A. $(a_1a_n)^{p+q-2} \geqq (a_2 \ldots a_{n-1})^2 \Rightarrow K_1' = a_1^{p+q}/(a_1 \ldots a_n)$. The inequality $K' \leqq K_1$ is evident if $a_1 \geqq 1$. But $a_1 < 1$ is absurde since then $a_n^{2(n-2)} > a_n^{p+q-2} > (a_1a_n)^{p+q-2} \geqq \geqq (a_2 \ldots a_{n-1})^2 \geqq a_n^{2(n-2)}$.

B. $(a_1a_n)^{p+q-2} < (a_2 \ldots a_{n-1})^2 \Rightarrow K_1' = (a_1 \ldots a_n)/a_n^{p+q}$. Next, suppose, to prove it is false, that $K' > K_1'$. Then, $a_1^{p-2}a_n^{p+q-2} > (a_2 \ldots a_{n-1})^2$. We must have $a_1 < 1$, since otherwise $(a_2 \ldots a_{n-1})^2 < a_1^{p-2}a_n^{p+q-2} \leqq (a_1a_n)^{p+q-2} < (a_2 \ldots a_{n-1})^2$, which is absurde. But, even in the case $a_1 < 1$, the inequality $K' > K_1'$ is not verified. Indeed, assume first that $p > 2$. Then, $a_1^{p-2}a_n^{p+q-2} > (a_2 \ldots a_{n-1})^2 \geqq a_n^{2(n-2)} \Rightarrow \Rightarrow a_1^{p-2} > a_n^{2(n-1)-(p+q)} \Rightarrow 2(n-1)-(p+q) > 0 \Rightarrow p+q < 2(n-1)$, contradicting the hypothesis $p' \geqq 2(n-1)$. In the case $p \leqq 2$, $a_1^{p-2}a_n^{p+q-2} > a_n^{2(n-2)} \Rightarrow a_n^{p+q-2(n-1)} > a_1^{2-p} \Rightarrow \Rightarrow p+q-2(n-1) < 2-p \Rightarrow p+p' < 2n$, contradicting $p+p' \geqq 2n$.

II. $K' = (a_1 \ldots a_n)/a_n^p \Rightarrow (a_1a_n)^p \leqq (a_1 \ldots a_n)^2$.

1. $a_1a_n \geqq 1 \Rightarrow a_1 \geqq 1$.

A. $(a_1a_n)^{p+q-2} \leqq (a_2 \ldots a_{n-1})^2 \Rightarrow K_1' = (a_1 \ldots a_n)/a_n^{p+q}$. The inequality $K' \leqq K_1'$ is evident if $a_n \leqq 1$. But, $a_n > 1$ is absurde since $a_1^{2(n-2)} \leqq a_1^{p+q-2} < (a_1a_n)^{p+q-2} \leqq \leqq (a_2 \ldots a_{n-1})^2 \leqq a_2^{2(n-2)} \leqq a_1^{2(n-2)}$.

B. $(a_1a_n)^{p+q-2} > (a_2 \ldots a_{n-1})^2 \Rightarrow K_1' = a_1^{p+q}/(a_1 \ldots a_n)$. Next, suppose, to prove it is false, that $K' > K_1'$. Then, $a_1^{p+q-2}a_n^{p-2} < (a_2 \ldots a_{n-1})^2$. We must have $a_n \geqq 1$, since otherwise $(a_2 \ldots a_{n-1})^2 \geqq a_1^{p+q-2}a_n^{p-2} \geqq (a_1a_n)^{p+q-2} > (a_2 \ldots a_{n-1})^2$, which is absurde. Now, assume first that $p > 2$. Then, $a_1^{p+q-2}a_n^{p-2} < (a_2 \ldots a_{n-1})^2 \leqq a_2^{2(n-2)} \leqq \leqq a_1^{2(n-2)} \Rightarrow a_1^{p+q-2(n-1)}a_n^{p-2} < 1 \Rightarrow p+q-2(n-1) < 0$, contradicting $p' \geqq 2(n-1)$. In the case $p \leqq 2$, $a_1^{p+q-2}a_n^{p-2} < a_1^{2(n-1)} \Rightarrow a_1^{p+q-2(n-1)} < a_n^{2-p} \Rightarrow p+q-2(n-1) < 2-p \Rightarrow p+ +p' < 2n$, contradicting $p+p' \geqq 2n$.

2. $a_1a_n < 1 \Rightarrow a_n < 1$, $(a_1a_n)^{p+q} \leqq (a_1 \ldots a_n)^2 \Rightarrow K_1' = (a_1 \ldots a_n)/a_n^{p+q} \Rightarrow K' \leqq K_1'$, as desired.

Then, arguing as in the last part of theorem 9, we obtain the inclusion of the class of the (K, p')-qc in the class of the (K, p)-qc, where $p \leqq p'$, $p+p' \geqq 2n$, $p' \geqq 2(n-1)$.

Now, let us show that if the hypotheses of the preceding theorem are not fulfilled, then, there are not inclusion relations between the different classes of (K, p)-qc (in Gehring sense).

Theorem 12. *For a fixed K, there are not inclusion relations between the classes of (K, p)-qc and (K, p')-qc (in Gehring's sense) with $p < p' \leqq 2(n-1)$ or $p+p' < 2n$.*

For the simplicity sake, we shall consider again the counter-example (19) of theorem 10. Let us denote, as above, $p' = p+q$ and suppose that $1 < p < p+q \leqq 2$. If $a_1 < 1$, then $(a_1a_2)^{2-p} \leqq (a_1a_2)^{2-(p+q)} \leqq 1 \Rightarrow K' = a_1^p/(a_1a_2)$, $K_1' = a_1^{p+q}/(a_1a_2)$ and $K' > K_1'$. If $a_1a_2 < 1$ and $a_1 > 1$, then, again $K' = a_1^p/(a_1a_2)$, $K_1' = a_1^{p+q}/(a_1a_2)$, but, this time, $K' < K_1'$.

Now, assume $1 < p < 2 \leqq p+q$. If $a_1a_2 < 1$, then (as above), $K' = = a_1^p/(a_1a_2)$, but $(a_1a_2)^{p+q-2} < 1$, so that $K_1' = (a_1a_2)/a_2^{p+q}$ and then, for any $a_1 > 1$, $a_1^{p-2}a_2^{p+q-2} < 1$ $\forall q > 0$ with $p+q \geqq 2 \Rightarrow K' < K_1'$. If $a_1 = a_2 = a < 1$, then, $a^{p+p+q-4} > > 1 \Rightarrow K' > K_1'$ for any p, p' with $p+p'-4 < 0 \Rightarrow p+p' < 4$, as desired.

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THE DINI PROPERTY: EQUIVALENCES, PRODUCTS AND COUNTEREXAMPLES

BY

J. L. LOPEZ ROSENDO

1. Introduction. R a h a and S r i v a s t a v a [5] proved that the Dini property and compacity are equivalent in Tychonoff spaces. Moreover, they characterize the spaces with such property just as those veryfing that every saturated completely regular filter is fixed. Previously, B a n a c h e w s k i [1] has shown the equivalence between this condition for filters and the S t o n e-W e i e r s t r a s s property in completely-Hausdorff spaces, (where the „saturated" character is superfluous). S t e p h e n s o n [3] established in these spaces the equivalence between the Stone-Weierstrass condition and the completely-Hausdorff complete character.

To complete these results, we first establish a diagram of equivalences by means of sufficient conditions, then we study the behaviour of these properties with respect to the product topology, and finally we propose three counterexamples in which several non equivalences are shown by removing some of the sufficient conditions.

2. Notation and definitions. The collection of all bounded continuous functions on the topological space X is denoted by $C^*(X)$. X_W stands for the set X, endowed with the weak topology induced by $C^*(X)$. A topological space X is said to satisfy *the Dini property* if every decrecent pointwise convergent net in $C^*(X)$ is uniformly convergent. A family $\mathcal{A}$, contained in $C^*(X)$, *separates points* if for every pair x, y of different points in X there is a function f in $\mathcal{A}$ with $f(x) \neq f(y)$. A topological space X is said to satisfy the *Stone—Weierstrass property* if every subalgebra of $C^*(X)$, which contains the constants and separates points is dense in $C^*(X)$.

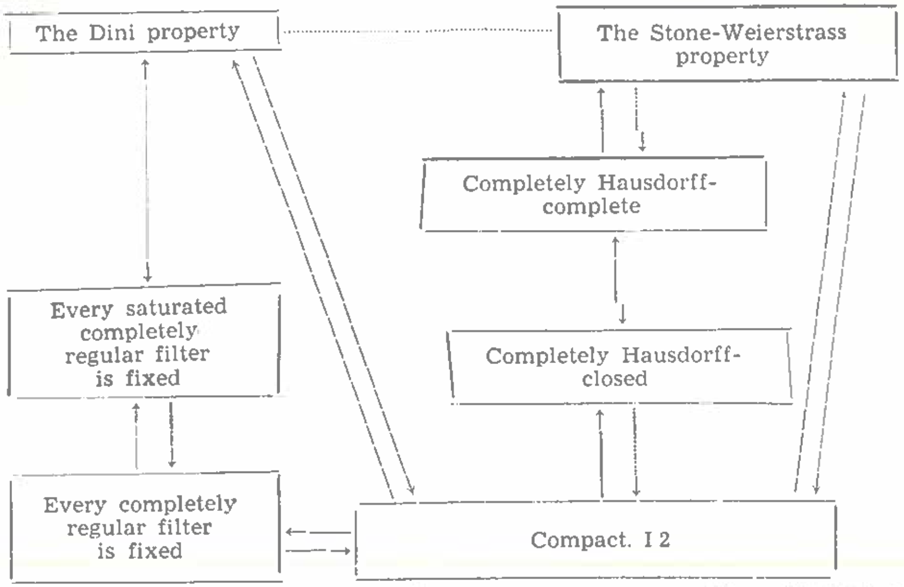
A filter $\mathcal{F}$ on X is called *completely regular* if $\mathcal{F}$ has a base β consisting of open sets such that for each $A \in \beta$ there exist $B \in \beta$ and $f \in C(X, [0, 1])$ verifying $f(B) = 0$, $f(X-A) = 1$. A completely regular filter $\mathcal{F}$ on X is said *saturated* if for each $B \in \beta$ and $x \in B$ the set $\{y \in X \mid f(x) = f(y), \forall f \in \in C^*(X)\}$ is contained in B.

If $C^*(X)$ separates points, then X is said *completely-Hausdorff*, (*C-H-space*). In this case, X will be called a *C-H-closed* space [5] if every image of X, by an embedding in any *C-H*-space, is closed. Moreover, X will be said a *C-H-complete* space [3] if there does not exist any dense embedding in a *C-H*-space.

Remark. A topological space X is C-H-closed if and only if it is C-H-complete.

Scheme of the proof of the diagram. A previous result is the following theorem, owed to B a n a c h e w s k i [1]:

3. Diagram of equivalences.



Remark. The continuous lines in the diagram suppose implications without any additional condition; the implications indicated by dots are true in $C-H$-spaces, whereas the discontinuous lines suppose that the condition of Tychonoff is verified.

Theorem [1]. *On a completely-Hausdorff space X, every completely regular filter is fixed if and only if X_W is compact.*

Proposition 2. *The compacity, the Dini property and the Stone-Weierstrass property are equivalent on uniform-T_2 spaces.*

Proof. A compact-T_2 space has the Dini property.

An uniform space is completely regular. If moreover it is a T_2-space and verifies the Dini property then it is compact by using [5].

On the other hand, a uniform-T_2 space is a C-H-space. So the Dini and Stone-Weierstrass properties are equivalent [5].

c.q.d.

The rest of the implications can be easily deduced using the theorem of Banachewski and the results mentioned in §1 and in propositions 1, 2.

From the previous diagram we conclude that *all the properties expressed there are equivalent on uniform-T_2 spaces.* Moreover, by means of the Tychonoff theorem we can assert that *a nonvoid product space has any of those properties if and only if each factor has the same property, provided that the spaces are uniform-T_2.*

Proposition 3. *The compacity is an strictly stronger condition than the Dini property on C-H-spaces.*

Proof. It is sufficient to find a non compact C-H-space X, such that X_W be compact. A precise example of this situation is the counterexample II.

4. **Products.** The result obtained in the previous section for uniform-T_2 spaces is extended to the completely-Hausdorff spaces. To prove this, we previously need the following lemma, whose proof is straightforward.

Lemma. *A nonempty product space is completely-Hausdorff iff each factor is a C-H-space.*

Theorem. *On completely-Hausdorff spaces a nonempty product space has the Dini property if and only if each factor has this property.*

Proof: $(X_\alpha)_W$ represents the space X_α, endowed with the weak topology induced by $C^*(X_\alpha)$. The topological spaces $(\pi X_\alpha)_W$ and $\pi(X_\alpha)_W$ agree because the product space has the initial topology induced by the projections. The compacity of $(X_\alpha)_W$ is followed, by the theorem of Banachewski, if the spaces X_α have de Dini property. Then $(\pi X_\alpha)_W$ is also compact and so πX_α has the Dini property.

Reciprocally, an analogous reasoning shows that each factor X_α preserves the Dini property if the product space πX_α has the same property.

Corollary. *The C-H-closed character is compatible with products.*

Proof. The last lemma shows that the completely-Hausdorff character is compatible with products. In these spaces the Dini property is equivalent to the condition of being C-H-closed. So, the result is concluded from the last theorem.

Remark. On C-H-spaces a similar result can be obtained for each property expressed in the diagram.

5. **Counterexamples.** *Counterexample I.* Let X be the Sierpinski space. Its topology consists of the open sets X, $\{a\}$, $\varnothing$. X is neither a C-H-space nor C-H-closed because it is not Hausdorff; but it is compact and verifies the Stone-Weierstrass property, because $C^*(X)$ is reduce to the constant functions. Moreover, completely regular filters do not exist on X.

If $\{F_\alpha\}$ is a decrecent pointwise convergent net in $C^*(X)$, it is obviously convergent. Then X has also the Dini propierty.

Counterexample II. If Λ is the open upper-half plane with the usual topology τ, we generate a topology on $\Gamma = \Lambda \cup \{(x, 0)/x \in \mathbf{R}\}$ by adding to τ all sets of the form $\{(x, 0)\} \cup (\Lambda \cap U)$, where U is an Euclidean neighborhood of $(x, 0)$ in the plane. P represents the same space Γ with the Euclidean topology. The x-axis and the unit square $[0, 1] \times [0, 1]$, endowed with the topology induced by Γ, will be denoted respectively by L and M.

i) First we claim $C^*(\Gamma) \equiv C^*(P)$. In fact, $C^*(P)$ is obviously contained in $C^*(\Gamma)$. On the other hand, let us consider $x \in L$, $f \in C^*(\Gamma)$, $a < f(x) < b$ and a closed interval $[a', b']$, veryfing $a < a' < f(x) < b' < b$. Then, there exists W, (a basic neighborhood of x, which can be supposed non euclidean), such that $f(W)$ is contained in the euclidean closed interval between a' and b'. Let V be the less euclidean neighborhood containing W. There is no point $m \in L \cap V$ veryfing $f(m) \notin [a'; b']$, because $f^{-1}(\{y \mid |y - f(m)| < \varepsilon\})$ must cut W. So, x has a neighborhood V in the space P, such that $f(V)$ is contained in $[a', b']$ and therefore in the open interval between a and b.

ii) M is no compact because $\{(x, 0)/0 \leqslant x \leqslant 1\}$ is a discret space. The usual topology of P, is both the topology induced by $C^*(\Gamma)$, and the weak one induced by $C^*(P)$, acording to „*i*)"; that is: $P = \Gamma_W$. M_W is obviously compact. This last fact and the theorem of Banachewski show that the space $M \subset \Gamma$, which is a C-H-space, *has the Dini property* but it is *no compact.*

Counterexample III. Let X be the product space $I \times Z$, where I represents the interval $[0, 1]$ with the usual topology, and let Z be the space

$\{x \in \mathbf{R} \mid 0 \leqslant x < 1\}$ endowed with the relative topology induced by the family $\{R_a\}_{a \in R}$; $R_a = \{x \in \mathbf{R} \mid a < x\}$. $p: X \to I$ denotes the first projection.

i) Every map $f: X \to I$ can be seen as $\tilde{f} \circ p$, $\tilde{f}$ being a map of $C(I)$. In fact, we shall see that f is constant on each fiber $\{(x_0, t) \mid 0 \leqslant t < 1\}$.

Fixed $\varepsilon > 0$, $f^{-1}(\{y \mid |y - f(x_0, 0)| < \varepsilon\})$ contains an open set $I_{x_0, \delta} \times Z$; where $I_{x_0, \delta}$ represents the open interval in I with center x_0 an radio δ. Let us suppose $f(x_0, t) \neq f(x_0, t')$, $t > t'$, and consider two disjoint open sets A and B containing respectively $f(x_0, t)$ and $f(x_0, t)$. $f^{-1}(A)$ and $f^{-1}(B)$ must contain a neighborhood of (x_0, t) and (x_0, t') is also a neighborhood of (x_0, t).

ii) We claim that every saturated completely regular filter on X is fixed.

In a saturated filter on $I \times Z$, all the fiber belongs to an element F of the filter if one of its points belongs to F.

The problem is reduced to the study of the finite intersection property for arbitrary filters on I. Obviously every filter on I has intersection non void because I is compact.

On the other hand X verifies the Stone-Weierstrass property because every net in $C^*(X)$ can be considered in $C(I, \mathbf{R})$.

The space X_W is just the product of I by Z; taking Z as a trivial space. So, we conclude that X_W is compact.

Remark. All the questions discussed in this paragraph are summarized in the following table:

	Counterexample I	Counterexample II	Counterexample III
C-H-Space	0	1	0
Dini	1	1	1
Stone-Weierstrass	1	1	1
Tychonoff	0	0	0
Compact	1	0	0
C-H Complete	0	1	0
Every completely regular filter is fixed	1	1	0
X_W Compact	1	1	1

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Departamento de Geometria y Topologia Facultad de Matematicas Universidad de Santiago Santiago de Compostela SPAIN

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ON NONOSCILLATORY BEHAVIOUR OF SOLUTIONS OF SECOND ORDER DIFFERENTIAL EQUATIONS

BY

N. PARHI

1. In this paper, we are concerned with nonoscillatory behaviour of solutions of

$$(r(t)y')' - q(t)(y')^{\alpha} - p(t)|y|^{\gamma} \operatorname{sgn} y = f(t) \tag{1}$$

and more general equation

$$(r(t)y')' - q(t)(y')^{\alpha} - p(t)g(y) = f(t), \tag{2}$$

where p, q, r, and $f \in C([0, \infty), R)$ such that $r(t) > 0$, $p(t) \geq 0$, $f(t) \geq 0$, and $q(t)$ does not change sign for large t, $q \in C(R, R)$ such that $yq(y) > 0$ for $y \neq 0$ and $g(0) = 0$ and $\alpha > 0$ is a quotient of odd integers and $\gamma > 0$. It is well-known that all solutions of (2) with $f(t) = 0$ are nonoscillatory. If γ is a ratio of odd integers then (1) is reduced to

$$(r(t)y')' - q(t)(y')^{\alpha} - p(t)y^{\gamma} = f(t). \tag{3}$$

In [1] sufficient conditions were given for nonoscillation of all solutions of (3) with $q(t) = 0$. In this paper sufficient conditions are obtained for nonoscillation of all solutions of (2). It is interesting to note that the conditions imposed here are milder than those in [1].

We are concerned with those solutions of (2) which exist on the half-line $[T, \infty]$, $T \geqslant 0$, and are non-trivial in any neighbourhood of infinity. We may recall that a solution $y(t)$ of (2) is said to be nonoscillatory if there exists a $t_1 \geq T$ such that $y(t) \neq 0$ for $t \geqslant t_1$; it is said to be oscillatory if for any $t_1 \geq T$ there exist t_2 and $t_3 (t_1 < t_2 < t_3)$ such that $y(t_2) > 0$ and $y(t_3) < 0$ and $y(t)$ is said to be of Z-type if it has arbitrarily large zeros but is ultimately non-negative or non-positive. Further, Eq. (2) is said to be nonoscillatory if all its solutions are nonoscillatory.

2. We begin with the following remark.

Remark 1. No solution of (2) is oscillatory or of non-negative Z-type.

Theorem 2. *If $f(t) > 0$, then (2) is nonoscillatory.*

Proof. In view of Remark 1 it is enough to show that no solution of (2) is of non-positive Z-type. If possible, let $y(t)$ be a non-positive Z-type solution of (2) with a double zero at 'a'. So $y(a) = 0 = y'(a)$ and $y(t) < 0$ and $y'(t) < 0$ for $t \in (a, a + \delta)$, $\delta > 0$. Since $(r(t)y'(t))'|_{t=a} = f(a) > 0$, from the continuity of $(r(t)y'(t))'$ it follows that $r(t)y'(t)$ is increasing on $(a, a + \epsilon)$, $0 < \epsilon <$