to see that (0,0) has at most a **F**-b.v.a. in every non-void convex subset (hence, every element has at most a **F**-b.v.a. in every non-empty convex set).

Remark 6. As in the classical case of normed linear spaces, the concepts of strict convexity introduced by Definition 4 and Definition 5 are not topological notions. For example, R^2 is vectorially strict convex (therefore, by Remark 4, R^2 is simultaneously strict convex) with respect to the usual norm, while R^2 is not simultaneously strict convex (consequently, again by Remark 4, R^2 is not vectorially strict convex) with respect to the equivalent norm $\|\cdot\|_1$ defined by $\|(x, y)\|_1 = |x| + |y|$ because $\|(0, 1)\|_1 = \|(1,0)\|_1 = 1$ and $\|[(0,1)+(1,0)]/2\|_1 = 1$.

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NEW CLASSES OF QUASICONFORMAL MAPPINGS IN R^n (II)*

BY

PETRU CARAMAN

§ 2. **(K, p)- quasiconformal mappings with** $p \neq n, \infty$. In this paragraph, we shall study the problem of the inclusion relations between different classes of (K, p)-qc for $K \in [1, \infty)$ fixed. In order to do this, let us remind first a few preliminary results and concepts.

Let us consider, according to F. G e h r i n g [9], the following subclasses of qc (quasiconformal mappings) obtained by means of the p-capacity.

A homeomorphism $f: D \rightleftarrows D^*$ is said to be (K, p)-qc in the sense of Gehring if $\text{cap}_p A/K \leq \text{cap}_p A^* \leq K \text{cap}_p A$ *for all spherical rings $A \subset\subset D$ (i.e. with $\bar{A} \subset D$), where $A^* = f(A)$ and*

$$\text{cap}_p A = \inf_u \int_A |\nabla u|^p dm, \tag{13}$$

the infimum being taken over the class of admissible functions.

A homeomorphism $f: D \rightleftarrows D^*$ is locally bi-Lipschitz with Lipschitz constant K if $\forall x \in D$, there is a neighbourhood $U_x \subset D$ such that $|x-y|/K \leq |f(x)-f(y)| \leq K|x-y| \; \forall y \in U_x$.

A homeomorphism $f: D \rightleftarrows D^*$ belongs to the class Lip (K) *if* $\overline{\lim\limits_{x \to x_0}} |f(x)-f(x_0)|/|x-x_0| \leq K, \; \forall x_0 \in D$.

It is easy to see that if $f: f^{-1} \in \text{Lip}(K)$, then f is locally bi-Lipschitz with Lipschitz constant $K + \varepsilon \; \forall \varepsilon > 0$.

Proposition 2. *If $f: D \rightleftarrows D^*$ is (K, p)-qc, then f is K_0-qc (according to Väisälä's geometric definition from the first part of the paper), where* $K_0 = K_0(K, n, p)$ (F. G e h r i n g [9], theorem 1).

Proposition 3. *If $f: D \rightleftarrows D^*$ is (K, p)-qc, then $f, f^{-1} \in \text{Lip}\, K_0$, where* $K_0 = K_0(K, n, p)$ (F. G e h r i n g [9], theorem 2).

Corollary. *In the hypothesis of the preceding proposition, f is locally bi-Lipschitz with Lipschitz constant $K_0 + \varepsilon \; \forall \varepsilon > 0$, where K_0 is the constant of the preceding proposition.*

Proposition 4. *If the homeomorphism $f: D \rightleftarrows D^*$ is locally bi-Lipschitz with Lipschitz constant $K > 1$, then*

$$M_p \Gamma / K^{(p+n)/(p-1)} \leq M_p \Gamma^* \leq K^{(p+n)/(p-1)} M_p \Gamma$$

holds for all arc families Γ contained in D (our paper [2], lemma 9, for the proof, see B. F u g l e d e [7], theorem 4).

* The first part of this paper (containing also the bibliography corresponding to this part) was published in An. şt. Univ. „Al. I. Cuza" Iaşi 29, s. I a Mat., f. 2 (1983), 45–52.

Remark. In our paper [2] and in Fuglede's one [7], instead of $K^{(p+n)/(p-1)}$ there is K^{p+n} since M_p is considered in the sense of Väisälä.

Proposition 5. *If $C_0, C_1 \subset R^n$ are 2 disjoint closed sets, where C_0 is assumed to contain the closure of the complement of a ball and $\Gamma(R^n, C_0, C_1)$ is the family of the arcs that intersect both C_0 and C_1, then* $\mathrm{cap}_p(R^n, C_0, C_1) = M_p\Gamma(R^n, C_0, C_1)$ $(p \geqq 1)$ (W. Ziemer [12], theorem 3.8).

Corollary 1. *If A is a ring, then* $\mathrm{cap}_p A = M_p \Gamma_A$.

Corollary 2. *If the homeomorphism $f: D \rightleftarrows D^*$ is locally bi-Lipschitz with Lipschitz constant $K > 1$, then*

$$\mathrm{cap}_p A / K^{(p+n)/(p-1)} \leqq \mathrm{cap}_p A^* \leqq K^{(p+n)/(p-1)} \,\mathrm{cap}_p A \qquad (p > 1),$$

for every ring $A \subset\subset D$, where

$$\mathrm{cap}_p A = \inf_u \left(\int_A |\nabla u|^p dm\right)^{1/p}, \tag{14}$$

the infimum being taken over the class of admissible functions.

This is a direct consequence of the preceding corollary and proposition 4.

Corollary 3. *In the hypothesis of the preceding corollary,* $\mathrm{cap}_\infty A/K \leqq \mathrm{cap}_\infty A^* \leqq K \mathrm{cap}_\infty A$, $\forall A \subset\subset D$.

Theorem 5. *If $f: D \rightleftarrows D^*$ is (p_0)-qc — i.e. (K, p_0)-qc with non-specified K — for $p_0 \neq n$, then f is (p)-qc for any p (including $p = n$ and $p = \infty$).*

Indeed, if f is (K, p_0)-qc for a certain $p_0 \neq n$, then, on account of proposition 2, f is qc according to Väisälä's geometric definition (by means of the module M, i.e. for $p = n$) and, on account of proposition 3, $f, f^{-1} \in \mathrm{Lip}\, K$, hence, by corollaries 2 and 3 of the preceding 2 propositions, we are allowed to conclude the (p)-qc of f for any p, including $p = \infty$.

And now, in order to be able to say something with respect to the inclusion relations between the different classes of (K, p)-qc for a fixed K and different p, let us remind some preliminary concepts and results.

Following N. V. Lam [10], let us give the following 2 definitions of the (K, p)-qc.

I. *A homeomorphism $f: D \rightleftarrows D^*$ is said to be (K, p)-qc according to the geometric definition if*

$$M_p\Gamma/K \leqq M_p\Gamma^* \leqq K M_p \Gamma, \tag{15}$$

$\forall \Gamma$ *of D, where $\Gamma^* = f(\Gamma)$ and M_p is given by* (1).

II. *A homeomorphism $f: D \rightleftarrows D^*$ is (K, p)-qc according to the analytic definition iff f is ACL in D, it is differentiable a.e. in D and, at each point of differentiability, it satisfies the double inequality*

$$\max |f'(x)|^p / K^{p-1} \leqq |J(x)| \leqq K^{p-1} \min |f'(x)|^p,$$

where

$$\max |f'(x)| = \max_{|\Delta x|=1} |f'(x)\Delta x|, \quad \min |f'(x)| = \min_{|\Delta x|=1} |f'(x)\Delta x|.$$

We remind that a mapping f is said to be ACL in a domain D if it is continuous in D and for every closed interval $I = \{x ; a^i \leqq x^i \leqq b^i\} \subset D$, f is absolutely continuous on almost every line segment in I, parallel to the coordinate axes.

III. *A homeomorphism $f: D \rightleftarrows D^*$ is (K, p)-qc if*

$$\mathrm{cap}_p(\Delta, E_0, E_1)/K \leqq \mathrm{cap}_p(\Delta^*, E_0^*, E_1^*) \leqq K\, \mathrm{cap}_p(\Delta, E_0, E_1),$$

where $\Delta \subset\subset D$ is an arbitrary spherical ring with the boundary spheres E_0, E_1 or an arbitrary cube with the bases E_0, E_1 and where $\Delta^* = f(\Delta)$, $E_i^* = f(E_i)$ $(i = 0, 1)$ and cap_p *is given* by (14).

Remark. The class of (K, p)-qc in the sense of Gehring is larger than that corresponding to the preceding definition.

Proposition 6. *Definitions* I *and* II *of the (K, p)-qc are equivalent (with the same p and K)* (N. V. Lam [10], theorem 1.4).

Now, let us remind that a topological cylinder (with respect to the euclidean metric) is a configuration (Z, E_0, E_1), where $Z \subset R^n$ is a domain, $E_0, E_1 \subset \partial Z$ and ∂Z the boundary of Z, if there exists a homeomorphism $\varphi: \bar{Z}_0 \rightleftarrows \bar{Z}$ such that $\varphi(E_i^0) = E_i (i = 0, 1)$, $Z_0 = \{x ; (x^1)^2 + \ldots + (x^{n-1})^2 < 1,\ 0 < x^n < 1\}$, $E_i^0 = \{x ; (x^1)^2 + .. + (x^{n-1})^2 \leqq 1,\ x^n = i\}$ $(i = 1, 2)$ E_0, E_1 are called the bases of the topological cylinder.

Proposition 7. *If $Z = (Z, E_0, E_1)$ is a topological cylinder and the restriction φ_Z of the homemorphism φ involved in its definition is qc, then*

$$M_p Z = \mathrm{cap}_p Z, \tag{16}$$

where $M_p Z = M_p \Gamma_Z$ (our paper [6], theorem 8).

Corollary. *If Z is a topological cylinder and $\varphi_{|Z}$ is (p)-qc (according to any of the definitions from above), then (16) holds.*

If $\varphi_{|Z}$ is (p)-*qc* according to definition I or II, then, $\varphi_{|Z}$ is (p)-qc according to definition III too and then, *a fortiori*, $\varphi_{|Z}$ is (p)-qc also in Gehring's sense, so that, on account of proposition 2, $\varphi_{|Z}$ is qc and the preceding proposition implies (16).

Proposition 8. *A (p)-qc is differentiable a.e. in D*(N. V. Lam [10]).

Proposition 9. *A homeomorphism $f: D \rightleftarrows D^*$ such that $M_p Q \leqq K M_p Q^*$ for every cube $Q \subset\subset D$, verifies, at any point of differentiability $x_0 \in D$, the inequality*

$$\max |f'(x_0)| \leqq K^{p-1} |J(x_0)| \tag{17}$$

(N. V. Lam [10], lemma 11).

Theorem 6. *The preceding 3 definitions of (K, p)-qc are equivalent (with the same K).*

The equivalence of definitions I and II of the (K, p)-qc is a direct consequence of proposition 6. Next, if f is (K, p)-qc according to definition I or II, then, on account of corollary 1 of proposition 5 and of the preceding corollary, f is (K, p)-qc also according to definition III (with the same K).

Now, let us prove the converse. Suppose f is (K, p)-qc according to definition III ; then, on account of propositions 3 and 4, f is (p)-qc according to definition I and also II (which is equivalent to I) ; in other words, there is a $K_0>1$ such that f is (K_0, p)-qc according to these definitions. Next, by proposition 7, $M_pQ^*=M_p\Gamma(Q^*, E_0^*, E_1^*)=\mathrm{cap}_p(Q^*, E_0^*, E_1^*)$, where $Q^*=f(Q)$, $\overline{Q}^*=f(\overline{Q})$, $E_i^*=f(E_i)$ $(i=0,1)$ and $Q\subset\subset D$ is a cube, so that, taking account of propositions 8,9 for f and f^{-1}, we obtain (17) for f and f^{-1}, hence, arguing as in proposition 6, it follows that the double inequality (15) holds, i.e. f is (K, p)-qc according to definition I and then also to definition II, as desired.

Now, assume x_0 is a point of differentiability. Then, the linear transformation $y=f'(x_0)\Delta x$ maps the unit ball $B(x_0, 1)$ onto an ellipsoid with the semi-axes $a_1(x_0)\geqq\ldots\geqq a_n(x_0)\geqq 0$, where

$$a_1(x_0)=\max|f'(x_0)|,\quad a_n(x_0)=\min|f'(x_0)|,\quad |J(x_0)|=a_1(x_0)\ldots a_n(x_0).$$

Proposition 10. *If a homeomorphism $f: D\rightleftarrows D^*$ is differentiable at a point $x_0\in D$ and satisfies the double inequality $M_2Z/K\leqq M_2Z^*\leqq KM_2Z$ for all topological cylinders $Z\subset\subset D$, where $Z^*=f(Z)$, then*

$$\frac{a_1(x_0)^2}{a_1(x_0)\ldots a_n(x_0)}\leqq K,\quad \frac{a_n(x_0)^2}{a_1(x_0)\ldots a_n(x_0)}\geqq\frac{1}{K}$$

(our paper [2], lemma 10).

Arguing as in the preceding proposition, we obtain

Lemma 4. *If a homeomorphism $f: D\rightleftarrows D^*$ is differentiable at a point $x_0\in D$ and satisfies $M_pZ/K\leqq M_pZ^*\leqq KM_pZ$, for all topological cylinders $Z\subset\subset D$, then*

$$\frac{a_1(x_0)^p}{a_1(x_0)\ldots a_n(x_0)}\leqq K^{p-1},\quad \frac{a_n(x_0)^p}{a_1(x_0)\ldots a_n(x_0)}\geqq\frac{1}{K^{p-1}}. \tag{18}$$

Corollary. *If f is (K, p)-qc (according to definitions* I, II *or* III*) then, at a point of differentiability (i.e. a.e. in D), (18) holds.*

With the above notations, we shall introduce the following new definition.

IV. *A homeomorphism $f: D\rightleftarrows D^*$ is called (K, p)-qc according to the metric definition if f is ACL in D, differentiable a.e. in D and, at each point of differentiability, $x_0\in D$,*

$$\max\left[\frac{a_1(x_0)^p}{a_1(x_0)\ldots a_n(x_0)},\ \frac{a_1(x_0)\ldots a_n(x_0)}{a_n(x_0)^p}\right]\leqq K^{p-1}.$$

Theorem 7. *Definitions* I, II, III, IV *are equivalent with the same p and K.*

Indeed, if f is (K, p)-qc according to I, II or III, it is (K, p)-qc according to II, implying the (K, p)-qc according to IV since the preceding inequality holds if we take into account the definition of $a_i(x_0)(i=1,\ldots,n)$. But, also conversely, if f is (K, p)-qc according to IV, then f is ACL in D and differentiable a.e. in D and, on account of the definition of $a_i(x_0)(i=1,\ldots, n)$, the preceding inequality implies the corresponding double inequality involved in the definition of II. The (K, p)-qc according to the other 2 definitions follows from the preceding theorem.

Theorem 8. *The class of (p)-qc is contained in the class of qc and the inclusion is stricte.*

Proposition 2 assures the inclusion, which is stricte since the inversion $x^*=x/|x|^2$ is conforme (i.e. 1-qc) and then has $a_1(x)=\ldots=a_n(x)=1/|x|^2$, hence $a_1(x)^p/[a_1(x)\ldots a_n(x)]=a_n(x)^p/[a_1(x)\ldots a_n(x)]=|x|^{n-p}$, so that there is no K $(1\leqq K<\infty)$ for which the inequalities (18) hold in a domain containing O[if $1<p<n$, the second inequality (18) can not be verified, while if $p>n$, the first one].

Theorem 9. *For a fixed K $(1\leqq K<\infty)$, the class of (K, p)-qc is contained in the class of (K, p')-qc with $1<p\leqq p'\leqq 2$ and if $n\geqq 3$, also with $p'>2$.*

Let us denote $p'=p+q$ and consider first the case $p+q\leqq 2$. If x_0 is a point of differentiability of an n-dimensional mapping f with $|J(x_0)|>0$ then, denoting $a_i(x_0)=a_i$ with $a_i(x_0)$ $(i=1,\ldots,n)$ from above, we have to consider the following possibilities:

I. $(a_1a_n)^{2-p}(a_2\ldots a_{n-1})^2\geqq 1$. Then, on account of (18), we have $K=\max[a_1/(a_2\ldots a_n)^{1/p-1}, (a_1\ldots a_{n-1})^{1/p-1}/a_n]=(a_1\ldots a_{n-1})^{1/p-1}/a_n$. Next, let us observe that $a_1\ldots a_{n-1}\geqq 1$ since otherwise, $a_2\ldots a_{n-1}<1/a_1$, hence

$$1\leqq(a_1a_n)^{2-p}(a_2\ldots a_{n-1})^2<a_n^{2-p}/a_1^p\Rightarrow 1\leqq a_1^p<a_n^{2-p}$$

($\Rightarrow$ means „implying") which is absurde.

$$1.\ K_1=\max\left[\frac{a_1}{(a_2\ldots a_n)^{1/(p+q-1)}},\ \frac{(a_1\ldots a_{n-1})^{1/(p+q-1)}}{a_n}\right]=\frac{(a_1\ldots a_{n-1})^{1/(p+q-1)}}{a_n}\Rightarrow K\geqq K_1.$$

$$2.\ K_1=\frac{a_1}{(a_2\ldots a_n)^{1/(p+q-1)}}\Rightarrow(a_1a_n)^{2-(p+q)}(a_2\ldots a_{n-1})^2\leqq 1,$$ whence, arguing as above for the inequality $a_1\ldots a_{n-1}\geqq 1$, we deduce that $a_2\ldots a_n\leqq 1\Rightarrow K=(a_1\ldots a_{n-1})^{1/(p-1)}/a_n\geqq a_1/(a_2\ldots a_n)^{1(p+q-1)}=K_1$, since this inequality comes to $(a_1a_n)^{2-p}(a_2\ldots a_{n-1})^2\geqq 1$ in the particular case $q=0$ and K_1 is non-increasing with respect to q.

II. $(a_1a_n)^{2-p}(a_2\ldots a_{n-1})^2<1$. Then, on account of (18), $K=a_1/(a_2\ldots a_n)^{1(p-1)}$. Next, arguing as above, it follows that $a_2\ldots a_n\leqq 1$.

$$1.\ K_1=\frac{a_1}{(a_2\ldots a_n)^{1(p+q-1)}}\Rightarrow K\geqq K_1.\qquad 2.\ K_1=\frac{(a_1\ldots a_{n-1})^{1/(p+q-1)}}{a_n}\Rightarrow$$

$(a_1a_n)^{2-(p+q)}(a_2\ldots a_{n-1})^2\geqq 1\Rightarrow a_1\geqq 1$ and, arguing as above, $a_1\ldots a_{n-1}\geqq 1$. But then, $K\geqq K_1$, since this inequality comes to $(a_1a_n)^{2-p}(a_2\ldots a_{n-1})^2\leqq 1$ for $q=0$ and K_1 is non-increasing with respect to q.

Now, let us consider also the case $p+q>2$. We shall discuss again succesively the different possibilities, where K and K_1 have the same meaning as in the first part of the proof.

I. $K=\dfrac{(a_1\ldots a_{n-1})^{1/(p-1)}}{a_n}\Rightarrow(a_1a_n)^{2-p}(a_2\ldots a_{n-1})^2\geqq 1$. Arguing as above (in the corresponding case), we deduce that $a_1\ldots a_{n-1}\geqq 1$.

$$1.\ K_1=\frac{(a_1\ldots a_{n-1})^{1(p+q-1)}}{a_n}\Rightarrow K\geqq K_1.$$

2. $K_1=\dfrac{a_1}{(a_2...a_n)^{1/(p+q-1)}}\Rightarrow(a_1a_n)^{p+q-2}\geqq(a_2...a_{n-1})^2.$

A. $a_2...a_n\leqq 1$. Let us show that $K\geqq K_1$. But this inequality comes to $a_1a_n\leqq(a_1...a_{n-1})^{1/(p-1)}a_2...a_n$ for $p+q=2$, hence $a_1^{2-p}(a_2...a_{n-1})^p\geqq 1$. In order to verify this last inequality, let us consider the following 2 possibilities: **a)** $a_2...a_{n-1}\leqq 1$ and **b)** $a_2...a_{n-1}>1$. But, in the case a), we have $a_1^{2-p}(a_2...a_{n-1})^p\geqq$ $\geqq(a_1a_n)^{2-p}(a_2....a_{n-1})^2\geqq 1$, which is true because we are in the case **I**; while in the case b), $a_1<a_1(a_2...a_{n-1})^2\leqq(a_1...a_{n-1})^{1/(p-1)}a_2...a_{n-1}\Rightarrow a_1a_n<$ $<(a_1...a_{n-1})^{1/(p-1)}a_2....a_n$. Thus, $K\geqq K_1$ is true in the particular case $p+q=2$ and taking into account that, K_1 is non-increasing with respect to q, it follows that $K\geqq K_1\forall q>0$ such that $p+q\geqq 2$.

B. $a_2...a_n>1$. Then, $a_2...a_{n-1}>1$ too. Letting $q\to\infty$ in the inequality $K\geqq K_1$, we get $a_1a_n\leqq(a_1...a_{n-1})^{1/(p-1)}\Rightarrow(a_1a_n)^{p-1}\leqq a_1...a_{n-1}\Rightarrow a_n^{p-1}\leqq a_1^{2-p}a_2...a_{n-1}$, which is evident if $a_n\leqq 1$, while if $a_n>1$, then, in the case $n\geqq 3$, $a_n^{p-1}\leqq a_n\leqq a_{n-1}\leqq$ $\leqq a_1^{2-p}a_2...a_{n-1}$. And now, because K_1 is non-decreasing with respect to q, it follows that since $K\geqq K_1$ for $q\to\infty$, it is *a fortiori* true for any q such that $p+q\geqq 2$.

II. $K=a_1/(a_2...a_n)^{1/(p-1)}\Rightarrow(a_1a_n)^{2-p}(a_2...a_{n-1})^2\leqq 1$. Arguing as in the first part of the proof (in the case II), we deduce that $a_2...a_n\leqq 1$.

1. $K_1=a_1/(a_2...a_n)^{1/(p+q-1)}\Rightarrow K\geqq K_1$.

2. $K_1=(a_1...a_{n-1})^{1/(p+q-1)}/a_n\Rightarrow(a_1a_n)^{p+q-2}\leqq(a_2...a_{n-1})^2$.

A. $a_1...a_{n-1}\geqq 1$. Let us show that $K\geqq K_1$. But, this inequality comes to $a_1a_n\geqq(a_2...a_n)^{1/(p-1)}a_1...a_{n-1}$ for $p+q=2$, hence, $a_n^{2-p}(a_2...a_{n-1})^p\leqq 1$. In order to verify this last inequality, we observe that if $a_2...a_{n-1}\geqq 1$, then, since $a_1\geqq 1$, we have $a_n^{2-p}(a_2...a_{n-1})^p\leqq(a_1a_n)^{2-p}(a_2...a_{n-1})^2\leqq 1$, while if $a_2...a_{n-1}<1$, then, $a_n>a_n(a_2...a_{n-1})^2\geqq(a_2...a_n)^{1/(p-1)}a_2...a_{n-1}\Rightarrow a_1a_n\geqq(a_2...a_n)^{1/(p-1)}a_1...a_{n-1}$. Thus, $K\geqq K_1$ is true for $p+q=2$ and since K_1 is non-increasing with respect to q, it follows that $K\geqq K_1\forall q>0$ such that $p+q\geqq 2$.

B. $a_1...a_{n-1}<1$. Then, $a_2...a_{n-1}<1$. too. Letting $q\to\infty$ in the inequality $K\geqq K_1$, we obtain $a_1a_n\geqq(a_2...a_n)^{1/(p-1)}\Rightarrow(a_1a_n)^{p-1}\geqq a_2...a_n\Rightarrow a_1^{p-1}\geqq a_n^{2-p}a_2...a_{n-1}$, which is evident if $a_1\geqq 1$, while, if $a_1<1$, then, in the case $n\geqq 3$, $a_1^{p-1}\geqq a_1\geqq$ $\geqq a_2\geqq a_n^{2-p}a_2...a_{n-1}$. And now, because K_1 is non-decreasing with respect to q, it follows that, since $K\geqq K_1$ is true for $q\to\infty$, it is *a fortiori* true for any q such that $p+q\geqq 2$.

Finally, using the notations

$$K(x_0)=\max[a_1/(a_2...a_n)^{1/(p-1)},\ (a_1...a_{n-1})^{1/(p-1)}/a_n],$$

$$K_1(x_0)=\max[a_1/(a_2...a_n)^{1/(p+q-1)},\ (a_1...a_{n-1})^{1/(p+q-1)}/a_n],$$

where $x_0\in D$ is an arbitrary point of differentiability with $|J(x_0)|>0$ of a homeomorphism $f: D\rightleftarrows D^*$ and $K=\sup K(x_0)$, $K_1=\sup K_1(x_0)$, where the supremum is taken over all such x_0, it follows that $K\geqq K_1$ and then, on account of theorem 7, if f is ACL in D and a.e. in D it is differentiable satisfying (18), we deduce that f is $(K_1,\ p+q)$-qc in D, and if $K>K_1$, it is not $(K_1,\ p)$-qc in D. However, f is $(K,\ p)$-qc and also $(K,\ p+q)$-qc. But, since this happens for any homeomorphism f ACL in D and a.e. in D differentiable and satisfying (18), we are allowed to conclude that the class of $(K,\ p)$-qc is contained

in the class of $(K,\ p+q)$-qc for $1<p\leqq p+q\leqq 2$ and, if $n\geqq 3$, also for $p+q>2$ as desired.

Theorem 10. *For a fixed K $(1\leqq K<\infty)$, there are not inclusion relations between the classes of $(K,\ p)$-qc and $(K,\ p+q)$-qc for $p\geqq 2$, $q>0$ and, in the case $n=2$, also if $3/2<p\leqq 2$, $\forall q>0$ with $p+q>2$.*

We shall prove this theorem by means of some counter examples. For simplicity sake, we shall take them in the case $n=2$. Let us consider, the affine transformation f given by

$$x^{*1}=a_1x^1,\quad x^{*2}=a_2x^2 \tag{19}$$

in the following cases:

I. $p\geqq 2$, $a_1\geqq a_2>1$. If f is $(K,\ p)$-qc, then (18) yields $K=a_1/a_2^{1/(p-1)}$. Clearly, $K_1=a_1/a_2^{1/(p+q-1)}$, hence f is $(K_1,\ p+q)$-qc and, since $K<K_1\forall q>0$, it follows that f is not $(K,\ p+q)$-qc, whence, we deduce that the class of $(K,\ p)$-qc is not contained in the class of $(K,\ p+q)$-qc and this $\forall q>0$.

II. $p\geqq 2$, $a_1\geqq 1>a_2$, $a_1a_2\geqq 1$. Again f is $(K,\ p)$-qc and $(K_1,\ p+q)$-qc with $K=a_1/a_2^{1/(p-1)}$, $K_1=a_1/a_2^{1/(p+q-1)}$. But, since in this case $K>K_1$, it follows that f is $(K_1,\ p+q)$-qc, but is not $(K_1,\ p)$-qc, allowing us to deduce that the class of $(K,\ p+q)$-qc is not contained in the class of $(K,\ p)$-qc and this $\forall$ $q>0$.

III. Suppose now that $1<p\leqq 2$, $p+q\geqq 2$, $a_1a_2\geqq 1$ and $a_1\geqq 1\geqq a_2$. Then $K=a_1^{1/(p-1)}/a_2$, $K_1=a_1/a_2^{1/(p+q-1)}$ hence $K>K_1$ since $a_1a_2<a_1^{1/(p-1)}/a_2^{1/(p+q-1)}$, which is equivalent to the evident inequality $a_1^{(2-p)/(p-1)}>a_2^{(p+q-2)/(p+q-1)}$ (for $1<$ $<p\leqslant 2$ and $p+q\geqq 2$).

IV. Now, assume that $3/2<p\leqq 2$, $p+q\geqq 2$, $a_1=a_2=a>1$. Then, again, $K=a^{1/(p-1)}/a$, $K_1=a/a^{1/(p+q-1)}$ and the inequality $K<K_1$ yields $a^{(2-p)/(p-1)}<$ $<a^{(p+q-2)/(p+q-1)}$, which is equivalent to $(2-p)/(p-1)<(p+q-2)/(p+q-1)$ implying $2(p-1)(2-p)<q(2p-3)$, whence $q>2(p-1)(2-p)/(2p-3)>0$, as desired.

If in the definition III of (K,p)-qc (from above), we change K by $K^{1/(p-1)}$, we obtain a class of (K,p)-qc corresponding to that considered by F. Gehring [9], where the p-capacity is given by (13). On account of theorem 7, this class of $(K,\ p)$-qc is equivalent to the other 3 classes corresponding to our definitions I, II and IV if we change in them K^{p-1} by K. For these classes too, we have analogous results with respect to the inclusion problem (as given by the preceding 2 theorems).

Theorem 11. *For a fixed K and $p\leqq p'$, $p+p'\geqq 2n$, $p'\geqq 2(n-1)$, the class of (K,p')-qc is contained in the class of $(K,\ p)$-qc (where the classes are in the sense considered by Gehring in [9]).*

Let us consider again (as in theorem 9) the different possibilities if $x_0\in D$ is a point of differentiability of $f: D\rightleftarrows D^*$ with $|J(x_0)|>0$. Let us denote $p'=p+q$, $K'=\max[a_1^p/(a_1...a_n),\ (a_1...a_n)/a_n^p]$, $K_1'=\max[a_1^{p+q}/(a_1...a_n),$ $(a_1...a_n)/a_n^{p+q})]$ and assume $q>0$.

I. $K'=a_1^p/(a_1...a_n)$. In this case, clearly, $(a_1a_n)^p\geqq(a_1...a_n)^2$.

1. $a_1a_n\geqq 1$. Then, $(a_1a_n)^{p+q}\geqq(a_1a_n)^p\geqq(a_1...a_n)^2$, hence, $K_1'=a_1^{p+q}/(a_1...a_n)$. Since $a_1\geqq 1$, it follows that $K'\leqq K_1'$.

2. Assume $a_1a_n<1$ and observe that $a_n<1$.

A. $(a_1a_n)^{p+q-2} \geqq (a_2 \ldots a_{n-1})^2 \Rightarrow K_1' = a_1^{p+q}/(a_1 \ldots a_n)$. The inequality $K' \leqq K_1$ is evident if $a_1 \geqq 1$. But $a_1 < 1$ is absurde since then $a_n^{2(n-2)} > a_n^{p+q-2} > (a_1a_n)^{p+q-2} \geqq$ $\geqq (a_2 \ldots a_{n-1})^2 \geqq a_n^{2(n-2)}$.

B. $(a_1a_n)^{p+q-2} < (a_2 \ldots a_{n-1})^2 \Rightarrow K_1' = (a_1 \ldots a_n)/a_n^{p+q}$. Next, suppose, to prove it is false, that $K' > K_1'$. Then, $a_1^{p-2}a_n^{p+q-2} > (a_2 \ldots a_{n-1})^2$. We must have $a_1 < 1$, since otherwise $(a_2 \ldots a_{n-1})^2 < a_1^{p-2}a_n^{p+q-2} \leqq (a_1a_n)^{p+q-2} < (a_2 \ldots a_{n-1})^2$, which is absurde. But, even in the case $a_1 < 1$, the inequality $K' > K_1$ is not verified. Indeed, assume first that $p > 2$ Then, $a_1^{p-2}a_n^{p+q-2} > (a_2 \ldots a_{n-1})^2 \geqq a_n^{2(n-2)} \Rightarrow$ $\Rightarrow a_1^{p-2} > a_n^{2(n-1)-(p+q)} \Rightarrow 2(n-1)-(p+q) > 0 \Rightarrow p+q < 2(n-1)$, contradicting the hypothesis $p' \geqq 2(n-1)$. In the case $p \leqq 2$, $a_1^{p-2}a_n^{p+q-2} > a_n^{2(n-2)} \Rightarrow a_n^{p+q-2(n-1)} > a_1^{2-p} \Rightarrow$ $\Rightarrow p+q-2(n-1) < 2-p \Rightarrow p+p' < 2n$, contradicting $p+p' \geqq 2n$.

II. $K' = (a_1 \ldots a_n)/a_n^p \Rightarrow (a_1a_n)^p \leqq (a_1 \ldots a_n)^2$.

1. $a_1a_n \geqq 1 \Rightarrow a_1 \geqq 1$.

A. $(a_1a_n)^{p+q-2} \leqq (a_2 \ldots a_{n-1})^2 \Rightarrow K_1' = (a_1 \ldots a_n)/a_n^{p+q}$. The inequality $K' \leqq K_1'$ is evident if $a_n \leqq 1$. But, $a_n > 1$ is absurde since $a_1^{2(n-2)} \leqq a_1^{p+q-2} < (a_1a_n)^{p+q-2} \leqq$ $\leqq (a_2 \ldots a_{n-1})^2 \leqq a_2^{2(n-2)} \leqq a_1^{2(n-2)}$.

B. $(a_1a_n)^{p+q-2} > (a_2 \ldots a_{n-1})^2 \Rightarrow K_1' = a_1^{p+q}/(a_1 \ldots a_n)$. Next, suppose, to prove it is false, that $K' > K_1'$. Then. $a_1^{p+q-2}a_n^{p-2} < (a_2 \ldots a_{n-1})^2$. We must have $a_n \geqq 1$, since otherwise $(a_2 \ldots a_{n-1})^2 \geqq a_1^{p+q-2}a_n^{p-2} \geqq (a_1a_n)^{p+q-2} > (a_2 \ldots a_{n-1})^2$, which is absurde. Now, assume first that $p > 2$. Then, $a_1^{p+q-2}a_n^{p-2} < (a_2 \ldots a_{n-1})^2 \leqq a_2^{2(n-2)} \leqq$ $\leqq a_1^{2(n-2)} \Rightarrow a_1^{p+q-2(n-1)}a_n^{p-2} < 1 \Rightarrow p+q-2(n-1) < 0$, contradicting $p' \geqq 2(n-1)$. In the case $p \leqq 2$, $a_1^{p+q-2}a_n^{p-2} < a_1^{2(n-1)} \Rightarrow a_1^{p+q-2(n-1)} < a_n^{2-p} \Rightarrow p+q-2(n-1) < 2-p \Rightarrow p+$ $+p' < 2n$, contradicting $p+p' \geqq 2n$.

2. $a_1a_n < 1 \Rightarrow a_n < 1$, $(a_1a_n)^{p+q} \leqq (a_1 \ldots a_n)^2 \Rightarrow K_1' = (a_1 \ldots a_n)/a_n^{p+q} \Rightarrow K' \leqq K_1'$, as desired.

Then, arguing as in the last part of theorem 9, we obtain the inclusion of the class of the (K, p')-qc in the class of the (K, p)-qc, where $p \leqq p'$, $p+p \geqq 2n$, $p' \geqq 2(n-1)$.

Now, let us show that if the hypotheses of the preceding theorem are not fulfilled, then, there are not inclusion relations between the different classes of (K, p)-qc (in Gehring sense).

Theorem 12. *For a fixed K, there are not inclusion relations between the classes of (K, p)-qc and (K, p')-qc (in Gehring's sense) with $p < p' \leqq 2(n-1)$ or $p+p' < 2n$.*

For the simplicity sake, we shall consider again the counter-example (19) of theorem 10. Let us denote, as above, $p' = p+q$ and suppose that $1 < p < p+q \leqq 2$. If $a_1 < 1$, then $(a_1a_2)^{2-p} \leqq (a_1a_2)^{2-(p+q)} \leqq 1 \Rightarrow K' = a_1^p/(a_1a_2)$, $K_1' = a_1^{p+q}/(a_1a_2)$and $K' > K_1'$. If $a_1a_2 < 1$ and $a_1 > 1$, then, again $K' = a_1^p/(a_1a_2)$, $K_1' = a_1^{p+q}/(a_1a_2)$, but, this time, $K' < K_1'$.

Now, assume $1 < p < 2 \leqq p+q$. If $a_1a_2 < 1$, then (as above), $K' =$ $= a_1^p/(a_1a_2)$, but $(a_1a_2)^{p+q-2} < 1$, so that $K_1' = (a_1a_2)/a_2^{p+q}$ and then, for any $a_1 > 1$, $a_1^{p-2}a_2^{p+q-2} < 1$ $\forall q > 0$ with $p+q \geqq 2 \Rightarrow K' < K_1'$. If $a_1 = a_2 = a < 1$, then, $a^{p+p+q-4} >$ $> 1 \Rightarrow K' > K_1'$ for any p, p' with $p+p'-4 < 0 \Rightarrow p+p' < 4$, as desired.

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THE DINI PROPERTY: EQUIVALENCES, PRODUCTS AND COUNTEREXAMPLES

BY

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1. Introduction. Raha and Srivastava [5] proved that the Dini property and compacity are equivalent in Tychonoff spaces. Moreover, they characterize the spaces with such property just as those veryfing that every saturated completely regular filter is fixed. Previously, Banachewski [1] has shown the equivalence between this condition for filters and the Stone-Weierstrass property in completely-Hausdorff spaces, (where the „saturated" character is superfluous). Stephenson [3] established in these spaces the equivalence between the Stone-Weierstrass condition and the completely-Hausdorff complete character.

To complete these results, we first establish a diagram of equivalences by means of sufficient conditions, then we study the behaviour of these properties with respect to the product topology, and finally we propose three counterexamples in which several non equivalences are shown by removing some of the sufficient conditions.

2. Notation and definitions. The collection of all bounded continuous functions on the topological space X is denoted by $C^*(X)$. X_W stands for the set X, endowed with the weak topology induced by $C^*(X)$. A topological space X is said to satisfy *the Dini property* if every decrecent pointwise convergent net in $C^*(X)$ is uniformly convergent. A family $\mathcal{A}$, contained in $C^*(X)$, *separates points* if for every pair x, y of different points in X there is a function f in $\mathcal{A}$ with $f(x) \neq f(y)$. A topological space X is said to satisfy the *Stone—Weierstrass property* if every subalgebra of $C^*(X)$, which contains the constants and separates points is dense in $C^*(X)$.

A filter $\mathcal{F}$ on X is called *completely regular* if $\mathcal{F}$ has a base β consisting of open sets such that for each $A \in \beta$ there exist $B \in \beta$ and $f \in C(X, [0, 1])$ verifying $f(B)=0$, $f(X-A)=1$. A completely regular filter $\mathcal{F}$ on X is said *saturated* if for each $B \in \beta$ and $x \in B$ the set $\{y \in X | f(x)=f(y), \forall f \in$ $\in C^*(X)\}$ is contained in B.

If $C^*(X)$ separates points, then X is said *completely-Hausdorff*, (*C-H-space*). In this case, X will be called a *C-H-closed* space [5] if every image of X, by an embedding in any *C-H*-space, is closed. Moreover, X will be said a *C-H-complete* space [3] if there does not exist any dense embedding in a *C-H*-space.

Remark. A topological space X is C-H-closed if and only if it is C-H-complete.

Scheme of the proof of the diagram. A previous result is the following theorem, owed to Banachewski [1]: