ANALELE ŞTIINŢIFICE ALE UNIVERSITĂŢII „AL. I. CUZA" DIN IAŞI
Tomul XXXI-1, s. I a. Matematică, 1985

ON SEVERAL CLASSES OF SETS

BY

HARRY I. MILLER and POLYCHRONIS J. XENIKAKIS

1. **Introduction.** If A is Lebesgue measurable and mA, the Lebesgue measure of A, is positive, then by a classical result of Steinhaus [10], $D(A)$ contains an open interval, where $D(A)$ is the set defined by the equation $D(A)=\{x-y : x, y \in A\}$. The set A is called a Baire set if A [6], can be written, in the form $A=(G \setminus P) \cup Q$, where G is an open set and P and Q are sets of the first category. Piccard [7] proved a Baire set analogue of the result of Steinhaus; namely he proved that $D(A)$ contains an open interval if A is a Baire set of the second Baire category.

Sets M and N will be called similar in case there exists a function $f, f : R \to R$, of the form $f(x)=ax+b$, $a \neq 0$, such that $f(M)=N$. Ruziewicz [9] showed that if $mA>0$ and E is any finite set, then A contains a subset E', such that E and E' are similar. Miller [3] proved the Baire set analogue of this result.

It is natural to consider the following classes:

$\mathcal{A}=\{A : D(A)$ contains an open interval$\}$;

$\mathcal{B}=\{B :$ for each finite set of reals E, B contains a subset E' such that E and E' are similar$\}$;

If we set

$m^{+}=\{A : A$ is measurable and $mA>0\}$ and

$B_a^{+}=\{A : A$ is a Baire set of the second Baire category$\}$, then by the four previously mentioned theorems we have:

$$m^{+} \cup B_a^{+} \subseteq \mathcal{A} \cap \mathcal{B}.$$

Let $\mathcal{N}$ denote the collection of all universal null sets, i.e. $N \in \mathcal{N}$ if and only if $\alpha N=0$ for each measure α induced by a nondecreasing continuous function.

In this paper several results about the five above mentioned collections are proved. Among the theorems presented here are the following. The collections $\mathcal{A}$ and $\mathcal{B}$ are incomparable, i.e. $\mathcal{A} \nsubseteq \mathcal{B}$ and $\mathcal{B} \nsubseteq \mathcal{A}$ and there exists a set $N \in \mathcal{N}$ such that N is a non Baire set of the second Baire category and $N \in \mathcal{A} \cap \mathcal{B}$. From this it follows that

$$m^{+} \cup B_a^{+} \subsetneq \mathcal{A} \cap \mathcal{B}.$$

2. **Results.** We first show (in Theorems 1 and 2), without using the continuum hypothesis, that $\mathcal{A}$ and $\mathcal{B}$ are incomparable.

Theorem 1. $\mathcal{A} \nsubseteq \mathcal{B}$. *Proof.* Consider a fixed subset of R with three elements, say $S=\{0, 1, 2\}$. We will, using transfinite induction, construct a subset A of R such that $D(A)=R$ and such that no subset of A is similar to S.

By the well ordering principle the set of positive real numbers can be written in the form $\{p_\alpha\}_{\alpha<\omega_c}$, where ω_c is the first ordinal having c (the cardinal of the continuum) predecessors.

Let A_1 be an arbitrary two point set, $A_1=\{a_{11}, a_{12}\}$, such that $a_{12}-a_{11}=p_1$. Suppose that β is an ordinal number, $\beta<\omega_c$, and that for each $\alpha<\beta$ a subset A_α of the real line has been defined such that :

a) $A_{\alpha_1} \subseteq A_{\alpha_2}$ if $\alpha_1<\alpha_2<\beta$.

b) A_α contains a pair of points $a_{\alpha 1}$, $a_{\alpha 2}$ such that $a_{\alpha 2}-a_{\alpha 1}=p_\alpha$ for each $\alpha<\beta$.

c) A_α contains no subset similar to S for each $\alpha<\beta$.

d) $\operatorname{card}(A_\alpha) \leqslant \max(\operatorname{card} \alpha, \aleph_0)$ for every $\alpha<\beta$.

Let $C_\beta=\bigcup_{\alpha<\beta} A_\alpha$. Then C_β contains no set similar to S and $\operatorname{card}(C_\beta) \leqslant \max(\operatorname{card} \beta, \aleph_0)$. Furthermore, $\operatorname{card}\{(a_{\beta 1}, a_{\beta 2}) \in R \times R : a_{\beta 2}-a_{\beta 1}=p_\beta\}=c$.

Given any two points, say x and y, in C_β, there exist at most 6 pairs, $(a^i_{\beta 1}, a^i_{\beta 2}) \in R \times R$, $i=1, 2, \ldots, 6$ such that $a^i_{\beta 2}-a^i_{\beta 1}=p_\beta$ for each $i=1, 2, \ldots, 6$ and such that $\{x, y, a^i_{\beta 1}, a^i_{\beta 2}\}$ contains a set which contains x and y and is similar to S for each $i=1, 2, \ldots, 6$. Also, given any point, say z, in C_β, there exist 3 pairs $(a^i_{\beta 1}, a^i_{\beta 2}) \in R \times R$, $i=1, 2, 3$, such that $a^i_{\beta 2}-a^i_{\beta 1}=p_\beta$ for each $i=1, 2, 3$ and such that the set $\{z, a^i_{\beta 1}, a^i_{\beta 2}\}$ is similar to S for each $i=1, 2, 3$. Therefore, since $\operatorname{card}(C_\beta)<c$, there exists a pair $a_{\beta 1}$, $a_{\beta 2}$ such that $a_{\beta 2}-a_{\beta 1}=p_\beta$ and such that $A_\beta=C_\beta \cup \{a_{\beta 1}, a_{\beta 2}\}$ contains no subset similar to S.

Hence, by transfinite induction there exists a transfinite sequence of subsets of R, $\{A_\alpha\}_{\alpha<\omega_c}$, satisfying the following conditions.

a') $A_{\alpha_1} \subseteq A_{\alpha_2}$ if $\alpha_1<\alpha_2<\omega_c$.

b') A_α contains a pair of points $a_{\alpha 1}$, $a_{\alpha 2}$ such that $a_{\alpha 2}-a_{\alpha 1}=p_\alpha$ for each $\alpha<\omega_c$.

c') A_α contains no subset similar to S for each $\alpha<\omega_c$.

d') $\operatorname{card}(A_\alpha) \leqslant \max(\operatorname{card} \alpha, \aleph_0)$ for every $\alpha<\omega_c$.

Let $A=\bigcup_{\alpha<\omega_c} A_\alpha$ and it is clear that $A \in \mathcal{A}$ and that $\mathcal{A} \notin \mathcal{B}$ since A contains no subset similar to S.

Remark 1. It is well known that $D(C)=[-1, 1]$, where C denotes the Cantor set. The current authors [4] have recently shown that $C \notin \mathcal{B}$ (and hence $\mathcal{A} \nsubseteq \mathcal{B}$).

Theorem 2. $\mathcal{B} \nsubseteq \mathcal{A}$. *Proof.* Let $\mathcal{S}$ denote the class of all finite non-empty subsets of the real line that do not contain the number zero. By the well ordering principle $\mathcal{S}$ can be written in the form $\{S_\alpha\}_{\alpha<\omega_c}$. We will construct, using transfinite induction, a set $B \in \mathcal{B}$ such that $B \notin \mathcal{A}$.

Suppose $S_1=\{a_{11}, a_{12}, \ldots, a_{1n(1)}\}$, where the elements of S_1 are listed in increasing order. Consider the family $\{sS_1\}_{s \in R}$, where $sS_1=\{sa_{11}, sa_{12}, \ldots, sa_{1n(1)}\}$.

If $1 \leqslant i<j \leqslant n(1)$, then the set $\{s \in R : s>0,\ s(a_{1i}-a_{1j})$ is rational is countable and therefore there exists $s_1 \in R$, $s_1>0$ such that the difference set $D(s_1S_1)=\{t-t' : t, t' \in s_1S_1\}$ contains no rational number other than zero. Set $B_1=s_1S_1$.

Now suppose β is an ordinal number, $\beta<\omega_c$, and that for each $\alpha<\beta$ we have defined a set B_α such that :

a) $B_{\alpha_1} \subset B_{\alpha_2}$ if $\alpha_1<\alpha_2<\beta$.

b) B_α contains a set similar to S_α, for every $\alpha<\beta$.

c) $\operatorname{card}(B_\alpha) \leqslant \max(\operatorname{card} \alpha, \aleph_0)$ for every $\alpha<\beta$.

d) $D(B_\alpha)$ contains no other rational number than zero for every $\alpha<\beta$.

Let $C_\beta=\bigcup_{\alpha<\beta} B_\alpha$. Then $\operatorname{card}(C_\beta) \leqslant \max(\operatorname{card} \beta, \aleph_0)$ and $D(C_\beta)$ contains no other rational number that zero.

If $S_\beta=\{a_{\beta 1}, a_{\beta 2}, \ldots, a_{\beta n(\beta)}\}$, then, as before, for each $1 \leqslant i<j \leqslant n(\beta)$, the set $\{s \in R : s>0,\ s(a_{\beta i}-a_{\beta j})$ is rational$\}$ is countable. Furthermore, since $a_{\beta i} \neq 0$ for each i, $1 \leqslant i \leqslant n(\beta)$, card $\{s \in R : s>0$, for which there exists an element $b \in C_\beta$, such that $b-sa_{\beta i}$ is a rational number other than zero for some $i \in \{1, 2, \ldots, n(\beta)\}\} \leqslant [\max(\operatorname{card}_\beta, \aleph_0)].\aleph_0<c$ (as $\mathcal{B}<\omega_c$). Therefore there exists an $s_\beta \in R$, $s_\beta>0$, such that $D(B_\beta)$ contains no rational number other than zero, where $B_\beta=C_\beta \cup s_\beta S_\beta$.

Therefore, by transfinite induction, there exists a transfinite sequence of subsets of R, $\{B_\alpha\}_{\alpha<\omega_c}$ with the following properties.

a') $B_{\alpha 1} \subset B_{\alpha 2}$ if $\alpha_1<\alpha_2<\omega_c$

b') B_α contains a subset similar to S_α, for every $\alpha<\omega_c$.

c') $D(B_\alpha)$ contains no rational number other than zero for each $\alpha<\omega_c$.

Let $B=\bigcup_{\alpha<\omega_c} B_\alpha$. Clearly $B \notin \mathcal{A}$ and by b') $B \in \mathcal{B}$, for if B contains a subset similar to each set in $\mathcal{S}$, then B contains a subset similar to each finite subset of R.

Remark 2. Actually a result stronger than Theorem 2 holds. Specifically $\mathcal{C} \nsubseteq \mathcal{A}$, where $\mathcal{C}$ denotes the collection of all subsets C of the real line with the property that for each at most countable subset E of the reals, C contains a subset C' that is similar to E.

In the introduction we mentioned that $m^+ \cup Ba^+ \subseteq \mathcal{A} \cap \mathcal{B}$. The next result shows that the symbol $\subseteq$ in the last sentence can actually be replaced by $\subsetneq$.

In the following, Q will denote the set of all rational numbers. A set D is said to be concentrated on Q [8] in case $D \setminus T$ is at most countable for each open set T containing Q.

Theorem 3. *Assuming the continuum hypothesis, there exists a set N concentrated on Q with $N \in \mathcal{A} \cap \mathcal{B}$.*

Proof. Assuming the continuum hypothesis, the cardinal of ω_c is $\aleph_1$ and therefore $\operatorname{card} \tau \leqslant \aleph_0$ for each $\tau<\omega_c$.

Let $\mathcal{R}$ denotes the class of all open sets containing Q, R^+ the set of positive real numbers and $\mathcal{S}$ the class of all non-empty finite subsets of R.

By the well ordering principle $\mathcal{R}$, R^+, and $\mathcal{S}$ can be written as transfinite sequences, namely we have :

$$\mathcal{R}=\{R_\tau\}_{\tau<\omega_c} \qquad R^+=\{t_\tau\}_{\tau<\omega_c} \qquad \mathcal{S}=\{S_\tau\}_{\tau<\omega_c}.$$

Take x_1, y_1 to be any two points of R_1 such that $x_1-y_1=t_1$. Clearly such points exist as $R_1=R\setminus(R\setminus R_1)$ and $R\setminus R_1$ is nowhere dense and $(t+(R\setminus M))\cap(R\setminus M)\neq\emptyset$ for every $t\in R$ if M is a set of the first Baire category, where $t+(R\setminus M)=\{t+u : u\in R\setminus M\}$.

Since R_1 is a Baire set of the second category, R_1 contains a subset, say T_1, similar to S_1 (see [3], Theorem 4).

Now suppose that $\tau<\omega_c$ and that x_σ, y_σ and T_σ have been chosen for all ordinals σ less than τ in a way that ensures:

i) $\{x_\sigma, y_\sigma\}\cup T_\sigma\subseteq \bigcap\limits_{\alpha\leqslant\sigma} R_\alpha$, for all $\sigma<\tau$.

ii) $x_\sigma-y_\sigma=t_\sigma$ and T_σ is similar to S_σ, for all $\sigma<\tau$.

We now proceed to select x_τ, y_τ and T_τ. Consider the set

$$E_\tau=\bigcup_{\sigma\leqslant\tau}(R\setminus R_\sigma).$$

E_τ is a countable union of no — where dense sets. By our previous remarks (in the case $\tau=1$) there exist x_τ, $y_\tau\in R\setminus E_\tau$ such that $x_\tau-y_\tau=t_\tau$. Furthermore, again by [3], Theorem 4, $R\setminus E_\tau$ contains a set, say T_τ, similar to S_τ.

Therefore by transfinite induction we have constructed transfinite sequences $\{x_\tau\}_{\tau<\omega_c}$, $\{y_\tau\}_{\tau<\omega_c}$, and $\{T_\tau\}_{\tau<\omega_c}$ satisfying the following properties.

i') $\{x_\alpha, y_\alpha\}\cup T_\alpha\subseteq\bigcap\limits_{\beta\leqslant\alpha} R_\beta$ for all $\alpha<\omega_c$

ii') $x_\alpha-y_\alpha=t_\alpha$ and T_α is similar to S_α for all $\alpha<\omega_c$.

Let $N=\bigcup\limits_{\alpha<\omega_c}(\{x_\alpha, y_\alpha\}\cup T_\alpha)$. Clearly $N\in\mathcal{A}\cap\mathcal{B}$.

If $\hat{R}$ is any open set that contains Q, then $\hat{R}=R_\alpha$ for some ordinal α, $\alpha<\omega_c$.

Then by i') we have:

$$\{x_\tau, y_\tau\}\cup T_\tau\subseteq R_\alpha \quad \text{for all } \alpha\leqslant\tau<\omega_c$$

and therefore the only points of N not in the set $\hat{R}=R_\alpha$ belong to the countable set

$$\bigcup_{\tau<\alpha}(\{x_\tau, y_\tau\}\cup T_\tau).$$

Therefore N is concentrated on Q.

Remark 3. If N is any set concentrated on Q, $\varepsilon>0$ and α is a measure induced by a non-decreasing continuous function, then clearly there exists a sequence (I_n) of open intervals, whose union contains Q and such that $\alpha(I_n)<\varepsilon/2^n$ for each n. $N\setminus\bigcup I_n$ is countable since $\bigcup I_n$ is open and contains Q and therefore has α measure zero. Hence the α-outer measure of N is less than ε, for each $\varepsilon>0$ and therefore $\alpha(N)=0$, which implies that N is a universal null set.

In addition, if N is a universal null set and $M\subseteq N$ then $M\notin B_a^+$. To see this observe that if $M\in B_a^+$ then it can be written in the form $M=G\cup P$, where G is an uncountable G_δ set and P is a set of the first category ([6], p. 20). There exists ([6], p. 23) a subset A of G and a non-decreasing

continuous function g which maps A onto the Cantor set ([6], p. 4). Let F denote the Cantor function. Let α denote the measure induced by the non-decreasing continuous composite function $F\circ g$. Then $\alpha(A)=1$. This contradicts the fact that N is a universal null set.

By the above, N, the set constructed in the proof of Theorem 3, is in $\mathcal{A}\cap\mathcal{B}$ and $N\notin m^+\cup B_a^+$. Therefore $m^+\cup B_a^+\nsubseteq\mathcal{A}\cap\mathcal{B}$.

The last mentioned proper inclusion can also be obtained without using the continuum hypothesis by constructing, using transfinite induction, a Bernstein set B (see [6]), satisfying $B\in\mathcal{A}\cap\mathcal{B}$ since any Bernstein set is non-measurable and lacks the property of Baire. Erdös and Kakutani [2] have constructed a bounded perfect set E of measure zero having the so-called S_ω property that is for every natural number n there exists a constant $u_n>0$ such that for every set of n points $\{a_1, a_2, \ldots, a_n\}$ of diameter $<u_n$ the set $\bigcap\limits_{s=1}^{n}(E+a_s)$ is non empty. It is a straight forward exercise to show that this implies $E\in\mathcal{A}\cap\mathcal{B}$, and therefore once again it follows that $m^+\cup B_a^+\subsetneq\mathcal{A}\cap\mathcal{B}$.

The set E_4 in the paper of Davies, Marstand and Taylor [1] also satisfies $E_4\in(\mathcal{A}\cap\mathcal{B})\setminus(m^+\cup B_a^+)$, if h, the measure function used in the construction of E_4, is taken to be the identity function.

Remark 4. The construction of the set N in the proof of Theorem 3 can be modified so as guarantee that N is a set of the second Baire category (in addition to having the other properties prescribed in Theorem 3).

Specifically, a set A is of the second Baire category if and only if $(R\setminus\bigcup\limits_{i=1}^{\infty}F_i)\cap A\neq\emptyset$ for each sequence $[F_i]_{i=1}^{\infty}$ of closed nowhere dense sets.

By the well ordering principle the collection of sets $\{R\setminus\bigcup\limits_{i=1}^{\infty}F_i$: each F_i is closed and nowhere dense$\}$ can be written in the form $\{V_\tau\}_{\tau<\omega_c}$.

Using transfinite induction, exactly as in the proof of Theorem 3, we can get $N=\bigcup\limits_{\alpha<\omega_c}(\{x_\alpha, y_\alpha, z_\alpha\}\cup T_\alpha)$, where x_α, y_α and T_α are the same as in the proof of Theorem 3 and $z_\alpha\in V_\alpha$ for every $\alpha<\omega_c$.

REFERENCES

1. Davies R., Marstand J., Taylor S. — *On intersections of transforms of linear sets*, Colloquium Matematicum, 2 (1960), 237–243.
2. Erdös P. and Kakutani S. — *On a perfect set*, Colloquium Matematicum 4 (1957), 195–196.
3. Miller H. — *Relationships between various gauges of the size of sets of real numbers (II)*, Akademija Nauka i Umjetnosti BiH, LIX (1976), 37–48.
4. Miller H. and Xenikukis P. — *Some results on the Cantor set*, (to appear).
5. Munroe M. — *Introduction to Measure and Integration*, Addison-Wesley, Reading Mass. 1953.

6. Oxtoby J. — *Measure and Category*, Springer-Verlag, New York, 1971.
7. Piccard S. — *Sur les ensembles de distances des ensembles de points d'un espace euclidien*, Mêm. Univ. Neuchâtel, Vol. 13, Secrétariat de l'Universite Neuchâtel (1939).
8. Rodgers C. — *Hausdorff Measures*, Cambridge University Press, Cambridge, 1970.
9. Ruziewicz M. — *Contribution à l'étude des ensembles de distances des points*, Fund. Math., 7 (1925), 141—143.
10. Steinhaus H.— *Sur les distances de points des ensembles de mesure positive*, Fund. Math., (1920), 93—104.

Received 20. IV. 1984

Department of Mathematics
University of Sarajevo Sarajevo, Yugoslavia

Department of Mathematics
University of Thessaloniki
Thessaloniki, Greece

ANALELE ŞTIINŢIFICE ALE UNIVERSITĂŢII „AL. I. CUZA" DIN IAŞI
Tomul XXXI-1, s. I a. Matematică, 1985

LOCALISATION OF *BI*-MAPPINGS BETWEEN STANDARD *H*-CONES OF FUNCTIONS

BY

EUGEN POPA

Let S and T be standard H-cones of functions on the sets X and Y. $\varphi : Y \to X$ is called a Bl-Mapping (from S to T) if, for each $s \in S_0$, $s \circ \varphi$ is a nearly continuous element from T. We give necessary and sufficient conditions for a Bl-Mapping to localize, in the following sense: for each naturally (resp. finely) open set $U \subseteq X$, $\varphi|_{\varphi^{-1}(U)}$ is a H-map from $\mathcal{S}'(U)$ to $\mathcal{T}'(\varphi^{-1}(U))$ (resp. from $\mathcal{S}(U)$ to $\mathcal{T}(\varphi^{-1}(U))$).

In the particular case of harmonic spaces, we obtain necessary and sufficient conditions for a *Bl*-Mapping to be a harmonic morphism (resp. a finely harmonic morphism in the sense of Fuglede). These results show the possibility to describe, even in the classical case, various properties of morphisms of H-cones, using commutation conditions with suitable balayages. They also clarify a hypothesis used in [13].

§ 1. Preliminaries. All the notions and notations about H-cones are as in [2]. For the definitions and results concerning harmonic spaces, see [4]. The notion of H-morphism is introduced in [9] or [2, p. 43]. Moreover, the following definitions and notations will be used. Let S and T be H-cones of functions on the sets X and Y. A map $\varphi : Y \to X$ is called a H-map (from S to T, see [9]) if, for any bounded $s \in S$, we have $s \circ \varphi \in T$. If S and T are standard H-cones of functions, φ is called a *Bl*-mapping if, for any $s \in S_0$, $s \circ \varphi$ is a nearly continuous element from T. Clearly, any *Bl*-mapping is a H-map. We remark also that if φ is a H-map from S to T, then φ^* exists and is semi-finite (cf. [9]).

Let S be a standard H-cone of functions on the nearly saturated space X. The next definitions were introduced in [1]. For each finely open set $G \subseteq X$, $\mathcal{S}^1(G)$ denotes the set of all functions $f : G \to \overline{\mathbb{R}}_+$, which are finite on a finely dense set, and for any finely open set $D \subseteq X$, which has the fine closure (denoted by fcl D) contained in G, we have $f|_D \in \mathcal{S}(D)$. X is said to be semi-saturated if any H-integral on S, dominated by a H-measure, is itself a H-measure.

The notion of harmonic morphism was introduced in [3]. We use the following definition: let X and Y be harmonic spaces and $\varphi : Y \to X$ be a continuous map. φ is called a harmonic morphism if, for any open set $U \subseteq X$ and any hyperharmonic function s on U, $s \circ \varphi$ is hyperharmonic on $\varphi^{-1}(U)$. By a regular set (in the case of H-cones of functions, as well as for harmonic