$$H(X)=H'(X_*)-3\|X\|^{-4}\|A_X\varphi X\|^2. \tag{7.12}$$

Proof. The equations follows from Theorem 7.1.

Corollary 7.5 a. *For an almost contact metric submersion* $f: M\to M'$

$$H(X)\leqslant H'(X_*).$$

Theorem 7.6. *If* $f: M\to M'$ *is a quasi-K-cosymplectic submersion, then*

(7.13) $H(V)=\hat{H}(V)+\|V\|^{-4}\{\|T_V V\|^2+\|T_V\varphi V\|^2\}$, (7.14) $H(X)=H'(X_*)$.

Proof. It follows from Theorems 7.2 and 5.1.

Corollary 7.6 a. *If* $f: M\to M'$ *is a quasi-K-cosymplectic submersion*, $H(V)\geqslant A(V)$, *and equality holds if and only if the fibres of* f *are totally geodesic.*

Corollary 7.6 b. *A quasi-K-cosymplectic submersion preserves the* φ*-sectional curvature on horizontal* φ*-sections.*

Corollary 7.6 c. *Let* $f: M\to M'$ *be a quasi-K-cosymplectic submersion and suppose that* M *is of constant* φ*-sectional curvature* c. *Then* M' *is a quasi-K-cosymplectic manifold of constant* φ*-sectional curvature* c.

Theorem 7.7. *Let* $f: M\to M'$ *be an nearly-K-cosymplectic submersion. Then*

$$H(V)=\hat{H}(V)+2\|V\|^{-4}\|T_V V\|^2.$$

Proof. It follows from Theorem 7.3.

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SEMI-INVARIANT SUBMANIFOLDS OF MANIFOLDS WITH A SASAKIAN 3-STRUCTURE

BY

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1. **Preliminaries.** Let N be a real $(4n+3)$-dimensional differentiable manifold. We call N a manifold with a *Sasakian 3-structure* if it is endowed with three Sasakian structures $(\varphi_a, \xi_a, \eta_a, g)$, $a=1, 2, 3$ such that ξ_1, ξ_2, ξ_3 are three vector fields mutually orthogonal to each other. For each $a=1, 2, 3$ we have denoted by φ_a a tensor field of type (1, 1) and by η_a a 1-form. The Riemannian metric on N is denoted by g. All these tensor fields are related by (see [3])

$$(\varphi_a)^2=-I+\eta_a\otimes\xi_a\,;\quad \varphi_a(\xi_a)=0\,;\quad \eta_a\circ\varphi_a=0\,;\quad \eta_a(\xi_a)=1, \tag{1.1}$$

$$\begin{cases}\eta_a(\xi_b)=0 \text{ for } a\neq b\,;\quad \varphi_a(\xi_b)=-\varphi_b(\xi_a)=\xi_c\,;\\ \eta_a\circ\varphi_b=-\eta_b\circ\varphi_a=\eta_c\,;\quad \varphi_a\circ\varphi_b-\xi_a\otimes\eta_b=-\varphi_b\circ\varphi_a+\xi_b\otimes\eta_a=\varphi_c,\end{cases} \tag{1.2}$$

where (a, b, c) is a cyclic permutation of (1, 2, 3),

$$\eta_a(X)=g(X,\xi_a),\quad g(\varphi_aX,\varphi_aY)=g(X,Y)-\eta_a(X)\eta_a(Y)\,; \tag{1.3}$$

$$(\tilde{\nabla}_X\varphi_a)Y=g(X,Y)\xi_a-\eta_a(Y)X, \tag{1.4}$$

for all vector fields X, Y on N, where I denotes the identity morphism on the tangent bundle to N and $\tilde{\nabla}$ is the Levi Civita connection on N. As it is well known (see [2]) we have

$$\tilde{\nabla}_X\xi_a=-\varphi_aX,\quad a=1, 2, 3, \tag{1.5}$$

for any vector field X tangent to N.

Y. Y. Kuo proved in [3] that K^{4n+3} and S^{4n+3} are examples of manifolds with a Sasakian 3-structure.

Now, let M be an $(m+3)$-dimensional submanifold of N tangent to the structure vector fields ξ_1, ξ_2, ξ_3. Of course, the geometry of M depends on the behaviour of the tangent space T_xM at each $x\in M$ to the actions of φ_a, $a=1, 2, 3$. One of the present authors studied in [1] generic submanifolds and proved that each real hypersurface of N inherites a natural structure of a generic submanifold. We shall give here a generalization of the concept of generic submanifold of N, and obtain fundamental results on the geometry of M.

We call M a *semi-invariant submanifold* if it is endowed with four distributions (D, D_1, D_2, D_3) satisfying the following conditions:

(i) $TM = D \oplus D_1 \oplus D_2 \oplus D_3 \oplus \{\xi_1\} \oplus \{\xi_2\} \oplus \{\xi_3\}$, where $\{\xi_a\}$ denotes the distribution spanned by ξ_a on M,

(ii) the distribution D is invariant by each φ_a, i.e., we have $\varphi_a(D) = D$, $a = 1, 2, 3$,

(iii) $\varphi_a(D_a) \subset TM^\perp$ for each $a = 1, 2, 3$ and $\varphi_a(D_b) = D_c$, where (a, b, c) is a cyclic permutation of $(1, 2, 3)$ and $TM^\perp$ is the normal bundle to M.

We denote by $\Gamma(S)$ the module of all differentiable sections of a vector bundle S on M. Now, we state

Proposition 1.1. *Let M be a semi-invariant submanifold of a manifold N with a Sasakian 3-structure. Then we have*

(a) $\varphi_1(D_1) = \varphi_2(D_2) = \varphi_3(D_3)$ *and*

(b) *the distribution $D, D_1, D_2, D_3, \{\xi_1\}, \{\xi_2\}, \{\xi_3\}$ are mutually orthogonal to each other.*

Proof. (a) Let $V \in \Gamma(\varphi_1(D_1))$, that is, there exists $X_1 \in \Gamma(D_1)$ such that $V = \varphi_1(X_1)$. Next, we note that φ_1 is an isomorphism on the complementary orthogonal distribution to $\{\xi_1\}$ on M. Hence by (iii) there exists $X_3 \in \Gamma(D_3)$ such that $X_1 = \varphi_2 X_3)$. Thus by (1.2) we get $V = \varphi_1(\varphi_2 X_3) = \varphi_3(X_3) \in \Gamma(\varphi_3(D_3))$. Therefore we have $\varphi_1(D_1) \subseteq \varphi_3(D_3)$. In a similar way follows the other inclusions $\varphi_3(D_3) \subseteq \varphi_1(D_1)$, $\varphi_1(D_1) \subseteq \varphi_2(D_2)$, $\varphi_2(D_2) \subseteq \varphi_1(D_1)$ and the proof of the assertion (a) is complete.

(b). For an arbitrary $X \in \Gamma(D)$ there exist $Y_a \in \Gamma(D)$ such that $\varphi_a(Y_a) = X$ since D is an invariant distribution with respect to all φ_a, $a = 1, 2, 3$. Thus by (1.1) and (1.3) we obtain $g(X, \xi_a) = g(\varphi_a Y_a, \xi_a) = -g(Y_a, \varphi_a \xi_a) = 0$, that is the distribution D is orthogonal to each of distributions $\{\xi_a\}$, $a = 1, 2, 3$. Now, let $X_a \in \Gamma(D_a)$. Then by the first part of (iii) there exists $V \in \Gamma(TM^\perp)$ such that $X_a = \varphi_a(V)$. Thus by (1.3) and (1.1) we get

$$g(X_a, \xi_a) = g(\varphi_a V, \xi_a) = -g(V, \varphi_a \xi_a) = 0. \tag{1.6}$$

On the other hand by the second part of (iii) for each $b \neq a$ there exists $X_c \in \Gamma(D_c)$ such that $X_a = \varphi_b(X_c)$, where (a, b, c) is a cyclic permutation of (1, 2, 3.) Hence we have

$$g(X_a, \xi_b) = g(\varphi_b(X_c), \xi_b) = -g(X_c, \varphi_b(\xi_b)) = 0. \tag{1.7}$$

Thus by (1.6) and (1.7) we proved that D_a is orthogonal to each distributions $\{\xi_1\}$, $\{\xi_2\}$, $\{\xi_3\}$.

Next, let $X \in \Gamma(D)$ and $X_a \in \Gamma(D_a)$. Then there exists $Y \in \Gamma(D)$ such that $\varphi_a(Y) = X$ and $V \in \Gamma(TM^\perp)$ such that $X_a = \varphi_a(V)$. Thus we have

$$g(X, X_a) = g(\varphi_a Y, \varphi_a V) = g(Y, V) = 0.$$

Hence the distribution D is orthogonal to each D_a, $a = 1, 2, 3$.

Finally, let $X_a \in \Gamma(D_a)$ and $X_b \in \Gamma(D_b)$ with $a \neq b$. Then, there exists $V_a, V_b \in \Gamma(TM^\perp)$ such that $X_a = \varphi_a(V_a)$ and $X_b = \varphi_b(V_b)$. Thus we have

$$g(X_a, X_b) = g(\varphi_a(V_a), \varphi_b(V_b)) = -g(V_a, (\varphi_a \circ \varphi_b)(V_b)) = -g(V_a, \varphi_c(V_b)) = 0,$$

where (a, b, c) is a cyclic permutation of $(1, 2, 3)$. Therefore D_a is orthogonal to D_b. The proof is complete.

By the assertion (a) of Proposition 1.1 we can put $\bar{D} = \varphi_a(D_a)$ for all $a = 1, 2, 3$. Hence we have a vector subbundle of the normal bundle $TM^\perp$. We denote by $\bar{D}^\perp$ the complementary orthogonal to $\bar{D}$ in $TM^\perp$. Then we see that for $\bar{D}^\perp = \{0\}$ we get a generic submanifold of N (see [1]).

As it was proven in [1] each real hypersurface of N is a non-trivial example of generic submanifold and thus it is also an example of semi-invariant submanifold of N. The geometry of real hypersurfaces in N is studied in [1].

2. **Integrability of distributions on a semi-invariant submanifold.** Let M be an $(m+3)$-dimensional semi-invariant submanifold of a $(4n+3)$-dimensional manifold N with a Sasakian 3-structure $(\varphi_a, \xi_a, \eta_a, g)$, $a = 1, 2, 3$. Denote by P, Q_1, Q_2 and Q_3 the projection morphisms of TM to distributions D, D_1, D_2 and respectively D_3. Then, for any vector field $X \in \Gamma(TM)$ we have

$$X = PX + \sum_{d=1}^{3} \{Q_d X + \eta_d(X) \xi_d\}. \tag{2.1}$$

Applying φ_a to (2.1) and taking account of (1.2) we obtain

$$\varphi_a X = \varphi_a PX + \varphi_a Q_b X + \varphi_a Q_c X + \eta_b(X) \xi_c - \eta_c(X) \xi_b + \varphi_a Q_a(X) \tag{2.2}$$

where (a, b, c) is a cyclic permutation of (1, 2, 3).

Now, we denote by $F_a X$ and $U_a X$ the tangent part and respectively the normal part of $\varphi_a X$. Then from (2.2) we have

$$F_a X = \varphi_a PX + \varphi_a Q_b X + \varphi_a Q_c X + \eta_b(X) \xi_c - \eta_c(X) \xi_b, \tag{2.3}$$

(2.4) $\qquad U_a X = \varphi_a Q_a X$ and $\qquad$ (2.5) $\varphi_a X = F_a X + U_a X$.

We take $X \in \Gamma(D)$ in (2.3) and obtain

$$F_a X = \varphi_a X, \quad a = 1, 2, 3. \tag{2.6}$$

Next, we denote by ∇ the Levi-Civita connection on M and by $\nabla^\perp$ the Riemannian connection on the normal bundle $TM^\perp$. Then the equations of Gauss and Weingarten are given by

(2.7) $\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y)$ and respectively $\quad$ (2.8) $\tilde{\nabla}_X V = -A_V X + \nabla^\perp_X V$,

for all $X, Y \in \Gamma(TM)$ and $V \in \Gamma(TM^\perp)$, where h is the second fundamental form of M and A_V is the fundamental tensor of Weingarten with respect to the normal section V. Moreover, we have

$$g(h(X, Y), V) = g(A_V X, Y). \tag{2.9}$$

For any $V \in \Gamma(TM^\perp)$ we put

$$\varphi_a V = B_a V + C_a V, \quad a = 1, 2, 3, \tag{2.10}$$

where $B_a V$ (resp. $C_a V$) is the tangent (resp. normal) component of $\varphi_a V$.

Lemma 2.1. *Let M be a semi-invariant submanifold of N. Then we have*

(2.11) $\nabla_Y F_a X - F_a(\Delta_Y X) = A_{U_a X} Y + B_a h(X, Y) + g(X, Y) \xi_a - \eta_a(X) Y$ *and*

$$h(Y, F_aX)+\nabla^{\perp}_{Y}U_aX=U_a(\nabla_YX)+C_ah(X, Y), \tag{2.12}$$

for all $X, Y \in \Gamma(TM)$.

Proof. We differentiate covariantly (2.5) and by using (1.4), (2.7) and (2.8) obtain

$$\begin{aligned}\nabla_YF_aX+h(Y, F_aX)-A_{U_aX}Y+\nabla^{\perp}_{Y}U_aX=F_a(\nabla_YX)+U_a(\nabla_YX)+\\ +B_ah(X, Y)+C_ah(X, Y)+g(X, Y)\xi_a-\eta_a(X)Y, \quad a=1, 2, 3.\end{aligned} \tag{2.13}$$

Then (2.11) and (2.12) follows from (2.13) by taking the tangential and normal components, respectively.

Lemma 2.2. *Let* M *be a semi-invariant submanifold of* N. *Then we have*

$$h(X, \xi_a)=0\ ;\ \nabla_X\xi_a=-\varphi_aX \quad \textit{for any} \quad X\in\Gamma(D), \tag{2.14}$$

$$h(Y, \xi_a)=-\varphi_aY\ ;\ \nabla_Y\xi_a=0 \quad \textit{for any} \quad Y\in\Gamma(D_a),\ a=1, 2, 3, \tag{2.15}$$

$$h(Z, \xi_a)=0\ ;\ \nabla_Z\xi_a=-\varphi_aZ \quad \textit{for any} \quad Z\in\Gamma(D_b),\ a\neq b. \tag{2.16}$$

The proof follows from (1.5) by using (2.7) and then taking the tangential and normal components.

Lemma 2.3. *Let* M *be a semi-invariant submanifold of* N. *Then we have*

$$\begin{aligned}g([X, \xi_a], Y)=g([Y, \xi_a], X)=0,\ a=1, 2, 3, \textit{ for any}\\ X\in\Gamma(D) \textit{ and } Y\in\Gamma(D_1\oplus D_2\oplus D_3)\ ;\end{aligned} \tag{2.17}$$

(2.18) $g([X, \xi_a], Y)=0$ *for any* $X\in\Gamma(D_a)$ *and* $Y\in\Gamma(D_b\oplus D_c\oplus\{\xi_b\}\oplus\{\xi_c\})$,

where (a, b, c) *is a cyclic permutation of* $(1, 2, 3)$ *and*

$$\begin{aligned}g([X, \xi_a], \xi_b)=g([Y, \xi_a], \xi_b)=0, \quad \textit{for } a\neq b, \quad X\in\Gamma(D) \textit{ and}\\ Y\in\Gamma(D_1\oplus D_2\oplus D_3).\end{aligned} \tag{2.19}$$

Proof. By using (1.5) and (2.7) we obtain

$$\begin{aligned}g([X, \xi_a], Y)=g(\tilde\nabla_X\xi_a-\tilde\nabla_{\xi_a}X, Y)=-g(\varphi_aX, Y)-\\ -g(\tilde\nabla_{\xi_a}X, Y)=g(X, \tilde\nabla_{\xi_a}Y),\end{aligned} \tag{2.20}$$

for any $X\in\Gamma(D)$ and $Y\in\Gamma(D_1\oplus D_2\oplus D_3)$. Suppose $Y\in\Gamma(D_a)$, $a=1, 2, 3$. Since D is invariant with respect to φ_a and φ_a is an isomorphism on D, there exists $Z\in\Gamma(D)$ such that $X=\varphi_aZ$. Then, using (1.3), (1.4), (2.8), (2.9), (2.14) and (2.20) we get $g([X, \xi_a], Y)=g(\varphi_aZ, \tilde\nabla_{\xi_a}Y)=-g(Z, \varphi_a(\tilde\nabla_{\xi_a}Y))=-g(Z, \tilde\nabla_{\xi_a}\varphi_aY)=g(Z, A_{\varphi_aY}\xi_a)=g(h(Z, \xi_a), \varphi_aY)=0$. In a similar way it follows the equalities (2.18) and (2.19) of the lemma.

A semi-invariant submanifold M of N is called *D-geodesic* if we have

$$h(X, Y)=0 \text{ for any } X, Y\in\Gamma(D). \tag{2.21}$$

Theorem 2.1. *Let* M *be a semi-invariant submanifold of* N. *Then the following assertions are equivalent to each other:*

(i) M *is D-geodesic,*

(ii) *the distribution* $D\oplus\{\xi_1\}\oplus\{\xi_2\}\oplus\{\xi_3\}$ *is integrable,*

(iii) *the second fundamental form* h *of* M *satisfies*

$$h(X, \varphi_aY)=h(Y, \varphi_aX), \textit{ for any } a=1, 2, 3 \textit{ and } X, Y\in\Gamma(D). \tag{2.22}$$

Proof. (i)⇒(ii). By (2.17), in order to prove the integrability of distribution $D\oplus\{\xi_1\}\oplus\{\xi_2\}\oplus\{\xi_3\}$ it suffices to prove

$$\begin{aligned}[X, Y]\in\Gamma(D\oplus\{\xi_1\}\oplus\{\xi_2\}\oplus\{\xi_3\}) \text{ and } [\xi_a, \xi_b]\in\\ \in\Gamma(D\oplus\{\xi_1\}\oplus\{\xi_2\}\oplus\{\xi_3\}),\end{aligned} \tag{2.23}$$

for any $X, Y\in\Gamma(D)$ and $a\neq b$. In (2.12) we take $X, Y\in\Gamma(D)$ and taking account of (2.6) obtain

$$h(Y, \varphi_aX)=U_a\nabla_YX+C_ah(X, Y). \tag{2.24}$$

Then, by using (2.21) in (2.23) we obtain $U_a(\nabla_YX)=0$ for each $a=1, 2, 3$. Thus from (1.3) it follows

$g(\nabla_XY, Q_aZ)=g(\varphi_a\nabla_XY, \varphi_aQ_aZ)=g(\varphi_a\nabla_XY, U_aZ)=0$ for all $Z\in\Gamma(TM)$.

Hence $\nabla_XY\in\Gamma(D\oplus\{\xi_1\}\oplus\{\xi_2\}\oplus\{\xi_3\})$ which implies the first relation of (2.23). On the other hand, by using (1.2) and (1.5) we obtain $[\xi_a, \xi_b]=\tilde\nabla_{\xi_a}\xi_b-\tilde\nabla_{\xi_b}\xi_a=-\varphi_b\xi_a+\varphi_a\xi_b=2\xi_c$, where (a, b, c) is a cyclic permutation of $(1, 2, 3)$. Thus the assertion (ii) is proved.

(ii)⇒(iii). Suppose $D\oplus\{\xi_1\}\oplus\{\xi_2\}\oplus\{\xi_3\}$ is integrable. Then we have

$$U_a([X, Y])=0 \text{ for any } X, Y\in\Gamma(D) \text{ and } a=1, 2, 3. \tag{2.25}$$

On the other hand, by (2.24) we get

$$h(Y, \varphi_aX)-h(X, \varphi_aY)=U_a([Y, X]), \tag{2.26}$$

and the assertion (iii) follows from (2.25) and (2.26).

(iii)⇒(i). Using (2.22) and (1.2) we obtain

$$\begin{aligned}h(\varphi_aX, Y)=h(X, \varphi_aY)=h(X, (\varphi_b\circ\varphi_c)Y)=h(\varphi_bX, \varphi_cY)=h((\varphi_c\circ\varphi_b)X, Y)=\\ =-h(\varphi_aX, Y),\end{aligned}$$

where (a, b, c) is a cyclic permutation of $(1, 2, 3)$, for any $X, Y\in\Gamma(D)$. Thus we have $h(\varphi_aX, Y)=0$ for any $X, Y\in\Gamma(D)$, which implies (2.21) since φ_a is an automorphism of D. The proof is complete.

Lemma 2.4. *Let* M *be a semi-invariant submanifold of* N. *Then we have*

$$A_{\varphi_aX}(Y)=A_{\varphi_aY}(X) \quad \textit{and} \tag{2.27}$$

$$g([X, Y], \xi_b)=0, \textit{ for any } X, Y\in\Gamma(D_a) \textit{ and } a, b,=1, 2, 3. \tag{2.28}$$

Proof. By using (1.1)., (1.3), (2.7), (2.8) and (2.9) we obtain

$$\begin{aligned}g(A_{\varphi_aX}Y, Z)=g(h(Y, Z), \varphi_aX)=g(\tilde\nabla_ZY, \varphi_aX)=-g(\varphi_a(\tilde\nabla_ZY), X)=\\ =-g(\tilde\nabla_Z(\varphi_aY), X)=g(A_{\varphi_aY}Z, X)=g(A_{\varphi_aY}X, Z), \quad a=1, 2, 3,\end{aligned}$$

for any $Z\in\Gamma(TM)$ and $X, Y\in\Gamma(D_a)$. Thus we have (2.27).

Next, since $\tilde{\nabla}$ is the Levi-Civita connection on N, by using (1.3), (1.5) and the definition of a semi-invariant submanifold, we get

$$g([X, Y], \xi_b) = g(\nabla_X Y - \nabla_Y X, \xi_b) = g(\tilde{\nabla}_Y \xi_b, X) - g(\tilde{\nabla}_X \xi_b, Y) = g(\varphi_b X, Y) -$$
$$- g(\varphi_b Y, X) = 2g(\varphi_b X, Y) = 0, \text{ for any } X, Y \in \Gamma(D_a),\ a, b = 1, 2, 3.$$

The proof is complete.

Theorem 2.2 *Let M be a semi-invariant submanifold of N. Then the distributions D_a, $a = 1, 2, 3$ are integrable.*

Proof. By using (2.5) and (2.11) we obtain

$$F_a(\nabla_Y X) = -A_{\varphi_a X} Y - B_a h(X, Y) - g(X, Y)\xi_a, \tag{2.29}$$

for all $a = 1, 2, 3$ and $X, Y \in \Gamma(D_a)$. By changing X and Y in (2,29) and then substracting the obtained relation from (2.29) and by taking account of (2.27) we get

$$F_a([Y, X]) = 0, \text{ for any } X, Y \in \Gamma(D_a). \tag{2.30}$$

Thus, (2.30) and (2.28) imply $[Y, X] \in \Gamma(D_a)$ for any $X, Y \in \Gamma(D_a)$ and the proof is complete.

From (2.18) and Theorem 2.2 we have

Corollary 2.1. *Let M be a semi-invariant submanifold of N. Then the distributions $D_a \oplus \{\xi_a\}$, $a = 1, 2, 3$ are integrable and we have*

$$g(\nabla_{\varphi_a V_1}(\varphi_a V_2), X) = g(\nabla_{\varphi_a V_2}(\varphi_a V_1), X), \tag{2.31}$$

for all $V_1, V_2 \in \Gamma(\tilde{D})$ and $X \in \Gamma(D)$.

Lemma 2.5. *Let M be a semi-invariant submanifold of N. Then we have*

$$g([X, \varphi_a X], \xi_a) \neq 0 \tag{2.32}$$

for any $X \in \Gamma(D)$, $X \neq 0$, $a = 1, 2, 3$ and

$$g([\varphi_a V, \varphi_b V], \xi_c) \neq 0, \tag{2.33}$$

for any $V \in \Gamma(\tilde{D})$, $V \neq 0$, where (a, b, c) is a cyclic permutation of (1, 2, 3).

Proof. By using (1.1), (1.3) and (1.5) we obtain
$g([X, \varphi_a X], \xi_a) = g(X, \tilde{\nabla}_{\varphi_a X}\xi_a) - g(\varphi_a X, \tilde{\nabla}_X \xi_a) = g(\varphi_a X, \varphi_a X) - g(X, \varphi_a^2 X) =$
$= 2g(X, X) \neq 0$, for any $X \in \Gamma(D)$, $X \neq 0$ and $a = 1, 2, 3$. Also, by a direct computation we obtain
$g([\varphi_a V, \varphi_b V], \xi_c) = 2g(V, V) \neq 0$, for any $V \in \Gamma(\tilde{D})$, $V \neq 0$, and any cyclic permutation (a, b, c) of (1, 2, 3). From Lemma 2.5 we have

Theorem 2.3. *Let M be a semi-invariant submanifold of N. Then the distributions D and $D_1 \oplus D_2 \oplus D_3$ are not integrable.*

In the sequel we denote by $D^\perp = D_1 \oplus D_2 \oplus D_3$ and by $\{\xi\} = \{\xi_1\} \oplus \{\xi_2\} \oplus \{\xi_3\}$. Now, we state

Theorem 2.4. *Let M be a semi-invariant submanifold of N. Then the distribution $D^\perp \oplus \{\xi\}$ is integrable if and only if we have*

$$B_a h(X, Y) = 0, \quad a = 1, 2, 3 \tag{2.34}$$

for any $X \in \Gamma(D)$ and $Y \in \Gamma(D^\perp)$.

Proof. Let (a, b, c) be a cyclic permutation of (1, 2, 3) and let $Y \in \Gamma(D_a)$, $Z \in \Gamma(D_b)$ and $U \in \Gamma(D)$. Then, by Proposition 1.1 and the definition of a semi-invariant submanifold, there exist $V_1, V_2 \in \Gamma(\tilde{D})$ and $X \in \Gamma(D)$ such that $Y = \varphi_a V_1$, $Z = \varphi_b V_2$ and $U = \varphi_c X$. Thus by using (1.2), (1.3), (1.4), (2.7) and (2.31) we obtain

$$g([Y, Z], U) = g(\nabla_{\varphi_a V_1}(\varphi_a V_2) + \nabla_{\varphi_b V_1}(\varphi_b V_2), X). \tag{2.35}$$

Now, taking account of (2.17), (2.35) and Theorem 2.2 we obtain that the distribution $D^\perp \oplus \{\xi\}$ is integrable if and only if we have

$$g(\nabla_{\varphi_a V_1}(\varphi_a V_2), X) = 0, \tag{2.36}$$

for all $a = 1, 2, 3$, $V_1, V_2 \in \Gamma(\tilde{D})$ and $X \in \Gamma(D)$.
On the other hand, by using (1.3), (1.4), (2.7), (2.8) and (2.10) we obtain

$$g(\nabla_{\varphi_a V_1}(\varphi_a V_2), X) = g(B_a h(\varphi_a V_1, \varphi_a X), \varphi_a V_2). \tag{2.37}$$

Thus the assertion of Theorem 2.4 follows from (2.36) and (2.37).

3. Geometry of leaves on a semi-invariant submanifold. In this section we investigate whether leaves of integrable distributions on M are totally geodesic immersed either in M or in N.

First we state

Theorem 3.1. *Let M be a semi-invariant submanifold of N. Suppose the distribution $D \oplus \{\xi_1\} \oplus \{\xi_2\} \oplus \{\xi_3\}$ is integrable. Then each leaf of $D \oplus \{\xi_1\} \oplus \{\xi_2\} \oplus \{\xi_3\}$ is totally geodesic immersed in N.*

Proof. Let M^* be a leaf of $D \oplus \{\xi_1\} \oplus \{\xi_2\} \oplus \{\xi_3\}$. Denote by A^*_V the fundamental tensor of Weingarten of the immersion of M^* in N. We note that the normal bundle to M^* is just the vector bundle $D^\perp \oplus TM^\perp$. Now, we take $Z_a \in \Gamma(D_a)$, $X \in \Gamma(D \oplus \{\xi_1\} \oplus \{\xi_2\} \oplus \{\xi_3\})$, $Y \in \Gamma(D)$ and by using the equation of Weingarten for the immersion of M^* in N, (1.3), (1.4) and Theorem 2.1 we obtain

$$g(A^*_{Z_a} X, Y) = 0. \tag{3.1}$$

On the other hand, by using the same equation of Weingarten for the immersion of M^* in N and (1.5) we get

$$g(A^*_{Z_a} X, \xi_b) = 0, \quad a, b = 1, 2, 3. \tag{3.2}$$

Finally, by using the equation of Weingarten for both immersions of M^* in N and of M in N, Lemma 2.2 and Theorem 2.1 we get

$$g(A^*_V X, Y) = 0, \tag{3.3}$$

for any $X, Y \in \Gamma(D \oplus \{\xi_1\} \oplus \{\xi_2\} \oplus \{\xi_3\})$ and $V \in \Gamma(TM^\perp)$.
Thus, by (3.1), (3.2) and (3.3) we have $A^*_W = 0$ for each W normal to M^*, that is M^* is totally geodesic immersed in N.

Theorem 3.2. *Let M a semi-invariant submanifold of N. If the distribution $D^\perp \oplus \{\xi\}$ is integrable then each leaf of it is totally geodesic immersed in M, but it is never totally geodesic immersed in N.*

Proof. Let M' be a leaf of the distribution $D^{\perp}\oplus\{\xi\}$ and denote by ∇' the Levi-Civita connection on M'. Then the equation of Gauss for the immersion of M' in M is given by

$$\nabla_X Y=\nabla'_X Y+h'(X, Y), \text{ for any } X, Y\in\Gamma(D^{\perp}\oplus\{\xi\}), \tag{3.4}$$

where h' is the second fundamental form of the immersion of M' in M. Now, we take $Y\in\Gamma(D_a)$. Then there exists $V\in\Gamma(\tilde{D})$ such that $Y=\varphi_a V$ and by using (1.3), (1.4), (2.7), (2.9) and (2.4) we obtain

$$g(h'(X, Y), Z)=g(h(X, \varphi_a Z), V)=g(B_a h(X, \varphi_a Z), Y) \tag{3.5}$$

for all $X\in\Gamma(D^{\perp})$ and $Z\in\Gamma(D)$. Taking account that $D^{\perp}\oplus\{\xi\}$ is supposed integrable, by (2.34) and (3.5) we get

$$g(h'(X, Y), Z)=0, \tag{3.6}$$

for any $X\in\Gamma(D^{\perp})$, $Y\in\Gamma(D_a)$ and $Z\in\Gamma(D)$.
On the other hand, by (1.5), (2.7) and (3.4) we obtain

$$g(h'(X, \xi_a), Z)=0, \tag{3.7}$$

for any $X\in\Gamma(D^{\perp}\oplus\{\xi\})$ and $Z\in\Gamma(D)$. Thus by (3.6) and (3.7) we infer that M' is totally geodesic immersed in M.
Next, we denote by h^* the second fundamental form for the immersion of M' in N. Then we have

$$h^*(X, Y)=h(X, Y)+h'(X, Y), \tag{3.8}$$

for any $X, Y\in\Gamma(D^{\perp}\oplus\{\xi\})$. We take a nonzero vector field $X\in\Gamma(D_a)$ and $Y=\xi_a$ in (3.8). Then, taking acoount that M' is totally geodesic immersed in M, by (2.15) we get $h^*(X, \xi_a)=h(X, \xi_a)=-\varphi_a X\neq 0$. Hence M' is not totally geodesic immersed in N and the proof of Theorem 3.2 is complete.

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ON NORMAL CURVATURE OF A HYPERSURFACE OF FINSLER SPACES

BY

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Introduction. A systematic study of geometry of hypersurfaces in Finsler spaces was recently given by Matsumoto [5]. He gave perfect proofs of Haimovici-Kikuchi-Rapcsak's results and, in particular, concluded: There exists a hyperplane of the 1-st kind (with respect to CΓ) which is characterized by the equation $H_\alpha=0$ at each hypersurface element of a Finsler space $F^n(n>2)$, if and only if F^n is projectively flat.

The main purpose of this paper is to give an answer of Professor Dr. Aurel Bejancu's conjecture. A totally geodesic hypersurface of a projectively flat Finsler space is also projectively flat*. This will be proved in §2. §1 is devoted to a study of a second fundamental tensor $\Omega_{\alpha\beta}$ defined by Berwald, and we consider the umbilical hypersurface with respect to the Cartan connection.

The terminologies and notations in the present paper are referred to M. Matsumoto's paper [5] unless otherwise stated.

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§1. Berwald's second fundamental tensor. We shall consider an n-dimensional Finsler space $F^n=(M^n, L(x, y))$, reffered to a local coordinate system (x^i), whose fundamental function $L(x, y)$ is (1) p-homogeneous in $y=(y^i)$. The metric tensor of the F^n is defined by

$$g_{ij}(x, y)=\partial^2(L^2/2)/\partial y^i\partial y^j.$$

A hypersurface M^{n-1} of the underlying differentiable manifold M^n may be parametrically represented by the equations

$$x^i=x^i(u^\alpha), \quad \alpha=1,\dots, n-1,$$

where u^α are Gaussian coordinates on M^{n-1}. We assume the matrix whose elements are

$$B^i_\alpha(u)=\partial x^i/\partial u^\alpha$$

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* This conjecture was proposed during the Romanian-Japanese Colloquium on Finsler Geometry in Romania, 1984.