On the other hand, it is known that the length of the second fundamental form h of the submanifold M is given by

$$(4.26)\qquad \|h\|^2=\sum_{\alpha,\beta=1}^{m}\|h(E_\alpha;E_\beta)\|^2=\sum_{\alpha=1}^{m}\sum_{a=1}^{2n+1-m}\|A_a E_\alpha\|^2,$$

where $\{E_1, \dots, E_{m-1}, E_m=\xi\}$ (resp. $\{N_1, \dots, N_{2n+1-m}\}$) is a local field of orthonormal frames on TM (resp. $TM^\perp$).

We choose the local field of orthonormal frames

$$(4.27)\qquad \{E_1, \dots, E_p, E_{p+1}=\varphi E_1, \dots E_{2p}=\varphi E_p, E_{2p+1}, \dots E_{2p+q}, E_m=\xi\}$$

on TM such that $E_i\in\Gamma(D)$ $(i=1, \dots, p)$ and $E_{2p+j}\in\Gamma(D^\perp)$ $(j=1, \dots, q)$. Also, we choose the local field of orthonormal sections

$$(4.28)\qquad \{N_1=\varphi E_{2p+1}, \dots, N_q=\varphi E_{2p+q}, V_1, \dots, V_s, V_{s+1}=\varphi V_1, \dots, V_{2s}=\varphi V_s\}$$

on $TM^\perp$, where $V_r\in\Gamma(\nu)$ $(r=1, \dots, s)$ and $2s=\dim.\ \nu_x$.

Then by using (4.25), (4.27) and (4.28) in (4.26) we obtain

$$(4.29)\qquad \|h\|^2=\sum_{i=1}^{2p}\sum_{j=1}^{q}\|A_jE_i\|^2+2\sum_{j=1}^{q}\|E_{2p+j}\|^2+\sum_{i=1}^{2p}\sum_{t=1}^{2s}\|PA_tE_i\|^2,$$

where $A_j=A_{\varphi E_{2p+j}}$ and $A_t=A_{V_t}$, $j=1, \dots, q$ and $t=1, \dots, 2s$. Thus, the assertion (i) of Theorem 4.5 and (4.29) imply

$$(4.30)\qquad \|h\|^2=2q(p+1)+\sum_{i=1}^{2p}\sum_{t=1}^{2s}\|PA_tE_i\|^2$$

and the assertion (ii) it follows from (4.30).

Also, from (4.30) we obtain that $\|h\|^2=2q(p+1)$ if and only if we have $PA_tE_i=0$ for all $i=1, \dots, 2p$ and $t=1, \dots, 2s$, i.e. $h(X, Y)\in\Gamma(\varphi D^\perp)$ for all $X, Y\in\Gamma(D)$. The proof of Theorem 4.5 is complete.

From Theorem 4.5 and Corollary 3.2 we obtain

Corollary 4.2. *Let M be a normal, generic and $D^\perp$-geodesic semi-invariant submanifold of $S^{2n+1}(1)$. Then we have*

(i) $\|A_{\varphi Z}X\|=1$ *for any unit vectors* $X\in\Gamma(D)$ *and* $Z\in\Gamma(D^\perp)$ *and*

(ii) $\|h\|^2=2q(p+1)$.

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Received 5. XII. 1983

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Tomul XXXI-1, s. I a. Matematică, 1985

ON ALMOST CONTACT METRIC SUBMERSIONS

BY

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B. Watson [13] and E. De la Torre [11] have examined the differential geometric properties of almost complex Riemannian submersions between almost Hermitian manifolds (almost Hermitian submersions). In this connection, the purpose of this paper is to study the properties of almost contact Riemannian submersions between almost contact metric manifolds (almost contact metric submersions). For it, we use the Gray-Hervella's classification of almost Hermitian manifolds [6] and the classification for almost contact metric structures obtained by J.A. Oubiña [8].

First of all we recall in §1 the definition and properties of an almost contact metric manifold and we give some results on invariant submanifolds of those manifolds. In §2, according to the different types of almost contact metric structures on a manifold M we study the properties of submersions of M onto an almost contact metric manifold M'. In §3 we study semi-cosymplectic and semi-Sasakian submersions and we relate the minimality of the fibres with the transference of the almost contact metric structure to the base manifold. In § 4, we study quasi-trans-Sasakian submersions. In § 5, we want to examine the influence of the almost contact metric nature of the submersion on its configuration tensors. In § 6 we prove that the horizontal distribution of an almost contact metric submersion whose total space is cosymplectic is completely integrable. Finally, the relations between the φ-sectional and φ-bisectional curvatures of the two manifolds of an almost contact metric submersion are studies in § 7.

1. **Preliminaries.** A $(2n+1)$-dimensional real differentiable manifold $\tilde M$ of class C^∞ is said to have a $(\tilde F, \tilde\xi, \tilde\eta)$-structure or an almost contact structure if it admits a field $\tilde F$ of endomorphisms of the tangent spaces, a vector field $\tilde\xi$, and a 1-form $\tilde\eta$ satisfying

$$(1.1)\qquad \tilde\eta(\tilde\xi)=1, \qquad\qquad (1.2)\qquad \tilde F^2=-\tilde I+\tilde\eta\otimes\tilde\xi,$$

where $\tilde I$ denote the identity transformation [9]. Then $\tilde F\tilde\xi=0$, $\tilde\eta\tilde F=0$ and the endomorphism $\tilde F$ has rang $2n$ [1].

If a manifold $\tilde M$ with a $(\tilde F, \tilde\xi, \tilde\eta)$-structure admits a Riemannian metric $\tilde g$ such that

$$\tilde{g}(\tilde{F}\tilde{X}, \tilde{F}\tilde{Y})=\tilde{g}(\tilde{X}, \tilde{Y})-\tilde{\eta}(\tilde{X})\tilde{\eta}(\tilde{Y}), \tag{1.3}$$

where $\tilde{X}, \tilde{Y}\in\chi(\tilde{M})$, then $\tilde{M}$ is said to have a $(\tilde{F}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$-structure or an almost contact metric structure and $\tilde{g}$ is called a compatible metric [9]. An immediate consequence is $\tilde{\eta}(\tilde{X})=\tilde{g}(\tilde{X}, \tilde{\xi})$. The 2-form $\tilde{\Phi}$ on $\tilde{M}$ defined by

$$\tilde{\Phi}(\tilde{X}, \tilde{Y})=\tilde{g}(\tilde{X}, \tilde{F}\tilde{Y}) \tag{1.4}$$

is called the fundamental 2-form of the almost contact metric structure.

For the properties and definitions of the different types of almost contact metric structures, we refer the reader to [8].

A $2r$-dimensional immersed submanifold M of a $(2n+1)$-dimensional almost contact metric manifold $\tilde{M}$ with structure $(\tilde{F}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ is said to be invariant if $\tilde{F}X$ is tangent to M for any tangent vector X to M.

Any invariant submanifold M with the structure (F, g) is an almost Hermitian manifold and $\tilde{\xi}$ is normal to M, where F and g are the restrictions of $\tilde{F}$ and $\tilde{g}$ in M, [14]. In this paragraph, M is a $2r$-dimensional invariant submanifold of a $(2n+1)$-dimensional manifold $\tilde{M}$ with an almost contact metric structure $(\tilde{F}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$.

Let $\tilde{\Phi}$, $\tilde{\nabla}$, $\tilde{\delta}$ (resp. Φ, ∇, δ) denote the fundamental 2-form of the structure, the Riemannian connection and the coderivative in $\tilde{M}$ (resp. M), and let $\chi(M)$ denote the Lie algebra of smooth vector fields on M an $\chi(M)^{\perp}$ the smooth vector fields orthogonal to M. If $\{X_1, \ldots, X_r, FX_1, \ldots, FX_r\}$ is a local orthonormal basis, defined on an open subset of M, the partial coderivatives of Φ and $\tilde{\eta}$ with respect to M are computed to be

$$\delta\tilde{\Phi}(\tilde{X})=-\sum_{i=1}^{r}((\tilde{\nabla}_{X_i}\tilde{\Phi})(X_i, \tilde{X})+(\tilde{\nabla}_{FX_i}\tilde{\Phi})(FX_i, \tilde{X})) \tag{1.5}$$

$$\delta\tilde{\eta}=-\sum_{i=1}^{r}((\tilde{\nabla}_{X_i}\tilde{\eta})X_i+(\tilde{\nabla}_{FX_i}\tilde{\eta})FX_i) \tag{1.6}$$

where $\tilde{X}\in\chi(\tilde{M})$, the Lie algebra of smooth vector fields on $\tilde{M}$.

Proposition 1.1. *For any vector field $X\in\chi(M)$*

$$\tilde{\delta}\tilde{\Phi}(X)=\delta\Phi(X). \tag{1.7}$$

Proposition 1.2. *If M is an invariant submanifold of a quasi-trans-Sasakian manifold $\tilde{M}$, then*

$$\tilde{\delta}\tilde{\Phi}(X)=[n/(r-1]\delta\tilde{\Phi}(X), \qquad X\in\chi(M), \tag{1.8}$$

$$\delta\tilde{\Phi}(N)=(n/r)\tilde{\delta}\tilde{\Phi}(N),\ N\in\chi(M)^{\perp}, \qquad (1.10)\qquad \tilde{\delta}\tilde{\eta}=(n/r)\delta\tilde{\eta}. \tag{1.9}$$

Proof. It follows from the definitions of partial coderivatives and of quasi-trans-Sasakian manifold, [8].

Also, we have

Proposition 1.3. *Let M be an invariant submanifold of an almost contact metric manifold $\tilde{M}$. Then*

$$\delta\tilde{\Phi}(N)=\tilde{g}(H, \tilde{F}N), \quad N\in\chi(M)^{\perp}, \qquad (1.12)\qquad \delta\tilde{\eta}=-\tilde{g}(H, \tilde{\xi}), \tag{1.11}$$

where H is the mean curvature vector field of the submanifold M, i.e. if α denotes the second fundamental form of M, $H=\sum_{i=1}^{r}(\alpha(X_i, X_i)+\alpha(FX_i, FX_i))$.

Proposition 1.4. *Let M be an invariant submanifold of an almost contact metric manifold $\tilde{M}$. Then M is a minimal submanifold if and only if $\delta\tilde{\Phi}(N)=0$ for all $N\in\chi(M)^{\perp}$ and $\delta\tilde{\eta}=0$.*

2. **Almost contact metric submersions.** Let $(M, \varphi, \xi, \eta, g)$ and $(M', \varphi', \xi', \eta', g')$ be almost contact metric manifolds, with $\dim M=2m+1$ and $\dim M'=2m'+1$. A smooth surjective mapping $f: M\to M'$ is called a Riemannian submersion [7] if:

(1) f has maximal rank, and (2) $f_*|(\ker f_*)^{\perp}$ is a lineal isometry.

We say that f is an almost contact metric submersion if f is a Riemannian submersion which, additionally, is

(3) An almost contact mapping (i.e. $\varphi'.f_*=f_*.\varphi$).

Vectors on M which are in the kernel of f_* are tangent to the fibres $(F_{m'}=f^{-1}(m'),\ m'\in M')$ and are called vertical vectors. Vectors which are orthogonal to the vertical distribution are said to be horizontal. We denote the vertical and horizontal distributions in the tangent bundle of the total space M by $V(M)$ and $H(M)$, respectively. Then $T(M)$ enjoys an orthogonal decomposition: $T(M)=V(M)\oplus H(M)$. The orthogonal projection mappings are denoted $v: T(M)\to V(M)$ and $h: T(M)\to H(M)$ respectively. Since the fibres of a Riemannian submersion are regular and closed submanifolds of the manifold M, the vertical distribution is completely integrable. In general, the horizontal distribution is not completely integrable.

Proposition 2.1. *Let $f: M\to M'$ be an almost contact metric submersion. Then the fibres $F_{m'}$ are invariant submanifolds of M of dimension $2(m-m')$.*

Proof. Let $W\in V(M)$. Since f is almost contact, $f_*(\varphi W)=\varphi' f_* W=0$, thus, $\varphi(V(M))\subseteq V(M)$. Since the fibres $F_{m'}$ are even-dimensional and invariant submanifold of M, $F_{m'}$ with the structure (φ, g) is almost Hermitian and the vector field ξ is horizontal (we shall suppose that $f_*\xi=\xi'$).

Also, we have

Proposition 2.2. *Let $f: M\to M'$ be an almost contact metric submersion. Then*

$$\varphi(V(M))=V(M) \text{ and } H(M)=\varphi(H(M))\oplus\xi.$$

Let ∇, ∇' denote the Riemannian connections on M and M', respectively.

A smooth vector field X on M is basic for the almost contact metric submersion $f: M\to M'$ if:

(1) X is horizontal, and (2) X is f-related to a vector field on M', which depends on X and is denoted X_*.

There is a one-to-one relationship between basic vector fields on M and vector fields on M'.

Proposition 2.3. *Let $f: M\to M'$ be an almost contact metric submersion, and let X and Y be basic vector fields on M. Then*

(a) $g(X, Y)=g'(X_*, Y_*).f$

(b) φX is the basic vector fields associated to $\varphi' X_*$.

(c) $h\nabla_X Y$ is the basic vector field associated to $\nabla'_{X_*} Y_*$.

(d) $h[X, Y]$ is the basic vector field associated to $[X_*, Y_*]$.

Proof. Assertions (a), (c) and (d) are properties of a Riemannian submersion [7], and (b) follows from Proposition 2.2 and from the fact that the map f is almost contact.

Now, we want to examine the influence of a given structure defined on the total space M on the determination of the corresponding structure on the fibre submanifolds and the manifold M'.

For the definition of the different types of almost contact metric (resp. almost Hermitian) structure, we refer the reader to [8] (resp. [6]).

Theorem 2.1. *Let $f: M \to M'$ be an almost contact metric submersion. If*

M is	then $F_{m'}$ is
(1) cosymplectic	(1) Kaehler and minimal
(2) nearly cosymplectic	(2) nearly Kaehler
(3) nearly-K-cosymplectic	(3) nearly Kaehler and minimal
(4) quasi-K-cosymplectic	(4) quasi-Kaehler and minimal
(5) almost cosymplectic	(5) almost Kaehler and minimal
(6) normal	(6) Hermitian
(7) quasi-Sasakian	(7) Kaehler
(8) G_1-Sasakian	(8) G_1-manifold
(9) G_2-Sasakian	(9) G_2-manifold
(10) trans-Sasakian	(10) Kaehler
(11) nearly-trans-Sasakian	(11) nearly Kaehler
(12) quasi-trans-Sasakian	(12) $W_1 \oplus W_2 \oplus W_4$-manifold
(13) almost-trans-Sasakian	(12) $W_2 \oplus W_4$-manifold.

Proof. Since the fibres $F_{m'}$ are invariant submanifolds of an almost contact metric manifold M (see [2]) we obtain the result.

Also,

Theorem 2.2. *Let $f: M \to M'$ be an almost contact metric submersion. Then M is not Sasakian, contact, K-contact, nearly Sasakian, an quasi-K-Sasakian either.*

Proof. It follows from the fact that $F_{m'}$ is an even-dimensional invariant submanifold of M, and this is not possible when M is Sasakian, contact, K-contact, nearly Sasakian or quasi-K-Sasakian [2].

Theorem 2.3. *Let $f: M \to M'$ be an almost contact metric submersion. If M is (a) cosymplectic, (b) nearly cosymplectic, (c) nearly-K-cosymplectic, (d) quasi-K-cosymplectic, (e) almost cosymplectic, (f) normal, (g) quasi-Sasakian, (h) G_1-Sasakian or (i) G_2-Sasakian, so is M'.*

Proof. Let X, Y, Z be basic vector fields on M, and let X, Y, Z, be their associated vector fields on M'. Then, by Proposition 2.3, the basic vector field associated to $(\nabla'_X \varphi')Y_*$ is $h((\nabla_X\varphi)Y)$. Thus, if M is cosymplectic, M' is also cosymplectic.

(b) is deduced in a similar way. (c) follows from (b) and because the basic vector field associated to $\nabla'_{X_*}\xi'$ is $h\nabla_X\xi$.

The basic vector field associated to $(\nabla'_{X_*}\varphi')Y_* + (\nabla'_{\varphi' X_*}\varphi')\varphi' Y_* - \eta'(Y_*)\nabla'_{\varphi' X_*}\xi'$ is $h((\nabla_X\varphi)Y + (\nabla_{\alpha X}\varphi)\varphi Y - \eta(Y)\nabla_{\alpha X}\xi)$. Thus we obtain (d).

We have $\Phi(X, Y) = f^*\Phi'(X, Y)$; $\eta(X) = f^*\eta'(X)$. Since f^* commutes with the exterior differential on differential forms, we see also that $d\Phi = f^* d\Phi'$ and $d\eta = f^* d\eta'$. Thus if M is almost cosymplectic, M' is also almost cosymplectic and therefore (e) is shown. (f) and (h) are deduced in a similar way to (d). (g) follows from (*f*) and (e). Finally, since $d\Phi = f^* d\Phi'$, $d\eta = f^* d\eta'$

and $\eta = f^*\eta'$ on basic vector fields, by Proposition 2.3, and from the definition of G_2-Sasakian structure [8] we deduce (*i*).

In view of Theorem 2.3, an almost contact metric submersion $f: M \to M'$ for which both M and M' are members of a class U of almost contact metric structure will be called a U-submersion.

3. **Semi-cosymplectic and semi-Sasaskian submersions.** As is seen in Theorem 2.1 and 2.3, the almost contact metric structure on M completely determines that structure on the fibres and the base manifold of the almost contact metric submersion $f: M \to M'$. The interrelation between the minimality of the fibres and the transference of the almost contact metric structure on the total space of the almost contact metric submersion to the base manifold is evident in the main Theorems of this section.

Theorem 3.1. *Let $f: M \to M'$ be an almost contact metric submersion. If M is a semi-cosymplectic, semi-cosymplectic normal, G_1-semi-cosymplectic or G_2-semi-cosymplectic manifold, then M' is semi-cosymplectic, semi-cosymplectic normal, G_1-semi-cosymplectic or G_2-semi-cosymplectic, respectively, if and only if the fibres $F_{m'}$ of f are minimal submanifolds.*

Proof. Let $\{E_1, \ldots, E_{m-m'}, \varphi E_1, \ldots, \varphi E_{m-m'}, F_1, \ldots, F_{m'}, \varphi F_1, \ldots, \varphi F_{m'}, \xi\}$ be a local φ-basis defined on an open subset of M, whose horizontal vector fields F_i are basic, and the vector fields E_i are vertical.

Let $m' \in M'$ and X be a basic vector field defined on $F_{m'}$. Then

$$\delta\Phi(X) = \sum_{i=1}^{m-m'} ((\nabla_{E_i}\Phi)(E_i, X) + (\nabla_{\varphi E_i}\Phi)(\varphi E_i, X)) - \sum_{i=1}^{m'} (\nabla_{F_i}\Phi)(F_i, X) +$$

$$+ (\nabla_{\varphi F_i}\Phi)(\varphi F_i, X)) - (\nabla_\xi\Phi)(\xi, X) = \bar{\delta}\Phi(X) - \sum_{i=1}^{m'} ((\nabla_{F_i}\Phi)(F_i, X) +$$

$$+ (\nabla_{\varphi F_i}\Phi)(\varphi F_i, X)) - (\nabla_\xi\Phi)(\xi, X).$$

On the other hand,

$$(\nabla_{F_i}\Phi)(F_i, X) = -g((\nabla_{F_i}\varphi)F_i, X) = -g(\nabla_{F_i}\varphi F_i, X) + g(\varphi\nabla_{F_i}F_i, X) =$$

$$= -g(h\nabla_{F_i}\varphi F_i, X) + g(h\varphi\nabla_{F_i}F_i, X) = -g'(\nabla'_{F_{i*}}\varphi' F_{i*}, X_*) \cdot f +$$

$$+ g'(\varphi'\nabla'_{F_{i*}} F_{i*}, X_*) \cdot f = (\nabla'_{F_{i*}}\Phi')(F_{i*}, X) \cdot f.$$

Analogously,

$$(\nabla_{\varphi F_i}\Phi)(\varphi F_i, X) = (\nabla'_{\varphi' F_{i*}}\Phi')(\varphi' F_{i*}, X_*) \cdot f \text{ and } (\nabla_\xi\Phi)(\xi, X) =$$

$$(\nabla'_{\xi'}, \Phi')(\xi', X_*) \cdot f.$$

Thus, $\delta\Phi(X) = \bar{\delta}\Phi(X) + \delta'\Phi'(X_*) \cdot f$.

In a similar way, we obtain $\delta\eta = \bar{\delta}\eta + \delta'\eta' \cdot f$.

Thus, from the definitions of the types of almost contact metric structures and from Proposition 1.4 it follows the assertion.

In a similar way to Theorem 3.1, we deduce :

Theorem 3.2. *Let M be a semi-Sasakian manifold and let $f: M \to M'$ be an almost contact metric submersion. Then the fibres $F_{m'}$ of f are minimal submanifolds of M if and only if $\eta' = [1/(2m)]\ \delta'\Phi'$.*

If now we consider in M' the almost contact metric structure $(\varphi', (m'/m)\xi', (m/m')\eta', (m/m')^2 g')$, we obtain:

Theorem 3.3. *Let $f: M \to M'$ be an almost contact metric submersion. If M is semi-Sasakian, semi-Sasakian normal, G_1-semi-Sasakian or G_2-semi-Sasakian then M' is semi-Sasakian, semi-Sasakian normal, G_1-semi-Sasakian or G_2-semi-Sasakian, respectively, if and only if the fibres $F_{m'}$ of f are minimal submanifolds.*

Eells & Sampson [3] studied harmonic maps $f: M \to M'$ between complete Riemannian manifolds. A map f is said to be harmonic if it minimizes a certain energy integral, or, equivalently, if it is the kernel of a generalized Laplacian operator defined by them. For instance, an immersion is harmonic if and only if it is minimal. A Riemannian submersion is harmonic if and only if its fibres are minimally immersed. Eells & Sampson, using the curvature content of the Laplacian operator, were able to prove several results which restrict the existence of harmonic mappings based of the Riemannian curvatures or Ricci curvatures of M and M'. These results obviously apply to almost contact metric submersions as we formulate in

Corollary 3.1. *Let M be a semi-cosymplectic or semi-Sasakian manifold and $f: M \to M'$ an almost contact metric submersion. Then M' is semi-cosymplectic or semi-Sasakian, respectively, if and only if f is harmonic.*

Theorems 3.1 and 3.3 allows us to say a great deal about the submanifolds of the two manifolds of an almost contact metric submersion for which both of the manifolds M and M' are semi-cosymplectic or semi-Sasakian. In fact, we gain information about the Riemannian submanifolds of M and M' without requiring that they have even dimension or be invariant submanifold.

In his thesis, Escobales [4] proved

Theorem 3.4. *Let $f: M \to M'$ be a Riemannian submersion with minimal fibres. Then a submanifold N of M' is minimal in M' if and only if $f^{-1}(N)$ is minimal in M.*

Thus we have

Theorem 3.5. *Let $f: M \to M'$ be a semi-cosymplectic or semi-Sasakian submersion. Then a Riemannian submanifold N of M' is minimally immersed in M' if and only if $f^{-1}(N)$ is minimally immersed in M.*

4. **Quasi-trans-Sasakian submersions.** We shall study the quasi-trans-Sasakian submersions, but previously we have the following result

Proposition 4.1. *Let $f: M \to M'$ be an almost contact metric submersion and let M be a quasi-trans-Sasakian manifold. Then for all basic vector field X,*

(4.1) $\bar{\delta}\Phi(X) = [(m-m')/m]\delta\Phi(X)$ (4.2) $\bar{\delta}\Phi(X) = [(m-m')]/m\,\delta'\Phi'(X_*).f$

(4.3) $\bar{\delta}\eta = [(m-m')/m]\delta\eta$ (4.4) $\bar{\delta}\eta = [(m-m'/m']\ \delta'\eta'.f$.

Proof. (4.1) and (4.3) follows from (1.9) and (1.10). In order to prove the formula (4.2), we choose a φ-basis in $\chi(M)$ as in Theorem 3.1. If X is a basic vector field, then

$$\delta\Phi(X) = \sum_{i=1}^{m} (g(F_i, X)\delta\Phi(F_i) + g(\varphi F_i, X)\delta\Phi(\varphi F_i)) + g(\xi, X)\delta\Phi(\xi).$$

Thus,

$$-\delta'\Phi'(X_*).f = \sum_{i=1}^{m'} ((\nabla'_{F_{i*}}\Phi')(F_{i*}, X_*).f + (\nabla'_{\varphi' F_{i*}}\Phi')(\varphi' F_{i*}, X_*).f) +$$
$$+ (\nabla'_{\xi'}\Phi')(\xi', X_*).f = \sum_{i=1}^{m'} ((\nabla_{F_i}\Phi)(F_i, X) + (\nabla_{\varphi F_i}\Phi)(\varphi F_i, X)) + (\nabla_\xi \Phi)(\xi, X) =$$
$$= -(1/m) \sum_{i=1}^{m'} (g(F_i, F_i)\delta\Phi(X) - g(F_i, X)\delta\Phi(F_i) + g(F_i, \varphi X)\delta\Phi(\varphi F_i)) -$$
$$-(1/m)(g(\xi, \xi)\delta\Phi(X) - g(\xi, X)\delta\Phi(\xi)) = -(m'/m)\delta\Phi(X).$$

Also, we have $\delta\Phi(X) = \bar{\delta}\Phi(X) + \delta'\Phi'(X_*).f$, thus,

$$[(m-m')/m']\delta'\Phi'(X_*).f = \bar{\delta}\Phi(X).$$

(4.4) is proved in a similar way.

By Proposition 1.4 and 4.1., we have

Theorem 4.1. *Let $f: M \to M'$ be an almost contact metric submersion and let M be a quasi-trans-Sasakian manifold. Then M' is a semi-cosymplectic manifold if and only if the fibres $F_{m'}$ of f are minimal submanifolds of M.*

Theorem 4.2. *Let $f: M \to M'$ be an almost contact metric submersion.*

(a) *If M is trans-Sasakian, nearly-trans-Sasakian, almost-trans-Sasakian or quasi-trans-Sasakian manifold then M' is trans-Sasakian, nearly-trans-Sasakian, almost-trans-Sasakian or quasi-trans-Sasakian, respectively.*

(b) *If M is trans-Sasakian, nearly-trans-Sasakian, almost-trans-Sasakian or quasi-trans-Sasakian manifold then the fibres $F_{m'}$ of f are minimal submanifolds of M if and only if M' is cosymplectic, nearly-K-cosymplectic, almost-cosymplectic or quasi-K-cosymplectic manifold, respectively.*

Proof. (a) If M is trans-Sasakian then, for all $X, Y, Z \in \chi(M)$:

(1) $$(\nabla_X\Phi)(Y, Z) = -\frac{1}{2m}(g(X, Y)\delta\Phi(Z) - g(X, Z)\delta\Phi(Y) + g(X, \varphi Y)\eta(Z)\delta\eta - g(X, \varphi Z)\eta(Y)\delta\eta),$$

(2) $$\delta\Phi(\varphi X) = 0.$$

For all X, Y, Z basic vector fields, we have $(\nabla_X\Phi)(Y, Z) = (\nabla'_{X_*}\Phi')(Y_*, Z_*).f$ $g(X, Y) = g'(X_*, Y_*).f$ and $\eta(X) = \eta'(X_*).f$, and by Proposition (4.1)

$$\delta\Phi(X) = (m/m')\delta'\Phi'(X_*).f \text{ and } \delta\eta = (m/m')\delta'\eta'.f.$$

Thus, substituting in (1) and (2), we obtain that M' is trans-Sasakian. The other relations are proved in a similar way.

(b) is deduced from (a) and Theorem 4.1.

Corollary 4.1. *Let M be a trans-Sasakian, nearly-trans-Sasakian, almost-trans-Sasakian or quasi-trans-Sasakian manifeld and $f: M \to M'$ an almost contact metric submersion. Then M' is cosymplectic, nearly-K-*

cosymplectic, almost-cosymplectic or quasi-K-cosymplectic manifold, respectively, if and only if f is harmonic.

Proof. It follows from the fact that a Riemannian submersion is harmonic if and only if its fibres are minimal submanifolds.

5. **The properties of the tensor fields T and A.** In this paragraph, we want to examine how the almost contact metric structure on M places restrictions on the configuration tensors.

Let ∇, ∇', $\hat{\nabla}$ denote the Riemannian connections on M, M' anf $F_{m'}$, respectively, and let E and F be vector fields on M. The O'Neill configuration tensors [7] of the Riemannian submersion $f: M \to M'$ are given by

(5.1) $$T_E F = h\nabla_{vE} vF + v\nabla_{vE} hF\,; \qquad A_E F = v\nabla_{hE} hF + h\nabla_{hE} vF.$$

For the properties of T and A, we refer the reader to [7] and [13].

Theorem 5.1. *Let $f: M \to M'$ be a quasi-K-cosymplectic submersion, let V and W be vertical vector filds, and let X and Y be horizontal vector fields. Then*

(5.2) $$T_V\varphi W = T_{\varphi V} W, \qquad (5.3)\quad T_{\varphi V}X = -\varphi T_V X,$$

(5.4) $$A_X\varphi X = 0, \qquad (5.5)\quad A_X\varphi Y = -A_Y\varphi X.$$

Proof. Assertion (5.2) follows from the similar result on the second fundamental form of the invariant submanifolds $F_{m'}$, [2]. To see (5.3) note that

$$g(T_{\varphi V}X, W) = -g(X, T_{\varphi V}W) = -g(X, T_V\varphi W) = g(T_V X, \varphi W) =$$
$$= -g(\varphi T_V X, W).$$

For a manifold which is quasi-K-cosymplectic we have ([8])

$$\nabla_X\varphi X - \nabla_{\varphi X} X = \varphi\nabla_X X + \varphi\nabla_{\varphi X}\varphi X - (\varphi X)(\eta(X))\xi.$$

Using the vertical part of this equation, we obtain $A_X\varphi X - A_{\varphi X}X = \varphi A_X X + \varphi A_{\varphi X}\varphi X = 0$, therefore $A_X\varphi X = 0$.

Assertion (5.5) follows from (5.4).

Theorem 5.2. *Let $f: M \to M'$ be a nearly cosymplectic submersion.* Then, for all vertical vectors V and W,

(5.6) $$2\varphi T_V W = T_V\varphi W + T_{\varphi V} W,$$

(5.7) $$2T_V\varphi X = \varphi T_V X - T_{\varphi V} X, \text{ for all horizontal } X.$$

Proof. (5.6) By direct calculation we have

$$T_V\varphi V = \nabla_V\varphi V - \hat{\nabla}_V\varphi V = (\nabla_V\varphi)V + \varphi(\nabla_V V) - (\hat{\nabla}_V\varphi)V - \varphi(\hat{\nabla}_V V) =$$
$$= \varphi(\nabla_V V - \hat{\nabla}_V V) = \varphi T_V V.$$

Using this equation, we obtain (5.6).

To see (5.7), we have

$$g(2T_V\varphi X, W) = -g(\varphi X, 2T_V W) = g(X, 2\varphi T_V W) = g(X, T_V\varphi W +$$
$$+ T_{\varphi V} W) = -g(T_V X, \varphi W) - g(T_{\varphi V}X, W) = g(\varphi T_V X - T_{\varphi V} X, W).$$

Theorem 5.3. *Let $f: M \to M'$ be a nearly-K-cosymplectic submersion. Then for all vertical vector V and W,*

(5.8) $$T_V\varphi W = \varphi T_V W = T_{\varphi V} W \qquad (5.9)\quad T_V\varphi X = \varphi T_V X, \text{ for all horizontal } X.$$

Proof. It follows from the fact that a nearly-K-cosymplectic manifold is nearly cosymplectic and quasi-K-cosymplectic, [8].

Theorem 5.4. *Let $f: M \to M'$ be a cosymplectic submersion. Then, for all E, $F \in \gamma(M)$*

(5.10) $$T_F\varphi E = \varphi T_F E, \qquad (5.11)\quad A_F\varphi E = \varphi A_F E.$$

Proof. By direct calculation, we have

$$T_F\varphi E = h\nabla_{vF} v\varphi E + v\nabla_{vF} h\varphi E = h(\nabla_{vF}\varphi)vE + h\varphi\nabla_{vF} vE + v(\nabla_{vF}\varphi)hE +$$
$$+ v\varphi\nabla_{vF} hE = \varphi T_F E.$$

(5.11) is proved in a similar way.

6. **Cosymplectic submersions.** Since all invariant submanifolds of cosymplectic (in fact, quasi-K-cosymplectic) manifolds are minimal [2], it is obvious that the fibres of a cosymplectic submersion are minimal. Moreover, we shall see that the horizontal distribution of a cosymplectic submersion is completely integrable. This obviously places severe restrictions on the existence of cosymplectic submersions.

Theorem 6.1. *The horizontal distribution of a cosymplectic submersion is completely integrable.*

Proof. Let X and Y be horizontal vector fields on M, and V vertical. Then

$$g(A_{\varphi X}Y, V) = g(A_X\varphi Y, V) = -g(\varphi Y, A_X V) = -g(\varphi Y, h\nabla_V X) =$$
$$= g(Y, h\varphi\nabla_V X) = g(Y, h\nabla_V\varphi X) = g(Y, A_{\varphi X} V) = -g(A_{\varphi X}Y, V).$$

Therefore $A_{\varphi X}Y = 0$. Moreover, $A_\xi Y = -A_Y\xi = -v\nabla_Y\xi = 0$, thus, $A_X Y = 0$, and from [13, Prop. 2.7 p. 154] it follows the assertion.

A smooth mapping $f: M \to N$ between smooth Riemann manifolds is said to be totally geodesic if f preserves parallel translation. Vilms [12] classified totally geodesic Riemannian submersions and showed that a Riemannian submersion $f: M \to N$ is totally geodesic if and only if both configuration tensors T and A vanish. He also proved that a Riemannian submersion of a complete manifold is totally geodesic if and only if it is a fibre bundle with flat connection and with the group of isometries of a fibre as structural group.

Corollary 6.1a. *A cosymplectic submersion with totally geodesic fibres is a totally geodesic mapping.*

Corollary 6.1b. *Let $f: M \to M'$ be a cosymplectic submersion with totally geodesic fibres. Then f is a smooth fibre bundle with flat connection and M is a locally product manifold.*

Proof. [12, Cor. 3.6.].

Corollary 6.10. *Let M be a simply connected cosymplectic manifold, and $f: M \to M'$ a cosymplectic submersion with totally geodesic fibres. Then M is a cosymplectic product space, and f is projection.*

7. **Curvature relations.** An almost Hermitian submersion $f: M\to M'$ is holomorphic sectional curvature increasing on horizontal holomorphic 2-planes, [13]. Similarly, we have found that an almost contact metric submersion is of φ-sectional curvature increasing on horizontal φ-sections (i.e., on plane sections where exists a vector $X\in H(M)$ orthogonal to ξ such that $\{X, \varphi X\}$ is an orthogonal basis in those planes). However, we prove that a quasi-K-cosymplectic submersion $f: M\to M'$ preserves the φ-sectional curvature on horizontal vectors, and, moreover, is φ-bisectional curvature decreasing on pairs of nonzero horizontal vectors. Restricting further to cosymplectic submersions, we find that such a submersion is φ-bisectional curvature preserving on pairs of horizontal vectors. This is a reflection of the complete integrability of the horizontal distribution, which was demonstrated in §3.

For the properties of the Riemannian sectional curvature for Riemannian submersions, we refer the reader to [7].

If X is a horizontal vector field, we shall denote the vector field f_*X on M' by X_*.

Let $(M, \varphi, \xi, \eta, g)$ be an almost contact metric manifold. The φ-bisectional curvature is defined for any pair of nonzero vectors E and F on M and orthogonals to ξ by $B(E,F)=\|E\|^{-2}\|F\|^{-2}g(R(E,\varphi E)F,\varphi F)$.

The φ-sectional curvature asigns to the nonzero vector E and orthogonal to ξ the value $H(E)=B(E,E)$.

Thus, $H(E)$ is the sectional curvature of the 2-plane orthogonal to ξ spanned by the orthogonal vectors E and φE.

We denote by B' and H' the φ-bisectional curvature and the φ-sectional curvature in M', respectively, and by $\hat B$ and $\hat H$ the holomorphic bisectional curvature and the holomorphic sectional curvature in the fibres $F_{m'}$, respectively.

Theorem 7.1. *Let $f: M\to M'$ be an almost contact metric submersion. Let X and Y be horizontal and orthogonals to ξ, and V and W vertical. Then the φ-bisectional curvatures satisfy*

$$B(V,W)=\hat B(V,W)+\|V\|^{-2}\|W\|^{-2}\{g(T_V\varphi W, T_{\varphi V}W)-g(T_VW, T_{\varphi V}\varphi W)\} \tag{7.1}$$

$$\begin{aligned} B(X,V)=\|X\|^{-2}\|V\|^{-2}\{&g((\nabla_VA)_X\varphi X,\varphi V)-g(A_X\varphi V, A_{\varphi X}V)+\\ &+g(A_XV, A_{\varphi X}\varphi V)-g((\nabla_{\varphi V}A)_X\varphi X, V)+g(T_{\varphi V}X, T_V\varphi X)-\\ &-g(T_VX, T_{\varphi V}\varphi X)\}, \end{aligned} \tag{7.2}$$

$$\begin{aligned} B(X,Y)=B'(X_*,Y_*)-\|X\|^{-2}\|Y\|^{-2}\{&2g(A_X\varphi X, A_Y\varphi Y)-\\ &-g(A_{\varphi X}Y, A_X\varphi Y)-g(A_YX, A_{\varphi X}\varphi Y)\}. \end{aligned} \tag{7.3}$$

Proof. (7.1) is just the Gauss equation for the fibre submanifolds. Relations (7.2) and (7.3) follows from certain equations of O'Neill concerning the covariant derivatives of the T and A tensors for a Riemannian submersion.

Specifically, (7.2) follows from equation (2′) and (7.3) from equation (4) of [7].

Theorem 7.2. *If $f: M\to M'$ is a quasi-K-cosymplectic submersion, then*

$$B(V,W)=\hat B(V,W)+\|V\|^{-2}\|W\|^{-2}\{\|T_V\varphi W\|^2+\|T_VW\|^2\}, \tag{7.4}$$

$$\begin{aligned} B(X,V)=\|X\|^{-2}\|V\|^{-2}\{&g((\nabla_VA)_X\varphi X,\varphi V)-g((\nabla_{\varphi V}A)_X\varphi X, V)-\\ &-2g(T_VX, T_{\varphi V}\varphi X)\}. \end{aligned} \tag{7.5}$$

$$B(X,Y)=B'(X_*,Y_*)+\|X\|^{-2}\|Y\|^{-2}\{\|A_X\varphi Y\|^2+\|A_XY\|^2\}. \tag{7.6}$$

Proof. It follows from Theorems 5.1 and 7.1.

Corollary 7. 2a. *If $f: M\to M'$ is a quasi-K-cosymplectic submersion, then, for all vertical vectors V and W $B(V,W)\geq \hat B(V,W)$ and equality holds if and only if the fibres of f are totally geodesic submanifolds of M.*

Corollary 7.2b. *If $f: M\to M'$ is a quasi-K-cosymplectic submersion, then, for all horizontal vectors X and Y, $B(X,Y)\geq B'(X_*,Y_*)$ and equality holds if and only if the horizo tal distribution is completely integrable.*

Theorem 7.3. *If $f: M\to M'$ is an nearly-K-cosymplectic submersion, then*

$$B(V,W)=\hat B(V,W)+2\|V\|^{-2}\|W\|^{-2}\|T_VW\|^2, \tag{7.7}$$

$$\begin{aligned} B(X,V)=\|X\|^{-2}\|V\|^{-2}\{&g((\nabla_VA)\varphi X,\varphi V)-g(A_X\varphi V, A_{\varphi X}V)+\\ &+g(A_XV, A_{\varphi X}\varphi V)-g((\nabla_{\varphi V}A)_X\varphi X, V)+2\|T_VX\|^2\}. \end{aligned} \tag{7.8}$$

Proof. It follows from Theorems 5.3 and 7.2.

Theorem 7.4. If $f: M\to M'$ is a cosymplectic submersion, then

$$B(X,V)=2\|X\|^{-2}\|V\|^{-2}\|T_VX\|^2, \tag{7.9}$$ $$B(X,Y)=B'(X_*,Y_*). \tag{7.10}$$

Proof. An immediate consequence of the previous Theorem and Theorems 6.1 and 7.2.

Corollary 7.4.a. *If M is a cosymplectic manifold of constant φ-bisectional curvature b, and $f: M\to M'$ is a cosymplectic submersion, then M' is o cosymplectic manifcld of constant φ-bisectional curvature b.*

Corollary 7.4.b. *For cosymplectic submersions $B(X,V)\geq 0$.*

Corollary 7.4.c. *If M is a cosymplectic manifold with strictly negative φ-bisectional curvature, then there do not exist any cosymplectic submersions $f: M\to M'$.*

Corollary 7.4d. *If M is a cosymplectic manifold with nonpositive φ-bisectional curvature, and $f: M\to M'$ is a cosymplectic submersion, then f is a totally geodesic mapping, and M is a locally product mapping.*

Proof. $B(E,F)\leq 0$ implies $B(X,V)=0$ for all horizontal X and all vertical V. Therefore $T_VX=0$. Thus T vanishes, and the assertion follow from Corollary 6.1a and 6.1b.

Now, we adapt Theorems 7.1 though 7.4 to the φ-sectional curvature tensor.

Theorem 7.5. *Let $f: M\to M'$ be an almost contact metric submersion, X a vector orthogonal to ξ, and V a vertical vector. Then*

$$H(V)=\hat H(V)+\|V\|^{-4}\{\|T_V\varphi V\|^2-g(T_VV, T_{\varphi V}\varphi V)\}, \tag{7.11}$$

$$H(X)=H'(X_*)-3\|X\|^{-4}\|A_X\varphi X\|^2. \tag{7.12}$$

Proof. The equations follows from Theorem 7.1.

Corollary 7.5 a. *For an almost contact metric submersion $f: M\to M'$*

$$H(X)\leqslant H'(X_*).$$

Theorem 7.6. *If $f: M\to M'$ is a quasi-K-cosymplectic submersion, then*

(7.13) $H(V)=\hat{H}(V)+\|V\|^{-4}\{\|T_V V\|^2+\|T_V\varphi V\|^2\}$, (7.14) $H(X)=H'(X_*)$;

Proof. It follows from Theorems 7.2 and 5.1.

Corollary 7.6 a. *If $f: M\to M'$ is a quasi-K-cosymplectic submersion, $H(V)\geqslant \hat{H}(V)$, and equality holds if and only if the fibres of f are totally geodesic.*

Corollary 7.6 b. *A quasi-K-cosymplectic submersion preserves the φ-sectional curvature on horizontal φ-sections.*

Corollary 7.6 c. *Let $f: M\to M'$ be a quasi-K-cosymplectic submersion and suppose that M is of constant φ-sectional curvature c. Then M' is a quasi-K-cosymplectic manifold of constant φ-sectional curvature c.*

Theorem 7.7. *Let $f: M\to M'$ be an nearly-K-cosymplectic submersion. Then*

$$H(V)=\hat{H}(V)+2\|V\|^{-4}\|T_V V\|^2.$$

Proof. It follows from Theorem 7.3.

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Received 10. 11. 1984

ANALELE ŞTIINŢIFICE ALE UNIVERSITĂŢII „AL. I. CUZA" DIN IAŞI
Tomul XXXI-1, s. I a. Matematică, 1985

SEMI-INVARIANT SUBMANIFOLDS OF MANIFOLDS WITH A SASAKIAN 3-STRUCTURE

BY

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1. Preliminaries. Let N be a real $(4n+3)$-dimensional differentiable manifold. We call N a manifold with a *Sasakian 3-structure* if it is endowed with three Sasakian structures $(\varphi_a, \xi_a, \eta_a, g)$, $a=1, 2, 3$ such that ξ_1, ξ_2, ξ_3 are three vector fields mutually orthogonal to each other. For each $a=1, 2, 3$ we have denoted by φ_a a tensor field of type (1, 1) and by η_a a 1-form. The Riemannian metric on N is denoted by g. All these tensor fields are related by (see [3])

$$(\varphi_a)^2=-I+\eta_a\otimes\xi_a;\ \varphi_a(\xi_a)=0;\ \eta_a\circ\varphi_a=0;\ \eta_a(\xi_a)=1, \tag{1.1}$$

$$\begin{cases} \eta_a(\xi_b)=0 \text{ for } a\neq b;\ \varphi_a(\xi_b)=-\varphi_b(\xi_a)=\xi_c;\\ \eta_a\circ\varphi_b=-\eta_b\circ\varphi_a=\eta_c;\ \varphi_a\circ\varphi_b-\xi_a\otimes\eta_b=-\varphi_b\circ\varphi_a+\xi_b\otimes\eta_a=\varphi_c, \end{cases} \tag{1.2}$$

where (a, b, c) is a cyclic permutation of (1, 2, 3),

$$\eta_a(X)=g(X, \xi_a),\ g(\varphi_a X, \varphi_a Y)=g(X, Y)-\eta_a(X)\eta_a(Y); \tag{1.3}$$

$$(\tilde{\nabla}_X\varphi_a)Y=g(X, Y)\xi_a-\eta_a(Y)X, \tag{1.4}$$

for all vector fields X, Y on N, where I denotes the identity morphism on the tangent bundle to N and $\tilde{\nabla}$ is the Levi Civita connection on N. As it is well known (see [2]) we have

$$\tilde{\nabla}_X\xi_a=-\varphi_a X,\quad a=1, 2, 3, \tag{1.5}$$

for any vector field X tangent to N.

Y. Y. Kuo proved in [3] that R^{4n+3} and S^{4n+3} are examples of manifolds with a Sasakian 3-structure.

Now, let M be an $(m+3)$-dimensional submanifold of N tangent to the structure vector fields ξ_1, ξ_2, ξ_3. Of course, the geometry of M depends on the behaviour of the tangent space T_xM at each $x\in M$ to the actions of φ_a, $a=1, 2, 3$. One of the present authors studied in [1] generic submanifolds and proved that each real hypersurface of N inherites a natural structure of a generic submanifold. We shall give here a generalization of the concept of generic submanifold of N, and obtain fundamental results on the geometry of M.