6. O x t o b y J. — *Measure and Category*, Springer-Verlag, New York, 1971.
7. P i c c a r d S. — *Sur les ensembles de distances des ensembles de points d'un espace euclidien*, Mém. Univ. Neuchâtel, Vol. 13, Secrétariat de l'Universite Neuchâtel (1939).
8. R o d g e r s C. — *Hausdorff Measures*, Cambridge University Press, Cambridge, 1970.
9. R u z i e w i c z M. — *Contribution à l'étude des ensembles de distances des points*, Fund. Math., 7 (1925), 141–143.
10. S t e i n h a u s H. — *Sur les distances de points des ensembles de mesure positive*, Fund. Math., (1920), 93–104.

Received 20. IV. 1984

Department of Mathematics
University of Sarajevo Sarajevo,
Yugoslavia
Department of Mathematics
University of Thessaloniki
Thessaloniki, Greece

ANALELE ŞTIINŢIFICE ALE UNIVERSITĂŢII „AL. I. CUZA" DIN IAŞI
Tomul XXXI-1, s. I a. Matematică, 1985

LOCALISATION OF *Bl*-MAPPINGS BETWEEN STANDARD *H*-CONES OF FUNCTIONS

BY

EUGEN POPA

Let S and T be standard H-cones of functions on the sets X and Y. $\varphi: Y\to X$ is called a Bl-Mapping (from S to T) if, for each $s\in S_0$, $s\circ\varphi$ is a nearly continuous element from T. We give necessary and sufficient conditions for a Bl-Mapping to localize, in the following sense: for each naturally (resp. finely) open set $U\subseteq X$, $\varphi|_{\varphi^{-1}(U)}$ is a H-map from $\mathcal{S}'(U)$ to $\mathcal{T}'(\varphi^{-1}(U))$ (resp. from $\mathcal{S}(U)$ to $\mathcal{T}(\varphi^{-1}(U))$).

In the particular case of harmonic spaces, we obtain necessary and sufficient conditions for a *Bl*-Mapping to be a harmonic morphism (resp. a finely harmonic morphism in the sense of Fuglede). These results show the possibility to describe, even in the classical case, various properties of morphisms of H-cones, using commutation conditions with suitable balayages. They also clarify a hypothesis used in [13].

§ **1. Preliminaries.** All the notions and notations about H-cones are as in [2]. For the definitions and results concerning harmonic spaces, see [4]. The notion of H-morphism is introduced in [9] or [2, p. 43]. Moreover, the following definitions and notations will be used. Let S and T be H-cones of functions on the sets X and Y. A map $\varphi: Y\to X$ is called a H-map (from S to T, see [9]) if, for any bounded $s\in S$, we have $s\circ\varphi\in T$. If S and T are standard H-cones of functions, φ is called a *Bl*-mapping if, for any $s\in S_0$, $s\circ\varphi$ is a nearly continuous element from T. Clearly, any *Bl*-mapping is a H-map. We remark also that if φ is a H-map from S to T, then φ^* exists and is semi-finite (cf. [9]).

Let S be a standard H-cone of functions on the nearly saturated space X. The next definitions were introduced in [1]. For each finely open set $G\subseteq X$, $\mathcal{S}'(G)$ denotes the set of all functions $f: G\to \overline{\mathbb{R}}_+$, which are finite on a finely dense set, and for any finely open set $D\subseteq X$, which has the fine closure (denoted by fcl D) contained in G, we have $f|_D\in\mathcal{S}(D)$. X is said to be semi-saturated if any H-integral on S, dominated by a H-measure, is itself a H-measure.

The notion of harmonic morphism was introduced in [3]. We use the following definition: let X and Y be harmonic spaces and $\varphi: Y\to X$ be a continuous map. φ is called a harmonic morphism if, for any open set $U\subseteq X$ and any hyperharmonic function s on U, $s\circ\varphi$ is hyperharmonic on $\varphi^{-1}(U)$. By a regular set (in the case of H-cones of functions, as well as for harmonic spaces) we understand a finely open set $U \subseteq X$, such that $b(X \setminus U) = X \setminus U$. Since the fine topology is regular, there exists a base $\mathcal{U}$ for the fine topology, which is formed of regular sets. Let X and Y be topological spaces; $\mathcal{F}$, and $\mathcal{G}$ be presheaves of numerical functions on X, resp. Y; and $\varphi : Y \to X$ be a continuous map. We call (in this paper) φ a morphism of presheaves if, for any open set $U \subseteq X$ and any bounded $f \in \mathcal{F}(U)$, we have $f\circ\varphi \in \mathcal{G}(\varphi^{-1}(U))$.

We recall some known results about the localisation of *H-cones*.

Proposition 1.1. *Let $(X, \mathcal{U})$ be a B-harmonic space with countable base (see [4]), such that $1 \in \mathcal{U}(X)$. We denote by S the set of positive, superharmonic functions of X.*

a) *S is a standard H-cone of functions on the semi-saturated space X; the topology of X is the same as the natural topology with respect to S.*

b) *For any open set $U \subseteq X$: $\mathcal{S}(U) = \mathcal{S}'(U)$ is the set of hyperharmonic functions on U, which are finite on a finely dense set (cf. [1], [12]). Particularly, $\mathcal{S}'$ is a complete presheaf for the natural topology.*

c) *If, moreover, X satisfies the axiom of domination, then for any finely open set $G \subseteq X$: $\mathcal{S}(G) = \mathcal{S}'(G)$ is an increasingly dense and solid convex subcone in the set of all the positive, finely hyperharmonic functions on U, in the sense of Fuglede (see [5], [8]). Particularly, $\mathcal{S}$ is a complete presheaf with respect to the fine topology.*

Remark. Let S be a standard H-cone of functions on the nearly saturated space X. If $\mathcal{S}$ is a natural complete presheaf, then $\mathcal{S}'$ is also a natural complete presheaf.

Proposition 1.2. a) *Let X and Y be B-harmonic spaces with countable bases, such that $1 \in \mathcal{U}(X)$, $1 \in \mathcal{V}(Y)$. We denote by S and T the set of positive superharmonic functions on X and Y. Let $\varphi : Y \to X$ be a H-map from S to T, such that: (i) φ is continuous; (ii) $\forall U \subseteq X$ open $\varphi|_{\varphi^{-1}(U)}$, is a H-map from $\varphi'(U)$ to $T'(\varphi^{-1}(U))$. Then φ is a harmonic morphism.*

b) *Suppose moreover that X and Y satisfy the domination axiom. Let $U \subseteq Y$ be a finely open set and $\varphi : U \to X$ be an H-map from S to $\mathcal{T}(U)$, such that: $\forall V \subseteq X$ naturally open, $\varphi|_{\varphi^{-1}(V)}$ is a H-map from $\mathcal{S}(V)$ to $\mathcal{T}(\varphi^{-1}(V))$. Then φ is a finely harmonic morphism in the sense of Fuglede* (cf. [6], [7]).

§ 2. **The Main Results. Theorem 2.1.** *Let S and T be standard H-cones of functions, on the nearly saturated spaces X and Y; $\varphi : Y \to X$ be a Bl-mapping. If the following conditions are fulfilled:*

(i) $\forall V \subseteq Y$ naturally open: $\mathrm{carr}_{(V, \tau_n)}\, 1 = \emptyset$ in $\mathcal{T}'(V)$.

ii) *$\mathcal{T}'$ is a natural complete presheaf on the semi-saturated space Y,*

(iii) *φ is naturally continuous,*

(iv) *There exists a base $\mathcal{B}$ of regular sets for the natural topology on X. Then, the following properties are equivalent:*

1) *φ is a morphism of natural presheaves from $\mathcal{S}'$ to $\mathcal{T}'$.*

2) $(B^{X\setminus U} s)\circ\varphi = B^{Y\setminus\varphi^{-1}(U)}(s\circ\varphi)$, $\forall s \in S$, $\forall U \in \mathcal{B}$.

The next result is the „fine" version of the preceeding theorem.

Theorem 2.2. *Let S and T be standard H-cones of functions on the nearly saturated spaces X and Y; $\varphi : Y \to X$ be a Bl-mapping. If the following*

conditions are fulfilled: (i) *$\forall V \subseteq Y$ finely open:* $\mathrm{carr}_{(V, \tau_f)}\, 1 = \emptyset$ *in* $\mathcal{T}(V)$; (ii) *$\mathcal{T}$ is a fine complete presheaf. Then, the following properties are equivalent:*

1) *φ is a morphism of fine presheaves from $\mathcal{S}$ to $\mathcal{T}$.*

2) $(B^{\beta(X\setminus U)} s)\circ\varphi = B^{\beta(Y\setminus\varphi^{-1}(U))}(s\circ\varphi)$, $\forall s \in S$, $\forall U \in \mathcal{U}$.

In the case of harmonic spaces, we get the following characterisations:

Corollary 2.3. *Let X and Y be harmonic spaces and $\varphi : Y \to X$ be a continuous map, with the properties:* (i) *X has countable base, is B-harmonic space and has a base $\mathcal{B}$ of regular sets;* (ii) *$\forall p$ continuous potential on X, $p\circ\varphi$ is a potential on Y. Then, the following properties are equivalent:*

1) *φ is a harmonic morphism.*

2) $(B^{X\setminus U} s)\circ\varphi = B^{Y\setminus\varphi^{-1}(U)}(s\circ\varphi)$, *$\forall s$ hyperharmonic on X and $\forall U \in \mathcal{B}$.*

The „fine" version is:

Corollary 2.4. *Let X and Y be harmonic spaces; $V \subseteq Y$ a finely open set and $\varphi : V \to X$ a map such that:* (i) *X has a countable base and is B-harmonic space;* (ii) *Y satisfies the axiom of domination;* (iii) *$\forall p$ finite, continuous potential on X, $p\circ\varphi$ is a fine potential (cf. [5, def. 10.5]). Then, the following properties are equivalent:*

1) *φ is a finely harmonic morphism (see [6], [7]).*

2) $(B^{X\setminus U} s)\circ\varphi = B^{V\setminus\varphi^{-1}(U)}(s\circ\varphi)$, *$\forall s$ hyperharmonic on X $\forall U \in \mathcal{U}$.*

§ 3. **The Proofs.** We give a slightly more general results than those announced in §2.

Proposition 3.1. *Let S and T be standard H-cones of functions on the nearly saturated spaces X and Y; let φ be a morphism of the natural presheaves $\mathcal{S}'$ and $\mathcal{T}'$. If* (i), (iii), (iv) *from th. 2.1. hold and moreover:*

(ii') *$\forall V \subseteq Y$ naturally open, for any naturally open covering (V_α) for V, such that* $\mathrm{fcl}\, V_\alpha \subseteq V$, *for any finite and nearly continuous $t \in T$, we have:*

$$B^{Y\setminus V} t = \bigwedge_{(\alpha_1, \dots, \alpha_n)} B^{Y\setminus V_{\alpha_1}} \dots B^{Y\setminus V_{\alpha_n}} t$$

then:

$$(B^{X\setminus U} s)\circ\varphi \leq B^{Y\setminus\varphi^{-1}(U)}(s\circ\varphi)$$

$\forall s \in S$, $\forall U \subseteq X$ naturally open.

Proof. Let $s \in S_0$, $s \leq 1$. By [1, th. 2.3] we have: $1 - B^{X\setminus U} s \in \mathcal{S}^f(U)$, hence: $1 - (B^{X\setminus U} s)\circ\varphi = (1 - B^{X\setminus U} s)\circ\varphi \in \mathcal{T}^f(\varphi^{-1}(U))$. From (i) it follows that there exists a naturally open covering (V_α) for $\varphi^{-1}(U)$ such that $\mathrm{fcl}\, V_\alpha \subseteq \varphi^{-1}(U)$ and: $\hat{B}^{\varphi^{-1}(U)\setminus V_\alpha} 1 = 1$, $\forall\alpha$ ($\wedge$ denotes the balayages with respect to $\mathcal{T}^f(\varphi^{-1}(U))$. But:

$$\hat{B}^{\varphi^{-1}(U)\setminus V_\alpha}[1 - (B^{X\setminus U} s)\circ\varphi] \leq 1 - (B^{X\setminus U} s)\circ\varphi$$

hence:

$$\hat{B}^{\varphi^{-1}(U)\setminus V_\alpha}[(B^{X\setminus U} s)\circ\varphi] = (B^{X\setminus U} s)\circ\varphi.$$

Now (iv) shows that we can apply (ii') in order to get:

$$B^{Y\setminus\varphi^{-1}(U)}(s\circ\varphi)=\bigwedge_{(\alpha_1,\ldots,\alpha_n)} B^{Y\setminus V_{\alpha_1}}\ldots B^{Y\setminus V_{\alpha}}(s\circ\varphi)\geqslant$$

$$\geqslant\bigwedge_{(\alpha_1,\ldots,\alpha_n)} B^{Y\setminus V_{\alpha_1}}\ldots B^{Y\setminus V_{\alpha_n}}[(B^{X\setminus U}s)\circ\varphi]=(B^{X\setminus U}s)\circ\varphi$$

on $\varphi^{-1}(U)$. But : $B^{Y\setminus\varphi^{-1}(U)}(s\circ\varphi)=s\circ\varphi\geqslant(B^{X\setminus U}s)\circ\varphi$ on $b(Y\setminus\varphi^{-1}(U))$. Since : $Y\setminus[\varphi^{-1}(U)\cup b(Y\setminus\varphi^{-1}(U))]$ is semi-polar, the inequality holds everywhere.

Proposition 3.2. *Let S and T be H-cones of functions on the sets X and Y. Let $\varphi: Y\to X$ be a H-map. For any $A\subseteq X$ and $s\in S$, we have :*

$$B^{\varphi^{-1}(b(A))}(s\circ\varphi)\leqslant(B^A s)\circ\varphi.$$

Proof. We have $b(A)\subseteq[B^A s=s]$, hence : $\varphi^{-1}(b(A))\subseteq\varphi^{-1}([B^A s=s])=$ $=[B^A s\circ\varphi=s\circ\varphi]$.

From the preceeding propositions, it follows that $1\Rightarrow 2$ from th. 2.1. holds, if we remark that condition (ii) from th. 2.1. implies (ii').

Proposition 3.3. a) *Let S be a standard H-cone of functions on the semi-saturated space X. If S^L is a natural complete presheaf, then (ii') holds.*

b) *Let S be a standard H-cone of functions on the nearly saturated space X. Let us suppose that S satisfies the axioms of convergence and that of nearly continuity. Then, for any balayages B, (B_n) such that $d(B)\subseteq\bigcup_n d(B_n)$, and for any universally bounded $s\in S$, we have :*

$$Bs\leqslant\bigwedge_{(i_1\ldots i_n)} B_{i_1}\ldots B_{i_n}s.$$

Proof. a) By [1, prop. 3.5. a)] follows that, for any naturally open covering (U_n) for X, and any finite and nearly continuous $s\in S$, we have

$$\bigwedge_{(i_1,\ldots,i_n)} B^{X\setminus U_{i_1}}\ldots B^{X\setminus U_{i_n}}s=0$$

We remark that : $(s-B^{X\setminus U}s)|_U$ is nearly continuous in $S^L(U)$. Indeed let $u\in S^{**}$ be a weak unit, such that s is u-continuous. $u|_U$ is a weak unit in $S^L(U)$. We prove that $(s-B^{X\setminus U}s)|_U$ is $u|_U$-continuous. Let (f_i) be a family from $S^L(U)$, increasing to $(s-B^{X\setminus U}s)|_U$. From $f_i\leqslant s-B^{X\setminus U}s$ it follows [1, th. 2.1.b)] that : $f_i=v_i-B^{X\setminus U}s$, where :

$$v_i=\sup_n(Vg_n^i-B^{X\setminus U}Vg_n^i+B^{X\setminus U}s)$$

and $Vg_n^i-B^{X\setminus U}Vg_n^i$ is increasing to f_i. We have $v_i\in S$, $v_i\leqslant s$ and $s=\sup_i v_i$ on U. By construction, it follows that $s=\sup_i v_i$ on $b(X\setminus U)$, and also that, (v_i) is increasing on $b(X\setminus U)\cup U$. Being increasing except a semi-polar set, it follows that (v_i) is increasing and $s=\Upsilon v_i$. Now : $\forall\varepsilon>0$ $\exists$ i_ε such that : $s\leqslant v_{i_\varepsilon}+\varepsilon u$, hence : $(s-B^{X\setminus U}s)|_U\leqslant f_{i_\varepsilon}+\varepsilon^U|_U$. It follows that :

$$\bigwedge_{(i_1,\ldots,i_n)} B^{U\setminus U_{i_1}}\ldots B^{U\setminus U_{i_n}}(s-B^{X\setminus U}s)=0$$

($\wedge$ denotes the balayages with respect to $S^L(U)$). If we remark that : $d(X\setminus U)\subseteq\mathrm{fcl}(U\setminus U_i)$, we get, from [1, prop. 3.3] :

$$\hat{B}^{U\setminus U_i}(s-B^{X\setminus U}s)=B^{X\setminus U_i}s-B^{X\setminus U}s\ \text{ on } U,$$

Since : $B^{X\setminus U}s=B^{X\setminus U}(B^{X\setminus U_i}s)$, the proof is complete.

b) Let us denote : $t=\bigwedge_{(i_1,\ldots,i_n)} B_{i_1}\ldots B_{i_n}s$. From the axiom of convergence, we obtain : $B_n t=t$, $\forall n$, hence : $\mathrm{carr}_{(X,\tau_f)}t\subseteq\bigcap_n b(B_n)$. Then : $\mathrm{carr}_{(X,\tau_f)}R(t-Bs)=\Phi$. By the axiom of nearly continuity, t is nearly continuous, hence $t\in\mathcal{P}(X,\tau_f)$. The same is true about $R(t-Bs)\leqslant t$, hence $R(t-Bs)=0$, so that $t\leqslant Bs$.

The implication $2\Rightarrow 1$ from th. 2.1. follows now from :

Proposition 3.4. *Let S and T be standard H-cones of functions on the nearly saturated spaces X and Y. Let $\varphi: Y\to X$ be a H-map. If $U\subseteq X$ is finely open and :*

(*) $$(B^{X\setminus U}s)\circ\varphi=B^{Y\setminus\varphi^{-1}(U)}(s\circ\varphi),\quad\forall s\in S$$

then $\varphi|_{\varphi^{-1}(U)}:\varphi^{-1}(U)\to U$ is a H-map from $S^L(U)$ to $\mathcal{T}^L(\varphi^{-1}(U))$. If, moreover, φ is naturally continuous and one of the following conditions hold :

— *(*) holds for any U from a base for the natural topology and $\mathcal{T}^L$ is a natural complete presheaf. Or :*

— *(*) holds for any naturally open U, then φ is a morphism of natural presheaves S^L, $\mathcal{T}^L$.*

The proof is direct.

We pass now to the proof of the th. 2.2. The implication $1\Rightarrow 2$ can be proved as follows : since there exists a base of regular, finely open sets, prop. 3.3. b) and prop. 3.2. show that the proof of the prop. 3.1., with obvious modifications, gives the desired result. The implication $2\Rightarrow 1$ follows immediately from a result entirely analogous to prop. 3.4.

We give now the proof of cor. 2.3. Let us remark that the condition for φ to be a Bl-mapping must be added in [10, prop. 6.2., cor. 6.3.].

$1\Rightarrow 2$. Let $U\subseteq X$ be regular. It follows that $B^{X\setminus U}s=s$ on $X\setminus U$, hence $(B^{X\setminus U}s)\circ\varphi=s\circ\varphi$ on $Y\setminus\varphi^{-1}(U)$, so that: $(B^{X\setminus U}s)\circ\varphi\geqslant B^{Y\setminus\varphi^{-1}(U)}(s\circ\varphi)$. Let us denote : $p=s\circ\varphi-B^{Y\setminus\varphi^{-1}(U)}(s\circ\varphi)$; $p'=s\circ\varphi-(B^{X\setminus U}s)\circ\varphi$. By [2, th. 5.1.9. c)] and [4, th. 8.3.2], it follows that $p\circ\varphi$ is a potential on $\varphi^{-1}(U)$. Since $p-p'$ is a harmonic, positive function on $\varphi^{-1}(U)$, we have $p=p'$ on $\varphi^{-1}(U)$. Since $B^{Y\setminus\varphi^{-1}(U)}(s\circ\varphi)=(B^{X\setminus U}s)\circ\varphi$ on $b(Y\setminus\varphi^{-1}(U))$, we conclude the equality everywhere.

$2\Rightarrow 1$. Since there exists a bounded, strictly positive, continuous superharmonic function (even a potential) f on X, replacing the two spaces by $f^{-1}\cdot\mathcal{U}$ resp. $(f\circ\varphi)^{-1}\cdot\mathcal{V}$, one can apply th. 2.1. (cf. [8] and [1]).

Cor. 2.4. represents the particularisation of th. 2.2. to harmonic spaces. Indeed, any $p\in S_0$ is a continuous, finite potential, hence $p\circ\varphi$ is a finite, finely potential. Now, from [5, def. 8.3] it follows that $p\in\mathcal{P}(V,\tau_f)$. Hence, p is nearly continuous.

§ 4. **Other Results.** We begin with complementary results to th. 2.1. and 2.2.

Proposition 4.1. *Let S and T be standard H-cones of functions on the nearly saturated spaces X and Y. Let $\varphi : Y \to X$ be a Bl-mapping. Suppose that conditions* (i), (iii) *from th. 2.1. and* (ii') *from prop. 3.1. hold, and moreover :*

(v) *The points of X are totally thin.*

Then the following properties are equivalent :

1') *φ is a morphism of the natural presheaves $\mathcal{S}'$ and $\mathcal{T}'$, and φ^{-1} preserves the totally thin sets.*

2') $(B^F s)\circ\varphi = B^{\varphi^{-1}(F)}(s\circ\varphi)$, $\forall s \in S$, $\forall F$ *naturally closed.*

Proof. 1'$\Rightarrow$2'. One of the inequalities is proved in prop. 3.1. ; the other one follows from the fact that φ^{-1} preserves the totally thin sets and from [11, th. 1].

2'$\Rightarrow$1'. Using prop. 3.4. we get that φ is a morphism between the natural presheaves $\mathcal{S}'$ and $\mathcal{T}'$, Applying 2') for $F = \{x\}$, it results that $\varphi^{-1}(\{x\})$ is totally thin. By [13, th. 1] it follows that φ^{-1} preserves the totally thin sets.

Proposition 4.2. *Let S and T be standard H-cones of functions on the nearly saturated spaces X and Y. Let $\varphi : Y \to X$ be a Bl-mapping. If condition* (i) *from th. 2.2. and :*

(ii') *$\forall V \subseteq Y$ finely open, for any finely open covering (V_α) of V, such that* fcl $V_\alpha \subseteq V$, *for any universally bounded $t \in T$:*

$$B^{\beta(Y\setminus V)}t = \bigwedge_{(\alpha_1,\dots,\alpha_n)} B^{Y\setminus V_{\alpha_1}} \dots B^{Y\setminus V_{\alpha_n}} t$$

hold, then the following properties are equivalent :

1') *φ is a morphism of fine presheaves $\mathcal{S}$ and $\mathcal{T}$ and φ^{-1} preserves the totally thin sets.*

2') $(B^{\beta(X\setminus U)}s)\circ\varphi = B^{\beta(Y\setminus\varphi^{-1}(U))}(s\circ\varphi)$, $\forall s \in S$, $\forall U \subseteq X$ *finely open.*

Proof, 1'$\Rightarrow$2'. One of the inequalities follows from an obvious modification of the proof of prop. 3.1. The other one is equivalent with the fact that φ^{-1} preserves the totally thin sets. Indeed, if φ^{-1} preserves the totally thin sets, then by [11, th. 1] : $\beta(\varphi^{-1}(A)) \subseteq \varphi^{-1}(\beta(A))$, $\forall A \subseteq X$. Hence :

$$B^{\beta(\varphi^{-1}(A))}(s\circ\varphi) \leqslant B^{\varphi^{-1}(\beta(A))}(s\circ\varphi) \leqslant B^{\varphi^{-1}(b(\beta(A)))}(s\circ\varphi) \leqslant (B^{\beta(A)}s)\circ\varphi$$

by prop. 3.2. and the fact that : $\beta(A) = \beta(\beta(A)) \subseteq b(\beta(A))$. Conversely if $A \subseteq X$ is totally thin, we have $\beta(A) = b(A) = \Phi$, hence $B^{\beta(\varphi^{-1}(A))}1 = 0$. It follows that $\beta(\varphi^{-1}(A))$ is a polar set, so that $\varphi^{-1}(A)$ is semipolar.

2'$\Rightarrow$1'. It is enough to remark that, from the above considerations and [11, th. 1] it follows that φ^{-1} preserves the totally thin sets.

Analogous results are valid in the case of harmonic spaces :

Proposition 4.3. *Let X and Y be harmonic spaces and $\varphi : Y \to X$ a continuous map, such that : for any positive hyperharmonic function s on X, $s\circ\varphi$ is hyperharmonic on Y. If the following conditions are satisfied :* (i) *X is a B-harmonic space with countable base and the points of X are totally thin ;* (ii) *Y satisfies the domination axiom and is a B-harmonic space, then the following properties are equivalent :*

1') *φ is a finely open morphism.*

2') $(B^{X\setminus U}s)\circ\varphi = B^{Y\setminus\varphi^{-1}(U)}(s\circ\varphi)$, *for any s positive hyperharmonic, for any open $U \subseteq X$.*

Proof. 1'$\Rightarrow$2'. φ being finely open, one of the inequalities follows from [11, th. 1]. The other one follows from cor. 2.3.

2'$\Rightarrow$1'. The fact that φ is a harmonic morphism is proved in cor. 2.3. The fact that φ is finely open follows from [13, th. 1].

Remark. If φ is finely open and $U \subseteq X$ is regular, then $\varphi^{-1}(U) \subseteq Y$ is regular, by [13, prop. 6].

Corollary 4.4 *Let X and Y be harmonic spaces, $V \subseteq Y$ finely open and $\varphi : V \to X$. Suppose that : (i) X is a B-harmonic space with countable base and the points of X are totally thin ; (ii) Y satisfies the axiom of domination ; (iii) for any finite continuous potential p on X, $p\circ\varphi$ is a fine potential on V. Then, the following properties are equivalent*

1) *φ is a finely open, fine harmonic morphism.*

2) $(B^{X\setminus U}s)\circ\varphi = B^{V\setminus\varphi^{-1}(U)}(s\circ\varphi)$, *for any s hyperharmonic on X, for any finely open $U \subseteq X$.*

Proposition 4.5. a) *Let S and T be standard H-cones ; B and B' pseudo-balayages on S and T ; $\varphi : S \to T$ a semi-finite morphism. If : $\varphi(Bs) = B'(\varphi(s))$, $\forall s \in D(\varphi)$, then, by : $\varphi_{B,B'}(s - Bs) = \varphi(s) - B'(\varphi(s))$, $\forall s \in D(\varphi)$ one defines a morphism of H-cones : $\varphi_{B,B'} : S_B \to T_{B'}$.*

b) *If, moreover, S and T are standard H-cones of functions on the sets X and Y ; $\varphi : Y \to X$ is a H-map and satisfies : $\varphi^{-1}(b(B)) \subseteq b(B')$, then : $\varphi|_{d(B')} : d(B') \to d(B)$ is a H-map from $\hat{S}_B$ to $\hat{T}_{B'}$.*

The proof is immediate.

Proposition 4.6. *Let S and T be standard H-cones of functions on the nearly saturated spaces X and Y. Let B and B' be balayages on S and T. Let $\varphi : Y \to X$ be a Bl-mapping. If :* (i) $\mathrm{carr}_{(d(B'),\tau_f)} 1 = \Phi$ *in* $\hat{T}_{B'}$; (ii) *for any finely open covering (A_i) for $d(B')$ such that* fcl $A_i \subseteq d(B')$ *and for any finite, nearly continuous $t \in T$, we have :*

$$B't = \bigwedge_{i_1,\dots,i_n} B^{Y\setminus A_{i_1}} \dots B^{Y\setminus A_{i_n}} t$$

then, if $\varphi^{-1}(b(B)) \subseteq b(B')$ and : $\varphi|d(B') : d(B') \to d(B)$ is an H-map from $\hat{S}_B$ to $\hat{T}_B$, then : $(Bs)\circ\varphi \leqslant B'(s\circ\varphi)$.

The proof is analogous to that of prop. 3.1.

We end with some results concerning the connection between the commutation of φ with balayages and the preservation of the carrier by φ. These represent a correction of [10, prop. 6.5.].

Proposition 4.7. *Let S and T be standard H-cones of functions on the nearly saturated spaces X and Y. Let τ and τ' be topologies on X and Y, finer than the natural topology and coarser than the fine topology. Let $\varphi : (Y, \tau') \to$ $\to (X, \tau)$ be a continuous Bl-mapping. If S satisfies the axioms of convergence and of nearly continuity, then the following properties are equivalent :*

1) $\mathrm{carr}_{(Y,\tau')}(s\circ\varphi) \subseteq \varphi^{-1}(\mathrm{carr}_{(X,\tau)}s)$, $\forall s \in S$.

2) $B^{Y\setminus\varphi^{-1}(U)}(s\circ\varphi) \geqslant (B^{X\setminus U}s)\circ\varphi$, $\forall s \in S$, $\forall U \in \tau$.

Proof. 1⇒2. Since $\mathrm{carr}_{(X,\tau)} B^{X\setminus U}s \subseteq X\setminus U$, by 1) we have: $\mathrm{carr}_{(Y,\tau')} [(B^{X\setminus U}s)\circ\varphi] \subseteq Y \setminus \varphi^{-1}(U)$. If $s\in S_0$, then: $(B^{X\setminus U}s)\circ\varphi \in \mathcal{P}_d(Y, \tau')$, which show that $B^{Y\setminus\varphi^{-1}(U)}[(B^{X\setminus U}s)\circ\varphi] = (B^{X\setminus U}s)\circ\varphi$.

2⇒1. Let $y\notin\varphi^{-1}(\mathrm{carr}_{(X,\tau)}s)$, hence there exists a τ-open neighbourhood W of $\varphi(y)$, such that $B^{X\setminus W}s = s$. By 2): $s\circ\varphi = (B^{X\setminus W}s)\circ\varphi = B^{Y\setminus\varphi^{-1}(W)}(s\circ\varphi)$ which shows that $y\notin \mathrm{carr}_{(Y,\tau')}(s\circ\varphi)$.

When τ and τ' are the fine, resp. natural topologies, stronger results are available:

Proposition 4.8. *Let S and T be standard H-cones of functions on the nearly saturated spaces X and Y. Let $\varphi: Y\to X$ be a Bl-mapping. If S satisfies the axiom of nearly continuity, then the following properties are equivalent:*

1) $\mathrm{carr}(s\circ\varphi) \subseteq \varphi^{-1}(\mathrm{carr}\, s)$, $\forall s\in S$ (*fine carrier*).

2) $(B^{X\setminus U}s)\circ\varphi = B^{Y\setminus\varphi^{-1}(U)}(s\circ\varphi)$, $\forall s\in S$ *and any regular fine open* U.

Proof. 1⇒2. Since the carrier is taken with respect to the fine topology: $\mathrm{carr}\, B^{X\setminus U}s \subseteq X\setminus U$. The converse inequality follows from the fact that U is regular.

2⇒1. Follows as in the proof of the preceeding proposition, since φ is finely continuous.

Remark. a) The hypotheses that S satisfies the axiom of nearly continuity can be replaced by: „T satisfies condition (ii′) from prop. 4.2."
b) 1) is also equivalent with: 2′) $Bs\circ\varphi = B^{\varphi^{-1}(b(B))}(s\circ\varphi)$, $\forall s\in S$, for any balayage B.

Proposition 4.9. *Let S and T be standard H-cones of functions on the nearly saturated spaces X and Y; $\varphi: Y\to X$ a naturally continuous Bl-mapping. If: $B^U(B^F s) = B^F s$, $\forall s\in S$, $\forall F$ naturally closed, $\forall U\supseteq F$ naturally open; $\forall V\subseteq Y$ naturally open, for any naturally open covering (V_α) of V, such that* fcl $V_\alpha \subseteq V$, *for any finite, nearly continuous $t\in T$:*

$$B^{Y\setminus V}t = \bigwedge_{(\alpha_1,\dots,\alpha_n)} B^{Y\setminus V_{\alpha_1}} \dots B^{Y\setminus V_{\alpha_n}} t$$

then the following properties are equivalent:

1) $\mathrm{carr}(s\circ\varphi) \subseteq \varphi^{-1}(\mathrm{carr}\, s)$, $\forall s\in S$ (*natural carrier*).

2) $B^{Y\setminus\varphi^{-1}(U)}(s\circ\varphi) \geq (B^{X\setminus U}s)\circ\varphi$, $\forall s\in S$, $\forall U$ *naturally open.*

Proof. 1⇒2. Since any naturally closed set is a G_δ-*set*, the first hypothesis shows that $\mathrm{carr}\, B^{X\setminus U}s \subseteq X\setminus U$. By 1) $\mathrm{carr}[(B^{X\setminus U}s)\circ\varphi] \subseteq Y\setminus\varphi^{-1}(U)$. We conclude the existence of a naturally open covering (V_α) for $\varphi^{-1}(U)$, such that fcl $V_\alpha \subseteq \varphi^{-1}(U)$ and: $B^{Y\setminus V_\alpha}[(B^{X\setminus U}s)\circ\varphi] = (B^{X\setminus U}s)\circ\varphi$. Hence:

$$B^{Y\setminus\varphi^{-1}(U)}(s\circ\varphi) = \bigwedge_{(\alpha_1,\dots,\alpha_n)} B^{Y\setminus V_{\alpha_1}} \dots B^{Y\setminus V_{\alpha_n}}(s\circ\varphi) \geq$$

$$\geq \bigwedge_{(\alpha_1,\dots,\alpha_n)} B^{Y\setminus V_{\alpha_1}} \dots B^{Y\setminus V_{\alpha_n}}[(B^{X\setminus U}s)\circ\varphi] = (B^{X\setminus U}s)\circ\varphi.$$

2.⇒1. Since φ is naturally continuous, the same proof as for prop. 4.7. holds.

REFERENCES

1. Boboc, N., Bucur Gh. — *Natural Sheaf Property in Standard H-cones of Functions.* Preprint INCREST 32/1982, București.
2. Boboc N., Bucur Gh., Cornea A. — *Order and Convexity in Potential Theory.* Lecture Notes in Math. vol. 853, Springer 1981.
3. Constantinescu, C., Cornea A. — *Compactifications of Harmonic Spaces.* Nagoya Math. J. 25, p. 1—57 (1963).
4. Constantinescu C., Cornea A. — *Potential Theory on Harmonic Spaces.* Grundlehren Band 158, Springer 1975.
5. Fuglede, B. — *Finely Harmonic Functions.* Lecture Notes in Math. vol. 289, Springer 1972.
6. Fuglede, B. — *Finely Harmonic Mappings and Finely Holomorphic Functions.* Ann. Acad. Sci. Fenn. 2 (1976), p. 113—127.
7. Fuglede, B. — *Invariant Characterisation of the Fine Topology in Potential Theory.* Math. Ann. 241, p. 187—192 (1979).
8. Fuglede, B. — *Localisation in Fine Potential Theory and Uniform Approximation by Subharmonic Functions* J. Funct. Anal. vol. 49 (1) 1982, p. 57—72.
9. Popa, E. — *Morphisms of H-cones.* An. șt. Univ. Iași, t. XXIX, f. 2, (1983), p. 53—62
10. Popa, E. — *Morphisms of H-cones.* II An. șt. Univ. Iași, t. XXX, f. 1, (1984), p. 3—12.
11. Popa, E. — *Morphisms of H-cones and Semi-polar Sets.* Preprint Series in Math. 2/1983, Iași (to appear in An. șt. Univ. Iași).
12. Popa, E. — *Natural Localisation of a Standard H-cone* in Lect. Notes in Math. vol. 1014, p. 220—227.
13. Popa, E. — *Finely Open Morphisms of H-cones.* Preprint Series in Math. 1/1984, Iași.
14. Popa, L. — *Fonctions C-inferieurement semi-continues.* An. șt. Univ. Iași, t. XXIX (1983), p. 21—26.

Received 11.V. 1984

Seminarul Matematic
Universitatea „Al. I. Cuza"
6600 — Iași, România