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STRICT CONVEXITY AND BEST APPROXIMATION SIMULTANEOUS AND VECTORIAL IN LINEAR SPACES

BY

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In the paper [2] we have established some characterizations of the elements of best vectorial approximation in a linear space with respect to a family of real functions defined on the space, a result on the existence of minimal elements for convex vectorial programs and we have introduced two concepts of strict convexity in linear spaces for the study of the uniqueness of the elements of best approximation simultaneous or vectorial by convex non-void subsets.

The present paper is devoted to the study of the connections between best simultaneous approximation, best vectorial approximation, the simultaneous strict convexity and the vectorial strict convexity.

First, we recall some definitions which we use in our considerations and useful characterizations of the simultaneous and vectorial proximinal linear subspaces.

Let X be a real or complex linear space. $\mathbf{F}=\{f_\alpha : \alpha\in A\}$ a family of real functions defined on X, G a non-void subset of X, $x_0\in X$ and the function $\rho_{\mathbf{F}} : X\to R^A$ defined by $[\rho_{\mathbf{F}}(x)](\alpha)=f_\alpha(x)$, $x\in X$, $\alpha\in A$. Let Y and D denote, respectively, a non-void set and a convex cone in R^A. The following definition is due to *Yu* [4].

Definition 1. *A point $y^0\in Y$ is a D-extreme point of Y if there exists no $y^1\in Y$, $y^1\neq y^0$ such that $y^0\in y^1+D$.*

We shall denote the set of all D-extreme points of Y by Ext $[Y/D]$ and we recall that if $f, g\in R^A$. then

(i) $f\leqslant g$ means $f(\alpha)\leqslant g(\alpha)$ for every $\alpha\in A$;

(ii) $f<g$ means $f\leqslant g$ and there exists $\alpha\in A$ such that $f(\alpha)<g(\alpha)$;

(iii) $R^A_+=\{f\in R^A : f\geqslant 0\}$;

If for every $g\in G$ we consider the set $S(g)=\{g'\in G : \rho_{\mathbf{F}}(x_0-g')=\rho_{\mathbf{F}}(x_0-g)\}$, then from [2] we have the following definitions.

Definition 2. *An element $g_0\in G$ is said to be a best simultaneous approximation of x_0 by the elements of G with respect to* $\mathbf{F}$ *(abbreviated g_0 is a* $\mathbf{F}$*—b.s.a. of x_0) if $\rho_{\mathbf{F}}(x_0-g_0)\leqslant\rho_{\mathbf{F}}(x_0-g)$ for all $g\in G$.*

We denote by $\mathcal{S}_G(\mathbf{F} ; x_0)$ the set of all the elements of best simultaneous approximation of x_0 by G with respect to $\mathbf{F}$.

Definition 3. *An element $g_0\in G$ is said to be a best vectorial approximation of x_0 by the elements of G with respect to* $\mathbf{F}$ *(abbreviated g_0 is a* $\mathbf{F}$*—b.v.a. of x_0) if it satisfies one of the following equivalent conditions :*

(a) *There exists no* $g \in G$ *such that* $\rho_{\mathbf{F}}(x_0-g) < \rho_{\mathbf{F}}(x_0-g_0)$;
(b) *For every* $g \in G \setminus S(g_0)$, *there exists* $f_\alpha \in \mathbf{F}$ *such that* $f_\alpha(x_0-g) > f_\alpha(x_0-g_0)$;
(c) $\bigcap_{\alpha \in A} \{g \in G \setminus S(g_0) : f_\alpha(x_0-g) \leqslant f_\alpha(x_0-g_0)\} = \Phi$ with Φ the empty set ;
(d) $\rho_{\mathbf{F}}(x_0-g_0) \in \mathrm{Ext}[\rho_{\mathbf{F}}(x_0-G)/R^A_+]$;

The set of all the elements of best vectorial approximation of x_0 by the elements of G with respect to $\mathbf{F}$ is denoted by $\mathcal{V}_G(\mathbf{F}\,;\,x_0)$.

Remark 1. It is clear that every $\mathbf{F}$—b.s.a. of x_0 by a non-void set $G \subseteq X$ is a F—b.v.a. of x_0 by the elements of G, while it is possible that the set of the elements of best simultaneous approximation of x_0 to be empty and the set of the elements of best vectorial approximation of x_0 to be infinite (see [2], Example 2.2 in which every element of the segment $[(0, 1), (1, 0)]$ is a best vectorial approximation of $(0, 0)$). It is also easy to see that if the set of all best simultaneous approximation of an element $x_0 \in X$ by a non-empty set $G \subseteq X$ is non-void, then all the elements of best vectorial approximation of x_0 by G coincide with the elements of best simultaneous approximation of x_0 by G, i.e., $\mathcal{S}_G(\mathbf{F}\,;\,x_0) = \mathcal{V}_G(\mathbf{F}\,;\,x_0)$.

Indeed, if $\mathcal{S}_G(\mathbf{F}\,;\,x_0)$ is non-void, then $\mathcal{S}_G(\mathbf{F}\,;\,x_0) \subseteq \mathcal{V}_G(\mathbf{F}\,;\,x_0)$ implies that $\mathcal{V}_G(\mathbf{F}\,;\,x_0)$ is non-empty. Let $g_0 \in \mathcal{V}_G(\mathbf{F}\,;\,x_0)$ and assume that $g_0 \notin \mathcal{S}_G(\mathbf{F}\,;\,x_0)$.
Then there exists $f_\alpha \in \mathbf{F}$ and $g^* \in G$ such that $f_\alpha(x_0-g) < f_\alpha(x_0-g_0)$.

Let $\bar{g} \in \mathcal{S}_G(\mathbf{F}\,;\,x_0)$. We have $f_\alpha(x_0-\bar{g}) < f_\alpha(x_0-g_0)$, because, in the contrary case by $\rho_{\mathbf{F}}(x_0-\bar{g}) \leqslant \rho_{\mathbf{F}}(x_0-g)$ for all $g \in G$ we obtain $f_\alpha(x_0-\bar{g}) \leqslant f_\alpha(x_0-g_0)$, hence $f_\alpha(x_0-\bar{g}) = f_\alpha(x_0-g_0)$ which implies $f_\alpha(x_0-g_0) \leqslant f_\alpha(x_0-g)$ for every $q \in G$, in particular, $f_\alpha(x_0-g_0) \leqslant f_\alpha(x_0-g^*)$, contradiction.

Therefore $f_\alpha(x_0-\bar{g}) < f_\alpha(x_0-g_0)$ and $f_\beta(x_0-\bar{g}) \leqslant f_\beta(x_0-g_0)$ for all $\beta \in A \setminus \{\alpha\}$, that is $\rho_{\mathbf{F}}(x_0-\bar{g}) < \rho_{\mathbf{F}}(x_0-g_0)$ in contradiction with $g_0 \in \mathcal{V}_G(\mathbf{F}\,;\,x_0)$.

Assume now that G is a linear subspace in X and denote by G_s the set of all the elements of X for which the origin is a $\mathbf{F}$—b.s.a. by G and by G_v the set of all the elements of X for which the origin is a $\mathbf{F}$—b.v.a. by the elements of G.i.e., $G_s = \{x \in X : \rho_{\mathbf{F}}(x) \leqslant \rho_{\mathbf{F}}(x-g)$ for every $g \in G\}$ and $G_v = \{x \in X :$ there exists no $g \in G$ such that $\rho_{\mathbf{F}}(x-g) < \rho_{\mathbf{F}}(x)\}$. In general, G_s and G_v are not linear subspaces of X and if G_v and G_s are non-void sets, then taking into account the Theorem 1.1. from [2] we have

Theorem 1. (a_1) *For each* $x \in X$ *there exists at least a best simultaneous approximation in* G *with respect to* $\mathbf{F}$ *if and only if* $G + G_s = X$;

(a_2) *For each* $x \in X$ *there exists only a best simultaneous approximation in* G *with respect to* $\mathbf{F}$ *if and only if* $G \oplus G_s = X$, i.e., *if and only if* X *is the direct sum of* G *and* G_s ;

(a_3) *For each* $x \in X$ *there exists at least a best vectorial approximation in* G *with respect to* $\mathbf{F}$ *if and only if* $G + G_v = X$;

(a_4) *For each* $x \in X$ *there exists only a best vectorial approximation in* G *with respect to* $\mathbf{F}$ *if and only if* $G \oplus G_v = X$. *i.e., if and only if* G *and* G_v *decompose in the direct sum the space* X ;

The proof follows the same line as the proof of the Theorem 1.1 from [2] and we omit it.

Definition 4. *A non-empty set* $M \subseteq X$ *is called simultaneously strict convex with respect to a family* $\mathbf{F} = \{f_\alpha : \alpha \in A\}$ *of real functions defined on* M *if* $x, y \in M$, $x \neq y$ *and* $\rho_{\mathbf{F}}(x) = \rho_{\mathbf{F}}(y)$ *implies*

(1) $$\rho_{\mathbf{F}}((1/2)\,(x+y)) < \rho_{\mathbf{F}}(x).$$

By virtue of the Theorem 2.1 from [2] we conclude that every Hausdorff H-locally convex space (this concept of linear topological space was introduced in [3]) is simultaneously strict convex with respect to the family of seminorms which induces the original topology of the space.

The following theorem shows the connection between the simultaneous strict convexity of a linear space with respect to a family of subadditive and homogeneous functionals defined on the space and the uniqueness of the elements of best simultaneous approximation.

Theorem 2. *A linear space* X *is simultaneously strict convex with respect to a family* $\mathbf{F}$ *of subadditive and homogeneous functionals defined on* X *if and only if every element* $x \in X$ *has at most a* $\mathbf{F}$—*b.s.a. in every closed segment.*

Proof. Let X be a linear space simultaneously strict convex with respect to a family $\mathbf{F}$ of subadditive and homogeneous functionals defined on X and assume that there exists $x_0 \in X$ which has at least two elements of $\mathbf{F}$—b.s.a g_1, g_2 $g_1 \neq g_2$ in a closed segment $[a, b]$.

We have

(2) $$\rho_{\mathbf{F}}(x_0-g_1) \leqslant \rho_{\mathbf{F}}(x_0-g_2) \leqslant \rho_{\mathbf{F}}(x_0-g_1),\ \text{i.e.,}\ \rho_{\mathbf{F}}(x_0-g_1) = \rho_{\mathbf{F}}(x_0-g_2).$$

Hence, by the simultaneous strict convexity of X with respect to we obtain

(3) $$\rho_{\mathbf{F}}(x_0-(g_1+g_2)/2) < \rho_{\mathbf{F}}(x_0-g_1)$$

with $(1/2)\,(g_1+g_2) \in [a, b]$, in contradiction with the assumption that g_1 is a $\mathbf{F}$—b.s.a. of x_0 by the elements of $[a. b]$.

We remark that for this part of the proof the hypotheses that every $f_\alpha \in \mathbf{F}$ is subadditive and homogeneous are not necessary, that is, in every linear space simultaneously strict convex with respect to a family $\mathcal{C}$ of real functionals defined on the space every element has at most a $\mathcal{C}$-b.s.a. in every non-void convex subset.

Conversely, assume that every element $x \in X$ has at most a $\mathbf{F}$—b.s.a. in every closed segment and that X is not simultaneously strict convex. Then, there exist $x, y \in X$, $x \neq y$ with $\rho_{\mathbf{F}}(x) = \rho_{\mathbf{F}}(y)$ and $\rho_{\mathbf{F}}([(x+y)/2]) = \rho_{\mathbf{F}}(x)$, that is, $\rho_{\mathbf{F}}(x+y) = \rho_{\mathbf{F}}(x) + \rho_{\mathbf{F}}(y)$.

Since every functionals of $\mathbf{F}$ is subadditive and homogeneous, from the comment at the beginning of the proof of Theorem 3 ([1], p. 326) it follows that

(4) $$\rho_{\mathbf{F}}(\lambda_x + (1-\lambda)y) = \lambda\rho_{\mathbf{F}}(x) + (1-\lambda)\rho_{\mathbf{F}}(y) = \rho_{\mathbf{F}}(x)\ \text{for every}\ \lambda \in [0,1]$$

which contradicts the assumption that the origin has at most a $\mathbf{F}$—b.s.a. in the segment $[x, y]$ and the theorem is proved.

Corollary. *A locally convex space* $(X, \mathcal{P} = \{p_\alpha : \alpha \in A\})$ *is simultaneously strict convex with respect to the family* $\mathcal{P}$ *of seminorms which induces the original* *topology if and only if every element $x \in X$ has at most a $\mathcal{D}$—b.s.a. in every closed segment.*

Remark 2. By virtue of this strong connection between the uniqueness of the elements of best simultaneous approximation and the simultaneous strict convexity we consider that every linear space X is simultaneously strict convex with respect to every family $\mathbf{F}$ of functionals defined on X even if there exist no $x, y \in X$, $x \neq y$ with $\rho_{\mathbf{F}}(x) = \rho_{\mathbf{F}}(y)$ because, in this case, every element $x \in X$ has at most a $\mathbf{F}$—b.s.a. in every non-empty subset. Indeed, if there exists $x_0 \in X$ and a non-void set $G \subseteq X$ in which x has at least two elements of $\mathbf{F}$—b.s.a. g_1, g_2 with $g_1 \neq g_2$, then $\rho_{\mathbf{F}}(x_0 - g_1) = \rho_{\mathbf{F}}(x_0 - g_2)$, in contradiction with the structure of X.

Remark 3. It is easy to see that if X is a linear space and $\mathbf{F}$ is a family of real functionals defined on X, then every element $x \in X$ has at most a $\mathbf{F}$—b.s.a. in every non-void subset if and only if the origin has at most a $\mathbf{F}$—b.s.a. in every non-empty subset. In general, if every element $x \in X$ has at most a $\mathbf{F}$—b.s.a. in every non-void convex subset, it is not true that the space is simultaneously strict convex. The following example serves to illustrate this fact.

Example 1. Let $X = R^2$ and $\mathbf{F} = \{f_1, f_2, f_3\}$ with $f_1(x, y) = x$, $f_2(x, y) = e^{-y^2}$, $f_3(x, y) = y^2$ for all $x, y \in R$. X is hot simultaneously strict convex since if $(x_1, y_1) \neq (x_2, y_2)$ and $\rho_{\mathbf{F}}(x_1, y_1) = \rho_{\mathbf{F}}(x_2, y_2)$ we have $x_1 = x_2$ and $y_1 = -y_2 \neq 0$, hence

$$(5) \quad \rho_{\mathbf{F}}(1/2)[(x_1, y_1) + (x_2, y_2)]\} = \rho_{\mathbf{F}}(x_1, 0) = (x_1, 1, 0) \nless (x_1, e^{-y_1^2}, y_1^2) = \rho_{\mathbf{F}}(x_1, y_1)$$

and (0,0) has at most a $\mathbf{F}$—b.s.a. in every non-void convex subset. Indeed, let us suppose that there exists a non-empty convex set $G \subseteq X$ in which (0,0) has at least two elements of $\mathbf{F}$—b.s.a. (x_1, y_1) and (x_2, y_2) with $(x_1, y_1) \neq (x_2, y_2)$.

We have

$$(6) \qquad \rho_{\mathbf{F}}(-x_1, -y_1) = \rho_{\mathbf{F}}(-x_2, -y_2), \text{ i.e., } x_1 = x_2 \text{ and } y_1 = -y_2 \neq 0.$$

Therefore

$$(7) \qquad \rho_{\mathbf{F}}(-[\lambda x_1 + (1-\lambda)x_1], -[\lambda y_1 + (1-\lambda)(-y_1)]) \geqslant \rho_{\mathbf{F}}(-x_1, -y_1) \quad \text{for every } \lambda \in [0,1]$$

which is equivalent with

$$(8) \qquad (-x_1, e^{-[(2\lambda-1)y_1]^2}, [(2\lambda-1)y_1]^2) \geqslant (-x_1, e^{-y_1^2}, y_1^2) \text{ for every } \lambda \in [0,1]$$

which implies

$$(9) \qquad [(2\lambda-1)y_1)]^2 \leqslant y_1^2 \; [(2\lambda-1)y_1]^2 \geqslant y_1^2 \text{ for all } \lambda \in [0,1] \text{ a contradiction.}$$

The following concept of strict convexity defined in [2] serves to study the uniqueness of the elements of best vectorial approximation.

Definition 5. *A non-void set $M \subseteq X$ is called vectorially strict convex with respect to a family $\mathbf{F} = \{f_\alpha : \alpha \in A\}$ of real functions defined on M if $x, y \in M$, $x \neq y$, $\rho_{\mathbf{F}}(x) \nless \rho_{\mathbf{F}}(y) \nless \rho_{\mathbf{F}}(x)$ implies*

$$(10) \qquad \rho_{\mathbf{F}}(1/2)\ (x+y)) < \rho_{\mathbf{F}}(x) \text{ or } \rho_{\mathbf{F}}(1/2(x+y)) < \rho_{\mathbf{F}}(y).$$

Example 2. Let $X = R$, $a, b \in R_+^*$ and the family $\mathbf{F} = \{f_1, f_2\}$ of real functions defined on X by $f_1(x) = a\,|\,x\,| + b$ and $f_2(x) = b\,|\,x\,| + a$, $x \in X$. If $\rho_{\mathbf{F}}(x) \nless \rho_{\mathbf{F}}(y) \nless \rho_{\mathbf{F}}(x)$ it follows that $|\,x\,| = |\,y\,|$. Since $x \neq y$ we have $x = -y \neq 0$, consequently

$$(11) \qquad \rho_{\mathbf{F}}(1/2)\ (x+y)) = (b, a) < (a\,|\,x\,| + b,\ b\,|\,x\,| + a) = \rho_{\mathbf{F}}(x).$$

Therefore R^2 is vectorially strict convex with respect to $\mathbf{F}$.

Remark 4. It is clear that every real or complex linear space X vectorially strict convex with respect to a family $\mathbf{F}$ of real functions defined on X is simultaneously strict convex with respect to $\mathbf{F}$ and that there exist simultaneously strict convex linear spaces which are not vectorially strict convex. In order to show this, we consider the following example.

Example 3. Let X be the linear space of all reall functions defined on the real field and $\mathbf{F} = \{f_\alpha; \alpha \in R\}$ defined by $f_\alpha(f) = |\,f(\alpha)\,|$, $f \in X$, $\alpha \in R$. X is simultaneously strict convex with respect to $\mathbf{F}$: let $f_1, f_2 \in X$, $f_1 \neq f_2$ and $\rho_{\mathbf{F}}(f_1) = \rho_{\mathbf{F}}(f_2)$.

Then $f_\alpha(f_1) = f_\alpha(f_2)$ for every $\alpha \in R$, that is, $|\,f_1(\alpha)\,| = |\,f_2(\alpha)\,|$ for $\alpha \in R$. $f_1 \neq f_2$, so there exists $\beta \in R$ such that $f_1(\beta) = -f_2(\beta) \neq 0$. Hence $|[(f_1 + f_2)/2](\beta)| = 0 < |\,f_1(\beta)\,| = |\,f_2(\beta)\,|$ and $\rho_{\mathbf{F}}(1/2(f_1+f_2)) < \rho_{\mathbf{F}}(f_1)$. But X is not vectorially strict convex. Indeed, let $f, g \in X$ defined by $f(x) = g(x)$ for every $x \in R \setminus \{1, 2\}$, $f(1) = g(2) = 1$ and $f(2) = g(1) = (1/2)$. We have $\rho_{\mathbf{F}}(f) \nless \rho_{\mathbf{F}}(g) \nless \rho_{\mathbf{F}}(f)$ and if we suppose that X is vectorially strict convex it follows that $\rho_{\mathbf{F}}(1/2)\ (f+g)) < \rho_{\mathbf{F}}(f)$ or $\rho_{\mathbf{F}}(1/2)\ (f+g)) < \rho_{\mathbf{F}}(g)$ which are impossible because $[(f+g)/2]\ (1) = [(f+g)/2]\ (2) = 3/4 \nless 1/2 = f(2) = g(1)$.

Remark 5. The theorems 2.2 and 2.3 of [2] together the Remark 1 of the present paper establish the immediate relationships between the elements of best simultaneous approximation and the elements of best vectorial approximation. Concerning the connections between the vectorial strict convexity of a linear space X with respect to a family $\mathbf{F}$ of real functions defined on X and the uniqueness of the elements of $\mathbf{F}$—b.v.a. we have shown (see [2], Theorem 2.5) that in every vectorially strict convex linear space every element has at most a best vectorial approximation in every non-empty convex subset. Taking into account the fact that every element of a linear space has at most a best vectorial approximation in every non-void subset if and only if the origin has at most a best vectorial approximation in every non-empty subset, we can prove by an example that, in general, the converse of the Theorem 2.5 from [2] is not valid.

Example 4. Let $X = \{0, y) : y \in R\}$ endowed with the usual structure of linear space over R and

$$\rho_{\mathbf{F}}(0, y) = \begin{cases} \min(|\,1-y\,|, |\,1+y\,|) & \text{if } y \neq 2, 3, 5/2, 0 \\ 2 & \text{if } y \in \{2, 3, 5/2\} \\ -1 & \text{if } y = 0 \end{cases}$$

We have $(0,2) \neq (0,3)$, $\rho_{\mathbf{F}}(0,2) = \rho_{\mathbf{F}}(0,3)$ and $\rho_{\mathbf{F}}(1/2[(0,2) + (0,3)]) = \rho_{\mathbf{F}}(0, 5/2) = 2 = \rho_{\mathbf{F}}(0,2) = \rho_{\mathbf{F}}(0,3)$.

Therefore X is not simultaneously strict convex (consequently, taking into account the (Remark 4, X is not vectorially strict convex) and it is easy to see that (0,0) has at most a **F**-b.v.a. in every non-void convex subset (hence, every element has at most a **F**-b.v.a. in every non-empty convex set).

Remark 6. As in the classical case of normed linear spaces, the concepts of strict convexity introduced by Definition 4 and Definition 5 are not topological notions. For example, R^2 is vectorially strict convex (therefore, by Remark 4, R^2 is simultaneously strict convex) with respect to the usual norm, while R^2 is not simultaneously strict convex (consequently, again by Remark 4, R^2 is not vectorially strict convex) with respect to the equivalent norm $\|\cdot\|_1$ defined by $\|(x, y)\|_1 = |x| + |y|$ because $\|(0, 1)\|_1 = \|(1,0)\|_1 = 1$ and $\|[(0,1)+(1,0)]/2\|_1 = 1$.

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NEW CLASSES OF QUASICONFORMAL MAPPINGS IN R^n (II)*

BY

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§ 2. (K, p)- quasiconformal mappings with $p \neq n, \infty$. In this paragraph, we shall study the problem of the inclusion relations between different classes of (K, p)-qc for $K \in [1, \infty)$ fixed. In order to do this, let us remind first a few preliminary results and concepts.

Let us consider, according to F. G e h r i n g [9], the following subclasses of qc (quasiconformal mappings) obtained by means of the p-capacity.

A homeomorphism $f: D \rightleftarrows D^*$ *is said to be* (K, p)*-qc in the sense of Gehring if* $\operatorname{cap}_p A/K \leq \operatorname{cap}_p A^* \leq K \operatorname{cap}_p A$ *for all spherical rings* $A \subset\subset D$ *(i.e. with* $\bar{A} \subset D$*), where* $A^* = f(A)$ *and*

$$\operatorname{cap}_p A = \inf_u \int_A |\nabla u|^p dm, \tag{13}$$

the infimum being taken over the class of admissible functions.

A homeomorphism $f: D \rightleftarrows D^*$ *is locally bi-Lipschitz with Lipschitz constant K if* $\forall x \in D$*, there is a neighbourhood* $U_x \subset D$ *such that* $|x-y|/K \leq |f(x)-f(y)| \leq K|x-y| \; \forall y \in U_x$.

A homeomorphism $f: D \rightleftarrows D^*$ *belongs to the class* Lip (K) *if* $\overline{\lim_{x \to x_0}} |f(x)-f(x_0)|/|x-x_0| \leq K$, $\forall x_0 \in D$.

It is easy to see that if $f: f^{-1} \in \operatorname{Lip}(K)$, then f is locally bi-Lipschitz with Lipschitz constant $K+\varepsilon \; \forall \varepsilon > 0$.

Proposition 2. *If* $f: D \rightleftarrows D^*$ *is* (K, p)*-qc, then* f *is* K_0*-qc (according to Väisälä's geometric definition from the first part of the paper), where* $K_0 = K_0(K, n, p)$ (F. G e h r i n g [9], theorem 1).

Proposition 3. *If* $f: D \rightleftarrows D^*$ *is* (K, p)*-qc, then* $f, f^{-1} \in \operatorname{Lip} K_0$, *where* $K_0 = K_0(K, n, p)$ (F. G e h r i n g [9], theorem 2).

Corollary. *In the hypothesis of the preceding proposition,* f *is locally bi-Lipschitz with Lipschitz constant* $K_0 + \varepsilon \; \forall \varepsilon > 0$, *where* K_0 *is the constant of the preceding proposition.*

Proposition 4. *If the homeomorphism* $f: D \rightleftarrows D^*$ *is locally bi-Lipschitz with Lipschitz constant* $K > 1$, *then*

$$M_p\Gamma / K^{(p+n)/(p-1)} \leq M_p\Gamma^* \leq K^{(p+n)/(p-1)} M_p\Gamma$$

holds for all arc families Γ *contained in* D (our paper [2], lemma 9, for the proof, see B. F u g l e d e [7], theorem 4).

* The first part of this paper (containing also the bibliography corresponding to this part) was published in An. şt. Univ. „Al. I. Cuza" Iaşi 29, s. I a Mat., f. 2 (1983), 45—52.