) we obtain

$$\nu(A)-\nu(E)<\sum_{i=1}^{n}\|\mu(A_i)\|+\varepsilon/2-\sum_{i=1}^{n}\|\mu(E_i)\|=\sum_{i=1}^{n}[\|\mu(A_i)\|-$$

$$-\|\mu(E_i)\|]+\varepsilon/2\leqslant\sum_{i=1}^{n}D(\mu(A_i),\ \mu(E_i))+\varepsilon/2<\varepsilon$$

which assures inner regularity of ν by virtue of proposition 2 of [1], p. 200. This completes the proof.

§ 4. Applications of the regular multimeasures. I) Theorems of Nikodym-type for multimeasures. Using the Diagonal Theorem, E. Pap [10] obtains two uniform-boundedness theorems for regular measures defined on the family $\mathcal{B}$ of Borel subsets of a compact Hausdorff topological space Ω taking values in a semigroup (An additive set function m on $\mathcal{B}$, taking

values in a semigroup S is regular in Pap's sense on $A\in\mathcal{B}$ iff for every $\varepsilon>0$ there exists a compact set K and an open set D such that $K\subset A\subset D$ and $|m|(D\setminus K)<\varepsilon$, where $|m|$ is the semi-variation of m.

In what follows we point out two results at the same type for regular multimeasures defined on the Borel subsets of a locally compact Hausdorff space Ω.

Let us denote by $\mathcal{B}$ the family of Borel sets of the locally compact Hausdorff space Ω, that is the σ-algebra generated by the open sets.

Theorem 4.1. *Let $(\mu_i)_{i\in I}$ be a family of regular f-super-additive multimeasures defined on $\mathcal{B}$. If for every open set $D\in\mathcal{T}$, $\{\mu_i(D)\,;\,i\in I\}$ is bounded, then the family $\{\mu_i\,;\,i\in I\}$ is uniformly bounded on $\mathcal{B}$, that is*

$$\sup\{f(\mu_i(A))\,;\,i\in I,\ A\in B\}<\infty.$$

The proof considerably uses the E. Pap's scheme given in [10] and therefore we do not reproduce it here. But we remark that in our case we use instead of the regularity in semi-variation of μ_i the following regularity property of $f\circ\mu_i$ (f is a triangle functional of the family $\mathcal{F}$ which generates the topology τ on $\mathcal{C}(X)$) obtained taking into account our regularity definition of the multimeasure $\mu_i (i\in I)$ and the continuity of f:

„if $A\in\mathcal{B}$ then for every $\varepsilon>0$ there exist an open set D and a compact set K such that $K\subset A\subset D$ and $|f(\mu(D))-f(\mu(K))|<\varepsilon$". We also precise that our condition on Ω to be a locally compact Hausdorff space does not essentialy modify Pap's proof. By virtue of theorem 4.1 exactly in the same way as in [10] we obtain a Nikodym-type theorem connected with the variation of the multimeasures.

Theorem 4.2. *Let $(\mu_i)_{i\in I}$ be a family of regular multimeasures defined on $\mathcal{B}$. If μ_i is of bounded variation on every open set of Ω for each $i\in I$, then $(\mu_i)_{i\in I}$ is uniformly bounded.*

Using our results from [11] concerning o-continuous multimeasures on a ring $\mathcal{R}$ of subsets of Ω and Pap's uniform boundedness theorems given in [8] we also obtain:

Theorem 4.3. *If $(\mu_n)_{n\in N}$ is a sequence of o-continuous multimeasures on the σ-ring $\mathcal{S}$ generated by the ring $\mathcal{R}$ and for every $A\in\mathcal{R}$ there exists a number $N(A)>0$ such that $f(\mu_n(A))\leqslant N(A)$, $\forall n\in N$, then $\{\mu_n\}_{n\in N}$ is uniformly bounded on $\mathcal{S}$ if and only if $\{\mu_n\}_{n\in N}$ satisfies the following two conditions:*

I) *for every sequence $\{A_n\}$ of pairwise disjoint sets of $\mathcal{R}$*

$$\sup_{n\in N}\{f(\mu_n(A_n))\}<\infty,\ \forall f\in\mathcal{F};$$

II) *for every $A\in\mathcal{S}$ there exists a set $T\in\{P\,;\,P=\bigcup_i A_i,\ A_i\in\mathcal{R}\}$ such that*

$$\sup_{n}\{f(\mu_n(A\setminus T))+f(\mu_n(T\setminus A))\}<\infty.$$

II) Locally measurable sets with respect to multimeasures. Let $\mathcal{R}$ be the σ-ring generated by the compact sets of locally compact Hausdorff space Ω and let X be a Banach space. Assume $\mu: R\to\mathcal{C}(X)$ a complete multimeasure with bounded variation and ν, ν^* the variation of μ and the outer measure induced by ν, respectively (with respect to a triangle functional f).

Definition 4.4. *A sequence $\{C_n\}$ of sets from $\mathcal{R}$ is said to be convergent to $x \in \Omega$ iff it satisfies the following two conditions:*

I) $x \in C_n$, for every $n \in N$ and

II) $\nu(C_n) \to 0$ for $n \to \infty$.

We denote by $\mathcal{E}(x)$ the family of all sequences convergent to x and by $\mathcal{E}$ the family of all sequences $\{C_n\}$ such that for each $x \in \Omega$ there exists at least one sequence in $\mathcal{E}(x)$.

Definition 4.5. *If $A \in \mathcal{P}$, the family of all sets of Ω, and $x \in \Omega$ we define by*

$$\overline{D}^*(A, x) = \sup\{\limsup_n \nu^*(A \cap C_n)/\nu(C_n)\ ;\ \{C_n\} \in \mathcal{E}(x)\} \tag{21}$$

the upper outer density of A at x and by

$$\underline{D}^*(A, x) = \inf\{\liminf_n \nu^*(A \cap C_n)/\nu(C_n)\ ;\ \{C_n\} \in \mathcal{E}(x)\} \tag{22}$$

the lower outer density of A at x, whenever

$$\nu(C_n) \neq 0 \quad \textit{and} \quad \nu^*(A \cap C_n)/\nu(C_n) = 0 \quad \textit{if} \quad \nu(C_n) = 0.$$

If $\overline{D}^*(A, x) = \underline{D}^*(A, x)$ we say that density of A exists at x and denote it by $D^*(A, x)$.

If $A \in \mathcal{R}$, then instead of $\overline{D}^*(A, x)$ or $\underline{D}^*(A, x)$ we put $\overline{D}(A, x)$, $\underline{D}(A, x)$ and in this case we speak of the upper and lower density of A at x.

If $\overline{D}(A, x) = \underline{D}(A, x)$, then we say that the density of A exists at x. The common value is denoted by $D(A, x)$. A point at which a set A has (outer) density one is called a *density point for* **A** and a point at which A has (outer) density zero a (outer) *dispersion point for* **A**.

As in [5] we can obtain the following properties of $\overline{D}^*(., x)$:

(I) if $A \subset B \Rightarrow \overline{D}^*(A, x) \leqslant \overline{D}^*(B, x)$;

(II) $\overline{D}^*(., x)$ is finitely subadditive;

(III) if $D(A, x)$ and $D(B, x)$ exist and $A \cap B = \varnothing$ then $D(A \cup B, x)$ exists and

$$D(A \cup B, x) = D(A, x) + D(B, x)\ ;$$

(IV) if $A \in \mathcal{R}$, then A has density at x iff

$$\underline{D}(A, x) + \overline{D}(\Omega \setminus A, x) = 1\ ;$$

(V) if a density theorem holds for $\mathcal{E}$, that is for every set A almost every point of A is an outer density point for A, then $A \in \mathcal{R}$ if and only if almost all points of $\Omega \setminus A$ are outer dispersion points for A.

(VI) Let Γ be the density topology of Ω. If a density theorem holds for $\mathcal{E}$, then $\Gamma \subset \mathcal{R}$.

Definition 4.6. *Let $A \subset \Omega$ and $x \in \Omega$. A is said to be locally measurable at x if there exist a F_σ-set F and a G_σ-set G such that $F \subset A \subset G$ and $D(G \setminus F, x) = 0$, where $D(G \setminus F, x)$ is the density of $G \setminus F$ at x.*

Let $\mathcal{M}(x)$ be the family of all locally measurable sets at x. If $A, B \in \mathcal{P}(\Omega)$ we write $A \in \mathcal{M}(B)$ if and only if $A \in \bigcup_{x \in B} \mathcal{M}(x)$.

Since in our case $\mathcal{R}$ is dense in $\mathcal{P}$ for the topology which has as a base the family $\mathcal{B} = \{[K, D]\ ;\ K \in \mathcal{K},\ D \in \mathcal{T}\}$ and μ is a regular multimeasure, by virtue of the considerations from § 3 the total variation ν of μ is regular on $\mathcal{R}$.

Taking into account this property of ν we immediately obtain from [4] the following results:

(I) The family $\mathcal{M}(x)$ of locally measurable sets at x is an algebra.

II) If $A \in \bigcup \mathcal{M}(x)$ for ν-almost every points of B, then $A \cap B \in \mathcal{R}$

III) If A, $B \in \mathcal{P}$ and $A \cap B \in \mathcal{R}$, then $A \in \mathcal{M}(B)$.

(IV) If $(B_n) \subset \mathcal{R}$, $\mu(B_n \Delta B) \to 0$ and $A \in \mathcal{M}(B_n)$ for every $n \in N$, then $A \in \bigcup \mathcal{M}(x)$ for ν-almost every point $x \in B$.

(V) $\mathcal{M}(B)$ is a σ-algebra.

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