Theorem 3. *Assume that the hypothes* (H_1) *and* (H_2) *hold. Then there exists a solution* $x(t)$ *of* BVP (10)—(11) *such that* $\alpha(t) \leq x(t) \leq \beta(t)$, *which may be computed by the method of successive approximations.*

Proof. The proof is similar to that of Theorem 8 in [5] but for the sake of completeness we shall outline the main steps It is easy to observe that the Grren's function $G(t, s)$ is nonpositive on $I \times I$. Define the operator T by

$$Tx(t) = \Phi(t) + \int_a^b G(t, s) f(s, x(s), \int_a^s k(s, \tau, x(\tau))d\tau)ds.$$

If $x(t) \leq z(t)$, then we obtain $Tx(t) \leq Tz(t)$ by hypothesis (H_2). Furthermore, since

$$T\beta''(t) = f(t, \beta(t), \int_a^t k(t, s, \beta(s)ds),$$

and since $T\beta(t)$ satisfies (11), the function $T\beta(t) - \beta(t)$ is a convex function which satisfies (16) with $A = B = 0$. From this we conclude that $T\beta(t) \leq \beta(t)$. Similarly we show that $\alpha(t) \leq T\alpha(t)$. Further we have $T\alpha(t) \leq T\beta(t)$ since T is monotone operator. We now define two sequences $\{x_n\}$ and $\{z_n\}$ by setting $x_n = Tx_{n-1}$, $z_n = Tz_{n-1}$, with $x_0 = \alpha$ and $z_0 = \beta$. It is then easy to see that $\alpha \leq x_1 \leq \ldots \leq x_n \leq z_n \leq \ldots \leq z_1 \leq \beta$.
Thus the sequences $\{x_n\}$ and $\{z_n\}$ are both bounded and monotone, and it is easy to verify the existence of the limits

$$\lim_{n \to \infty} x_n(t) = x(t) \leq z(t) = \lim_{n \to \infty} z_n(t),$$

and further that $Tx(t) = x(t)$, $Tz(t) = z(t)$. This completes the proof of the theorem.

Remark 3. In the case when $x(t) = z(t)$, then there exists exactly one solution whose graph stays between the graphs of α and β. We also note that existence and bounds on the solutions of BVP (3)—(4) by reducing it to two systems of more general forms of first order integrodifferential equations is established by the present author in [4] by using monotone method under slightly different conditions on the functions involved therein (see, also [3]).

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ON NORMAL SEMI-INVARIANT SUBMANIFOLDS IN SASAKIAN SPACE FORMS

BY

NECULAI PAPAGHIUC

1. **Introduction.** The geometry of classes of submanifolds in Sasakian manifolds is one of the most interesting topic of the differential geometry. Very recently, A. Bejancu and the present author introduced and studied in [2], [3] the semi-invariant submanifolds of a Sasakian manifold.

The purpose of the present paper is to obtain results on geometry of normal semi-invariant submanifolds in Sasakian space forms. In § 2 we give definitions and fundamental formulas for semi-invariant submanifolds of Sasakian manifolds. In § 3 there are given fundamental results on normal semi-invariant submanifolds of a Sasakian manifold. The main results of the paper are given in § 4 and they are concerned with normal semi-invariant submanifolds in Sasakian space forms (see Theorems 4.1—4.5).

2. **Preliminaries.** Let $\tilde{M}$ be a $(2n+1)$ — dimensional Sasakian manifold with the Sasakian structure (φ, ξ, η, g), where φ is a tensor field of type (1.1.) on $\tilde{M}$, ξ is a vector field, η is a 1-form and g is the Riemannian metric on $\tilde{M}$. These tensor fields are related by

(2.1) $\varphi^2 X = -X + \eta(X)\xi$; $\varphi\xi = 0$; $\eta(\varphi X) = 0$; $\eta(\xi) = 1$.

(2.2) $g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y)$ and

(2.3) $(\tilde{\nabla}_X \varphi)Y = g(X, Y)\xi - \eta(Y)X$

for any vector fields X, Y tangent to $\tilde{M}$, where $\tilde{\nabla}$ denotes the Riemannian connection on $\tilde{M}$. From (2.3) we also have

(2.4) $\tilde{\nabla}_X \xi = -\varphi X$ for any vector field X tangent to $\tilde{M}$.

Let M be an m-dimensional Riemannian manifold isometrically immersed in $\tilde{M}$ and suppose that the structure vector field ξ of $\tilde{M}$ is tangent to M. We denote by TM and $TM^\perp$ the tangent bundle to M and respectively the normal bundle to M. The submanifold M of $\tilde{M}$ is called a *semi-invariant submanifold* if it is endowed with the pair of distributions $(D, D^\perp)$ satisfying conditions :

(i) $TM = D \oplus D^\perp \oplus \{\xi\}$, where $\{\xi\}$ denotes the distribution spanned by ξ on M,

(ii) the distribution D is invariant by φ, that is, $\varphi(D_x) = D_x$ for each $x \in M$,

(iii) the distribution $D^\perp$ is anti-invariant by φ, that is, $\varphi(D_x^\perp)\subseteq T_x M^\perp$ for each $x\in M$, [2]. By using (2.1), (2.2) and the conditions (ii) and (iii) from the definition of a semi-invariant submanifold we obtain that the distributions D, $D^\perp$ and $\{\xi\}$ are mutually ortogonal to each other.

Throughout this paper we denote by $2p$ (resp. q) the dimension of D_x (resp. $D_x^\perp$). The semi-invariant submanifold M is called a *proper semi-invariant submanifold* if it is neither an invariant nor an anti-invariant submanifold, i.e. we have $p\neq 0$ and $q\neq 0$. A *generic semi-invariant submanifold* is a semi-invariant submanifold characterized by the condition $\dim T_x M^\perp = q$, for each $x\in M$, [2], [6]. We denote by the same symbol g both Riemannian metrics on M and $\tilde M$. The projection morphisms of TM to D and $D^\perp$ are denoted respectively by P and Q. For any vector bundle H on M we denote by $\Gamma(H)$ the module of all differentiable sections of H.

The Gauss and Weingarten formulas for the submanifold M of $\tilde M$ are given respectively by

(2.5) $\tilde\nabla_X Y=\nabla_X Y+h(X, Y)$ and (2.6) $\tilde\nabla_X N=-A_N X+\nabla_X^\perp N$

for any $X, Y\in\Gamma(TM)$ and $N\in\Gamma(TM^\perp)$, where ∇ is the Levi-Civita connection on M, $\nabla^\perp$ is the linear connection induced by $\tilde\nabla$ on $TM^\perp$, h is the second fundamental form of M and A_N is the fundamental tensor of Weingarten with respect to the normal section N. The fundamental tensor A_N is related to h by

(2.7) $g(h(X, Y), N)=g(A_N X, Y)$ for any $X, Y\in\Gamma(TM)$ and $N\in\Gamma(TM^\perp)$.

For all $X\in\Gamma(TM)$ we have

(2.8) $X=PX+QX+\eta(X)\xi$, where $\eta(X)=g(X, \xi)$. Then, by applying φ to (2.8) we obtain

$$\varphi X=\varphi PX+\varphi QX, \text{ for all } X\in\Gamma(TM). \tag{2.9}$$

For any $N\in\Gamma(TM^\perp)$ we put

$$\varphi N=BN+CN \tag{2.10}$$

where BN and CN are the tangential and normal components of φN, respectively.

Now, suppose $\tilde M(k)$ be a Sasakian space form of constant φ-sectional curvature k. Then, the curvature tensor $\tilde R$ of $\tilde M(k)$ is given by ([4], p. 97),

$$\begin{aligned}\tilde R(X, Y)Z=&\frac{k+3}{4}\{g(Y, Z)X-g(X, Z)Y\}+\frac{k-1}{4}\{\eta(X)\eta(Z)Y-\\ &-\eta(Y)\eta(Z)X+g(X, Z)\eta(Y)\xi-g(Y, Z)\eta(X)\xi+g(Z, \varphi Y)\varphi X-\\ &-g(Z, \varphi X)\varphi Y+2g(X, \varphi Y)\varphi Z\}\end{aligned} \tag{2.11}$$

for any vector fields Y, X, Z tangent to $\tilde M(k)$.

We denote by $R^{2n+1}(-3)$ the Sasakian space form with constant φ-sectional curvature $k=-3$ and with standard Sasakian structure in the Euclidean space R^{2n+1} (see [4], p. 99). By $S^{2n+1}(1)$ we denote a $(2n+1)$-dimensional unit sphere endowed with the natural Sasakian structure of constant φ-sectional curvature $k=1$ (see [4], p. 99).

For the second fundamental form h of a submanifold M in $\tilde M$, we define the covariant differentiation $\bar\nabla$ by

$$(\bar\nabla_X h)(Y, Z)=\nabla_X^\perp h(Y, Z)-h(\nabla_X Y, Z)-h(Y, \nabla_X Z), \text{ for any } X, Y, Z\in\Gamma(TM). \tag{2.12}$$

The equations of Gauss and Codazzi are then given respectively by

$$R(X, Y; Z, W)=\tilde R(X, Y; Z, W)+g(h(X, W), h(Y, Z))-g(h(X, Z), h(Y, W)) \text{ and} \tag{2.13}$$

$$(\tilde R(X, Y)Z)^\perp=(\bar\nabla_X h)(Y, Z)-(\bar\nabla_Y h)(X, Z) \tag{2.14}$$

for any $X, Y, Z, W\in\Gamma(TM)$, where $R(X, Y; Z, W)=g(R(X, Y)Z, W)$ and $(\tilde R(X, Y)Z)^\perp$ is the normal component of $\tilde R(X, Y)Z$.

For a semi-invariant submanifold M of the Sasakian manifold $\tilde M$ we denote by ν the orthogonal complementary vector bundle to $\varphi(D^\perp)$ in $TM^\perp$. Thus, the normal bundle to M has the decomposition

$$TM^\perp=\varphi(D^\perp)\oplus\nu, \text{ where } g(\varphi(D^\perp), \nu)=0. \tag{2.15}$$

3. Normal semi-invariant submanifolds of a Sasakian manifold. In [3], A. Bejancu and the present author have defined and studied the normal semi-invariant submanifolds of a Sasakian manifold. We recall here the definition and some fundamental results which we need in the sequel.

Let M be a proper semi-invariant submanifold of the Sasakian manifold $\tilde M$. We define the tensor field F on M of type (1, 1) by $FX=\varphi PX$ for any $X\in\Gamma(TM)$. Also, we define the non-null normal bundle valued 1-form u on M by $uY=\varphi QY$, for any $Y\in\Gamma(TM)$. Then, the triplet (F, u, η) is called *the semi-invariant structure* on M. By means of this structure, we define the tensor field S on M by

$$S(X, Y)=N_F(X, Y)-2Bdu(X, Y)+2d\eta(X, Y) \tag{3.1}$$

for all $X, Y\in\Gamma(TM)$, where N_F denotes the Nijenhuis tensor field of F given by

$$N_F(X, Y)=F^2[X, Y]+[FX, FY]-F[FX, Y]-F[X, FY] \tag{3.2}$$

and du and $d\eta$ are the exterior differentials of u and η, respectively. The tensor field S is called *the torsion tensor* of the semi-invariant stucture (F, u, η). The semi-invariant submanifold M of the Sasakian manifold $\tilde M$ is called a *normal semi-invariant submanifold* if M is a proper semi-invariant submanifold and the torsion tensor S vanishes identically on M.

We recall for later use the following fundamental results from [3].

Theorem 3.1. *Let M be a proper semi-invariant submanifold of the Sasakian manifold $\tilde M$. Then M is a normal semi-invariant submanifold if and only if we have*

(3.3) $\quad A_{uY}(FX)=F(A_{uY}X)$, *forr all* $X\in\Gamma(D)$ *and* $Y\in\Gamma(D^{\perp})$.

Corollary 3.1. *A proper semi-invariant submanifold M of a Sasakian manifold $\tilde{M}$ is a normal semi-invariant submanifold if and only if we have*

(3.4) $\quad h(X, FY)+h(FX, Y)\in\Gamma(\nu)$ *for all* $X, Y\in\Gamma(D)$ *and*

(3.5) $\quad h(X, Z)\in\Gamma(\nu)$ *for all* $X\in\Gamma(D)$ *and* $Z\in\Gamma(D^{\perp})$.

Corollary 3.2. *A generic semi-invariant submanifold is normal if and only if we have*

(3.6.) $\quad h(X, FY)+h(FX, Y)=0$ *for all* $X,Y\in\Gamma(D)$ *and*

(3.7) $\quad h(X, Z)=0$ *for any* $X\in\Gamma(D)$ *and* $Z\in\Gamma(D^{\perp})$.

We say that the semi-invariant submanifold M of a Sasakian manifold $\tilde{M}$ is a *semi-invariant product* if the distribution $D\oplus\{\xi\}$ is integrable and locally M is a Riemannian product $M_1\times M_2$ where M_1 (resp. M_2) is a leaf of $D\oplus\{\xi\}$ (resp. $D^{\perp}$).

Corollary 3.3. [3]. *Each semi-invariant product of a Sasakian manifold is a normal semi-invariant submanifold.*

Theorem 3.2. [3]. *A normal semi-invariant submanifold of a Sasakian manifcld is a semi-invariant product if and only if the distribution $D\oplus\{\xi\}$ is integrable.*

4. Normal semi-invariant submanifolds in Sasakian space forms.

Suppose $\tilde{M}(k)$ be a Sasakian space form of constant φ-sectional curvature k and M be a semi-invariant submanifold of $\tilde{M}(k)$. Then, from (2.11) and (2.14) we obtain that the equation of Codazzi for the semi-invariant submanifold M of $\tilde{M}(k)$ is given by

$$(\bar{\nabla}_X h)(Y, Z)-(\bar{\nabla}_Y h)(X, Z)=\frac{k-1}{4}\, g(Z, FY)uX-g(Z, FX)uY+ \qquad (4.1)$$

$$+2g(X,FY)uZ\}$$

for any $X, Y, Z\in\Gamma(TM)$, where $\bar{\nabla}_X h$ is defined by (2.12).

First, we recall the following lemmas:

Lemma 4.1. [2]. *Let M be a semi-invariant submanifold of a Sasakian manifold $\tilde{M}$. Then we have*

(4.2) $\quad P\nabla_X FY=F\nabla_X Y+PA_{uY}X-\eta(Y)PX,$

(4.3) $\quad Q\nabla_X FY=QA_{uY}X+Bh(X, Y)-\eta(Y)QX,$

(4.4.) $\quad \eta(\nabla_X FY)=\eta(A_{uY}X)+g(\varphi X, \varphi Y),$

(4.5) $\quad h(X, FY)+\nabla^{\perp}_X uY=Ch(X, Y)+u\nabla_X Y$

for amy $X,Y\in\Gamma(TM)$.

Lemma 4.2. [2]. *Let M be a semi-invariant submanifold of a Sasakian manifold $\tilde{M}$. Then we have*

(4.6) $\quad h(X, \xi)=-uX$ *and* $\nabla_X\xi=-FX$ *for any* $X\in\Gamma(TM)$ *and*

(4.7) $\quad A_{\varphi X}Y=A_{\varphi Y}X$ *for any* $X, Y\in\Gamma(D^{\perp})$.

Now, we state the following fundamental result:

Lemma 4.3. *Let M be a normal semi-invariant submanifold of the Sasakian manifold $\tilde{M}$. Then, the curvature tensor $\tilde{R}$ of M satisfies*

$$\tilde{R}(X, \varphi X\,; Z, \varphi Z)=-2\|h(X, Z)\|^2+2g(Bh(X, X), Bh(Z, Z))- \qquad (4.8)$$

$$-2\|A\varphi_Z X\|^2+2$$

for all unit vectors $X\in\Gamma(D)$ *and* $Z\in\Gamma(D^{\perp})$.

Proof. By using (2.12) and (2.14) we have

$$\tilde{R}(X, \varphi X\,; Z, \varphi Z)=g(\nabla^{\perp}_X h(\varphi X,Z)-\nabla^{\perp}_{\varphi X}h(X, Z), \varphi Z)+ \qquad (4.9)$$

$$+g(h([\varphi X, X], Z),\varphi Z)+g(h(X, \nabla_{\varphi X}Z)-h(\varphi X, \Delta_X Z), \varphi Z)$$

for all $X\in\Gamma(D)$ and $Z\in\Gamma(D^{\perp})$. From (2.3) and (2.5) we get

(4.10) $\quad h(\varphi X, Z)=\tilde{\nabla}_Z\varphi X-\nabla_Z\varphi X=\varphi\tilde{\nabla}_Z X-\nabla_Z\varphi X=\varphi\nabla_Z X+\varphi h(X, Z)-\nabla_Z\varphi X.$

By taking the components from ν in (4.10) and by using (3.5) we get

(4.11) $\quad h(\varphi X, Z)=\varphi h(X, Z)$ for all $X\in\Gamma(D)$ and $Z\in\Gamma(D^{\perp})$.

Now, by using (2.3), (2.5), (2.6), (3.5) and (4.11) we obtain

$$g(\nabla^{\perp}_X h(\varphi X, Z)-\nabla^{\perp}_{\varphi X}h(X, Z), \varphi Z)=g(h(X, Z), \tilde{\nabla}_{\varphi X}\varphi Z)- \qquad (4.12)$$

$$-g(h(\varphi X,Z), \tilde{\nabla}_X\varphi Z)=g(h(X,Z),\varphi h(\varphi X,Z))-g(h(\varphi X,Z),\varphi h(X,Z))=$$

$$=-2\|h(X, Z)\|^2$$

for all $X\in\Gamma(D)$ and $Z\in\Gamma(D^{\perp})$. Also, by using (2.8), (4.11)), (3.5), (4.3) and (4.6) we get

$$g(h([\varphi X, X], Z), \varphi Z)=g(h(Q[\varphi X, X]+\eta([\varphi X, X])\xi, Z), \varphi Z)= \qquad (4.13)$$

$$=-g(h(Bh(\varphi X, \varphi X)+Bh(X, X), Z), \varphi Z)+2$$

for all unit vectors $X\in\Gamma(D)$ and $Z\in\Gamma(D^{\perp})$. In (3.4) we take $Y=\varphi X\in\Gamma(D)$ and obtain

(4.14) $\quad Bh(\varphi X, \varphi X)=Bh(X, X)$ for all $X\in\Gamma(D)$.

Substituting (4.14) in (4.13) we have

(4.15) $\quad g(h([\varphi X, X], Z), \varphi Z)=-2g(h(Bh(X, X). Z), \varphi Z)+2$

for all unit vectors $X\in\Gamma(D)$ and $Z\in\Gamma(D^{\perp})$. On the other hand, by using (2.7) and (4.3) we have

(4.16) $\quad g(h(Bh(X, X), Z), \varphi Z)=g(QA_{\varphi Z}Z, Bh(X, X))=-g(Bh(Z, Z), Bh(X, X))$

for any $X \in \Gamma(D)$ and $Z \in \Gamma(D^{\perp})$. Then, substituting (4.16) in 4.15) we obtain

$$g(h([\varphi X, X], Z), \varphi Z) = 2g(Bh(X, X), Bh(Z, Z)) + 2 \tag{4.17}$$

for all unit vectors $X \in \Gamma(D)$ and $Z \in \Gamma(D^{\perp})$.

Finally, by using (3.4), (3.5), (2.7), (4.2) and (3.3) we get

$$\begin{aligned} g(h(X, \nabla_{\varphi X} Z) - h(\varphi X, \nabla_X Z), \varphi Z) &= g(h(X, \varphi P \nabla_X Z + P \nabla_{\varphi X} Z), \varphi Z) = \\ &= g(A_{\varphi Z} X, \varphi P \nabla_X Z + P \nabla_{\varphi X} Z) = g(A_{\varphi Z} X, -PA_{\varphi Z} X + P \nabla_{\varphi X} Z) = \\ &= g(A_{\varphi Z} X, -PA_{\varphi Z} X - \varphi^2 P \nabla_{\varphi X} Z) = g(A_{\varphi Z} X, -PA_{\varphi Z} X + \\ &+ \varphi P A_{\varphi Z} \varphi X) = g(A_{\varphi Z} X, -2PA_{\varphi Z} X) = -2\|PA_{\varphi Z} X\|^2 = -2\|A_{\varphi Z} X\|^2 \end{aligned} \tag{4.18}$$

for all $X \in \Gamma(D)$ and $Z \in \Gamma(D^{\perp})$. Thus, (4.8) follows from (4.9), (4.12), (4.17) and (4.18) and the proof is complete.

In the particular case of Sasakian space forms, from (2.14), (4.1) and Lemma 4.3 we have the following

Proposition 4.1. *Let M be a normal semi-invariant submanifold of a Sasakian space form $\tilde{M}(k)$. Then we have*

$$\|h(X, Z)\|^2 + \|A_{\varphi Z} X\|^2 - g(Bh(X, X), Bh(Z, Z)) = (k+3)/4. \tag{4.19}$$

for all unit vectors $X \in \Gamma(D)$ and $Z \in \Gamma(D^{\perp})$.

We say that a semi-invariant submanifold M of a Sasakian manifold $\tilde{M}$ is $(D, D^{\perp})$-*geodesic* respectively $D^{\perp}$-*geodesic* if we have

$$h(X, Z) = 0 \text{ for all } X \in \Gamma(D) \text{ and } Z \in \Gamma(D^{\perp}), \text{ respectively} \tag{4.20}$$

$$h(Y, Z) = 0 \text{ for all } Y, Z \in \Gamma(D^{\perp}). \tag{4.21}$$

From (4.21) and Proposition 4.1 we have

Theorem 4.1. *Let M be a normal semi-invariant submanifold of a Sasakian space form $\tilde{M}(k)$ and suppose M be $D^{\perp}$-geodesic. Then we have $k \geqslant -3$.*

We recall the following

Lemma 4.4, [2]. *Let M be a semi-invariant submanifold of a Sasakian manifold $\tilde{M}$. Then the distribution $D \oplus \{\xi\}$ is integrable if and only if we have*

$$h(X, \varphi Y) - h(\varphi X, Y) \in \Gamma(\nu) \text{ for any } X, Y \in \Gamma(D). \tag{4.22}$$

Now, we state

Theorem 4.2. *Let M be a normal semi-invariant submanifold of a Sasakian space form $\tilde{M}(k)$. If the distribution $D \oplus \{\xi\}$ is integrable then $k \geqslant -3$ and M is a semi-invariant product.*

Proof. Since M is a normal semi-invariant submanifold and the distribution $D \oplus \{\xi\}$ is integrable, from (3.4) and (4.22) we obtain that $h(X, Y) \in \Gamma(\nu)$ for any $X, Y \in \Gamma(D)$, i.e. we have $Bh(X, X) = 0$ for any $X \in \Gamma(D)$. Then (4.19) implies $k \geqslant -3$. On the other hand Theorem 3.2 implies that M is a semi-invariant product and the proof is complete.

Theorem 4.3. *Let M be a normal semi-invariant submanifold in $R^{2n+1}(-3)$ and suppose that M is $(D, D^{\perp})$-geodesic and $D^{\perp}$-geodesic. Then M is a semi-invariant product.*

Proof. From (4.20), (4.21) and (4.19) it follows that $A_{\varphi Z} X = 0$ for any $X \in \Gamma(D)$ and $Z \in \Gamma(D^{\perp})$, which implies that the distributions $D \oplus \{\xi\}$ is integrable and the assertion follows from Theorem 4.2.

From Theorem 4.3 and Corollary 3.2 we have

Corollary 4.1. *Let M be a normal, generic and $D^{\perp}$-geodesic semi-invariant submanifold of $R^{2n+1}(-3)$. Then M is a semi-invariant product.*

Theorem 4.4. *Let M be a normal semi-invariant submanifold of $S^{2n+1}(1)$ and suppose the invariant distribution $D \oplus \{\xi\}$ is integrable. Then the following assertions hold:*

(a) *M is a semi-invariant product $M_1 \times M_2$,*

(b) *$\|h(X, Z)\| = 1$ for any unit vectors $X \in \Gamma(D)$ and $Z \in \Gamma(D^{\perp})$,*

(c) *$n \geqslant p + q + pq$, where $2p = \dim. D_x$ and $q = \dim. D_x^{\perp}$,*

(d) *if $n = p + q + pq$ then M_1 is an invariant submanifold totally geodesic immersed in $S^{2n+1}(1)$.*

(e) *$\|h\|^2 \geqslant 2q(2p+1)$,*

(f) *$\|h\|^2 = 2q(2p+1)$ if and only if M_1 and M_2 are totally geodesic immersed in $S^{2n+1}(1)$, M_1 is a Sasakian space form of constant φ-sectional curvature $k = 1$ and M_2 is a real space form of constant sectional curvature $c = 1$,*

(g) *if the submanifold M is minimal, then the scalar curvature satisfies $\rho \leqslant 4p^2 + q^2 + 2p - q$ and the equality sign holds if and only if $\|h\|^2 = 2q(2p+1)$.*

Proof. The assertion (a) follows from Theorem 4.2. Since M is a semi-invariant product, from Corollary 1 of [1] we have

$$A_{\varphi Z} X = 0 \text{ for all } X \in \Gamma(D) \text{ and } Z \in \Gamma(D^{\perp}). \tag{4.23}$$

Also, since M is a normal semi-invariant submanifold and the distribution $D \oplus \{\xi\}$ is integrable, from (3.4) and (4.22) we obtain $h(X, \varphi Y) \in \Gamma(\nu)$ for any $X, Y \in \Gamma(D)$, i.e. we have

$$Bh(X, Y) = 0 \text{ for any } X, Y \in \Gamma(D). \tag{4.24}$$

Then the assertion (b) follows from (4.23), (4.24) and (4.19) for $k = 1$. The assertions (c), (d), (e), (f) and (g) are then consequences of Theorems 4.1, 4.2, 4.4 and 4.5 from [5].

Theorem 4.5. *Let M be a $(D, D^{\perp})$-geodesic and $D^{\perp}$-geodesic normal semi-invariant submanifold of $S^{2n+1}(1)$. Then we have*

(i) *$\|A_{\varphi Z} X\| = 1$ for all unit vectors $X \in \Gamma(D)$ and $Z \in \Gamma(D^{\perp})$,*

(ii) *$\|h\|^2 \geqslant 2q(p+1)$,*

(iii) *$\|h\|^2 = 2q(p+1)$ if and only if $h(X, Y) \in \Gamma(\varphi D^{\perp})$ for any $X, Y \in \Gamma(D)$.*

Proof. Since M is a normal, $(D. D^{\perp})$-geodesic and $D^{\perp}$-geodesic semi-invariant submanifold and $k = 1$, the assertion (i) is an immediate consequence of (4.19). Now, since M is $(D, D^{\perp})$-geodesic and $D^{\perp}$-geodesic, by using (2.7), (4.6), (4.20) and (4.21) we get

$$\begin{aligned} &A_{\varphi Z} \xi = -Z \,;\; PA_{\varphi Z} Y = 0 \,;\; QA_{\varphi Z} Y = 0 \,;\; \eta(A_{\varphi Z} Y) = -g(Y, Z) \,; \\ &QA_{\varphi Z} X = 0 \,;\; \eta(A_{\varphi Z} X) = 0 \,;\; QA_N Y = 0 \,;\; \eta(A_N Y) = 0 \end{aligned} \tag{4.25}$$

for any $X \in \Gamma(D)$, $Y, Z \in \Gamma(D^{\perp})$ and $N \in \Gamma(\nu)$.

On the other hand, it is known that the length of the second fundamental form h of the submanifold M is given by

$$\|h\|^2=\sum_{\alpha,\beta=1}^{m}\|h(E_\alpha : E_\beta)\|^2=\sum_{\alpha=1}^{m}\sum_{a=1}^{2n+1-m}\|A_aE_\alpha\|^2, \tag{4.26}$$

where $\{E_1, \dots, E_{m-1}, E_m=\xi\}$ (resp. $\{N_1, \dots, N_{2n+1-m}\}$) is a local field of orthonormal frames on TM (resp. $TM^\perp$).

We choose the local field of orthonormal frames

$$\{E_1, \dots, E_p, E_{p+1}=\varphi E_1, \dots E_{2p}=\varphi E_p, E_{2p+1}, \dots E_{2p+q}, E_m=\xi\} \tag{4.27}$$

on TM such that $E_i\in\Gamma(D)$ $(i=1, \dots, p)$ and $E_{2p+j}\in\Gamma(D^\perp)$ $(j=1, \dots, q)$. Also, we choose the local field of orthonormal sections

$$\{N_1=\varphi E_{2p+1}, \dots, N_q=\varphi E_{2p+q}, V_1, \dots, V_s, V_{s+1}=\varphi V_1, \dots, V_{2s}=\varphi V_s\} \tag{4.28}$$

on $TM^\perp$, where $V_r\in\Gamma(\nu)$ $(r=1, \dots, s)$ and $2s=\dim.\, \nu$.

Then by using (4.25), (4.27) and (4.28) in (4.26) we obtain

$$\|h\|^2=\sum_{i=1}^{2p}\sum_{j=1}^{q}\|A_jE_i\|^2+2\sum_{j=1}^{q}\|E_{2p+j}\|^2+\sum_{i=1}^{2p}\sum_{t=1}^{2s}\|PA_tE_i\|^2, \tag{4.29}$$

where $A_j=A_{\varphi E_{2p+j}}$ and $A_t=A_{V_t}$, $j=1, \dots, q$ and $t=1, \dots, 2s$. Thus, the assertion (i) of Theorem 4.5 and (4.29) imply

$$\|h\|^2=2q(p+1)+\sum_{i=1}^{2p}\sum_{t=1}^{2s}\|PA_tE_i\|^2 \tag{4.30}$$

and the assertion (ii) it follows from (4.30).

Also, from (4.30) we obtain that $\|h\|^2=2q(p+1)$ if and only if we have $PA_tE_i=0$ for all $i=1, \dots, 2p$ and $t=1, \dots, 2s$, i.e. $h(X, Y)\in\Gamma(\varphi D^\perp)$ for all $X, Y\in\Gamma(D)$. The proof of Theorem 4.5 is complete.

From Theorem 4.5 and Corollary 3.2 we obtain

Corollary 4.2. *Let M be a normal, generic and $D^\perp$-geodesic semi-invariant submanifold of $S^{2n+1}(1)$. Then we have*

(i) $\|A_{\varphi Z}X\|=1$ *for any unit vectors* $X\in\Gamma(D)$ *and* $Z\in\Gamma(D^\perp)$ *and*

(ii) $\|h\|^2=2q(p+1)$.

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ON ALMOST CONTACT METRIC SUBMERSIONS

BY

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B. Watson [13] and E. De la Torre [11] have examined the differential geometric properties of almost complex Riemannian submersions between almost Hermitian manifolds (almost Hermitian submersions). In this connection, the purpose of this paper is to study the properties of almost contact Riemannian submersions between almost contact metric manifolds (almost contact metric submersions). For it, we use the Gray-Hervella's classification of almost Hermitian manifolds [6] and the classification for almost contact metric structures obtained by J.A. Oubiña [8].

First of all we recall in §1 the definition and properties of an almost contact metric manifold and we give some results on invariant submanifolds of those manifolds. In §2, according to the different types of almost contact metric structures on a manifold M we study the properties of submersions of M onto an almost contact metric manifold M'. In §3 we study semi-cosymplectic and semi-Sasakian submersions and we relate the minimality of the fibres with the transference of the almost contact metric structure to the base manifold. In § 4, we study quasi-trans-Sasakian submersions. In § 5, we want to examine the influence of the almost contact metric nature of the submersion on its configuration tensors. In § 6 we prove that the horizontal distribution of an almost contact metric submersion whose total space is cosymplectic is completely integrable. Finally, the relations between the φ-sectional and φ-bisectional curvatures of the two manifolds of an almost contact metric submersion are studies in § 7.

1. **Preliminaries.** A $(2n+1)$-dimensional real differentiable manifold $\tilde M$ of class C^∞ is said to have a $(\tilde F, \tilde\xi, \tilde\eta)$-structure or an almost contact structure if it admits a field $\tilde F$ of endomorphisms of the tangent spaces, a vector field $\tilde\xi$, and a 1-form $\tilde\eta$ satisfying

$$\tilde\eta(\tilde\xi)=1, \tag{1.1}$$

$$\tilde F^2=-\tilde I+\tilde\eta\otimes\tilde\xi, \tag{1.2}$$

where $\tilde I$ denote the identity transformation [9]. Then $\tilde F\tilde\xi=0$, $\tilde\eta\tilde F=0$ and the endomorphism $\tilde F$ has rang $2n$ [1].

If a manifold $\tilde M$ with a $(\tilde F, \tilde\xi, \tilde\eta)$-structure admits a Riemannian metric $\tilde g$ such that