$\{x \in \mathbf{R} \mid 0 \leqslant x < 1\}$ endowed with the relative topology induced by the family $\{R_a\}_{a \in R}$; $R_a = \{x \in \mathbf{R} \mid a < x\}$. $p : X \to I$ denotes the first projection.

i) Every map $f : X \to I$ can be seen as $\tilde{f} \circ p$, $\tilde{f}$ being a map of $C(I)$. In fact, we shall see that f is constant on each fiber $\{(x_0, t) \mid 0 \leqslant t < 1\}$.

Fixed $\varepsilon > 0$, $f^{-1}(\{y \mid |y - f(x_0, 0)| < \varepsilon\})$ contains an open set $\dot{I}_{x_0, \delta} \times Z$; where $\dot{I}_{x_0, \delta}$ represents the open interval in I with center x_0 an radio δ. Let us suppose $f(x_0, t) \neq f(x_0, t')$, $t > t'$, and consider two disjoint open sets A and B containing respectively $f(x_0, t)$ and $f(x_0, t)$. $f^{-1}(A)$ and $f^{-1}(B)$ must contain a neighborhood of (x_0, t) and (x_0, t') is also a neighborhood of (x_0, t).

ii) We claim that every saturated completely regular filter on X is fixed.

In a saturated filter on $I \times Z$, all the fiber belongs to an element F of the filter if one of its points belongs to F.

The problem is reduced to the study of the finite intersection property for arbitrary filters on I. Obviously every filter on I has intersection non void because I is compact.

On the other hand X verifies the Stone-Weierstrass property because every net in $C^*(X)$ can be considered in $C(I, \mathbf{R})$.

The space X_W is just the product of I by Z; taking Z as a trivial space. So, we conclude that X_W is compact.

Remark. All the questions discussed in this paragraph are summarized in the following table:

Counterexample I		Counterexample II	Counterexample III
C-H-Space	0	1	0
Dini	1	1	1
Stone-Weierstrass	1	1	1
Tychonoff	0	0	0
Compact	1	0	0
C-H Complete	0	1	0
Every completely regular filter is fixed	1	1	0
X_W Compact	1	1	1

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ON NONOSCILLATORY BEHAVIOUR OF SOLUTIONS OF SECOND ORDER DIFFERENTIAL EQUATIONS

BY

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1. In this paper, we are concerned with nonoscillatory behaviour of solutions of

$$(r(t)y')' - q(t)y')^{\alpha} - p(t)|y|^{\gamma} \operatorname{sgn} y = f(t) \tag{1}$$

and more general equation

$$(r(t)y')' - q(t)(y')^{\alpha} - p(t)q(y) = f(t), \tag{2}$$

where p, q, r, and $f \in C([0, \infty), R)$ such that $r(t) > 0$, $p(t) \geq 0$, $f(t) \geq 0$, and $q(t)$ does not change sign for large t, $q \in C(R, R)$ such that $yq(y) > 0$ for $y \neq 0$ and $g(0) = 0$ and $\alpha > 0$ is a quotient of odd integers and $\gamma > 0$. It is well-known that all solutions of (2) with $f(t) = 0$ are nonoscillatory. If γ is a ratio of odd integers then (1) is reduced to

$$(r(t)y')' - q(t)(y')^{\alpha} - p(t)y^{\gamma} = f(t). \tag{3}$$

In [1] sufficient conditions were given for nonoscillation of all solutions of (3) with $q(t) = 0$. In this paper sufficient conditions are obtained for nonoscillation of all solutions of (2). It is interesting to note that the conditions imposed here are milder than those in [1].

We are concerned with those solutions of (2) which exist on the half-line $[T, \infty]$, $T \geqslant 0$, and are non-trivial in any neighbourhood of infinity. We may recall that a solution $y(t)$ of (2) is said to be nonoscillatory if there exists a $t_1 \geq T$ such that $y(t) \neq 0$ for $t \geqslant t_1$; it is said to be oscillatory if for any $t_1 \geq T$ there exist t_2 and $t_3 (t_1 < t_2 < t_3)$ such that $y(t_2) > 0$ and $y(t_3) < 0$ and $y(t)$ is said to be of Z-type if it has arbitrarily large zeros but is ultimately non-negative or non-positive. Further, Eq. (2) is said to be nonoscillatory if all its solutions are nonoscillatory.

2. We begin with the following remark.

Remark 1. No solution of (2) is oscillatory or of non-negative Z-type.

Theorem 2. *If $f(t) > 0$, then (2) is nonoscillatory.*

Proof. In view of Remark 1 it is enough to show that no solution of (2) is of non-positive Z-type. If possible, let $y(t)$ be a non-positive Z-type solution of (2) with a double zero at 'a'. So $y(a) = 0 = y'(a)$ and $y(t) < 0$ and $y'(t) < 0$ for $t \in (a, a + \delta)$, $\delta > 0$. Since $(r(t)y'(t))'|_{t=a} = f(a) > 0$, from the continuity of $(r(t)y'(t))'$ it follows that $r(t)y'(t)$ is increasing on $(a, a + \in)$, $0 < \in <$

$<\delta$. This in turn implies that $y'(t)>0$ for $t\in(a, a+\in)$, a contradiction. Hence the theorem.

Remark 3. In what follows an attempt has been made to relax positivity condition on f.

Theorem 4. *If $f(t)\geq p(t)$ such that $f(t)\not\equiv p(t)$ over any subinterval of $[0, \infty)$, then (2) is nonoscillatory.*

Proof. Let $q(t)\geq 0$ for $t\geq t_0$. To complete the proof of the theorem it is enough to show that no solution of (2) is of nonpositive Z-type. If possible, let $y(t)$ be a non-positive Z-type solution of (2) with consecutive double zeros at a and b $(t_0<a<b)$. From the continuity of $g(y)t))$ it follows that there exists a $\delta>0$ such that $g(y(t))>-1$ for $t\in[a, a+\delta]$. Also there exists a $c_1\in(a, b)$ such that $y'(t)<0$ for $t\in(a, c_1]$. Setting $c=\min c_1, a+\delta]$ and integrating (2) from a to c, we obtain

$$0>r(c)y'(c)=\int_a^c f(t)dt+\int_a^c p(t)g(y(t))dt+\int_a^c q(t)(y'(t))^\alpha dt\geq\int_a^c [f(t)-p(t)]dt>0,$$

a contradiction.

Suppose that $q(t)\geq 0$ for $t\geq t_0$. It is possible to find a $c\in(a, b)$ such that $g(y(t))>-1$ and $y'(t)>0$ for $t\in[c, b)$. Now integrating (2) from c to b yields

$$0>-r(c)y'(c)=\int_c^b f(t)dt+\int_c^b p(t)g(y(t))dt+$$
$$+\int_c^b q(t)(y'(t))^\alpha dt\geq\int_c^b [f(t)-p(t)]dt>0,$$

a contradiction again.

This completes the proof of the theorem.

Theorem 5. *If $f(t)+p(t)\neq 0$ for $t\in[0, \infty)$, then (2) is nonoscillatory.*

Proof. Let $q(t)\geq 0$ for $t\geq t_0$. If possible, let $y(t)$ be a non-positive Z-type solution of (2) with consecutive double zeros at a and $b(t_0<a<b)$. There exists a $c\in(a, b)$ such that $y'(t)>0$ and $g(y(t))>-1$ for $t\in[c, b)$. Now integrating (2) from t to b, $t\in[c, b)$, yields

$$-r(t)y'(t)=\int_t^b f(s)ds+\int_t^b p(s)g(y(s))ds+\int_t^b q(s)(y'(s))^\alpha ds\geq-\int_t^b p(s)ds.$$

So

$$\lim_{t\to b-}\int_t^b p(s)ds/r(t)y'(t)\geq 1.$$

But

$$\lim_{t\to b-}\int_t^b p(s)ds/r(t)y'(t)=\lim_{t\to b-}\ -p(t)/(r(t)y'(t))'=-p(b)/f(b)\geq 0,$$

a contradiction.

Similar argument holds when $q(t)\leq 0$ for $t\geq t_0$. In this case we take limit as $t\to a+$ to get a contradiction. Hence the theorem.

In the following we give two examples to illustrate Theorems 4 and 5.

Example 1. Consider

$$y''-(1+\sin t)y(1+y^2)=2(1+\sin t),\quad t\geq 0. \tag{4}$$

Since all the conditions of Theorem 4 are satisfied, (4) is non-oscillatory. In particular, $y(t)=-1$ is a nonoscillatory solution of (4). We may note that Theorem 5 cannot be applied to (4).

Example 2. From Theorem 5 it follows that all solutions of

$$y''-y=2(1+\sin t)$$

are nonoscillatory. In particular, $y(t)=-(2+\sin t)$ is a non-oscillatory solution. It may be noted that Theorem 4 cannot be applied to this example.

Theorem 6. *If $r(t)$ is differentiable, then (2) is nonoscillatory.*

Proof. In view of Remark 1 it is enough to show that (2) does not admit a non-positive Z-type solution. If possibls, let $y(t)$ be a non-positive Z-type solution of (2) with consecutive double zeros at a, b and $a'(a<b<a')$. So there exist $c\in(a, b)$ and $d\in(b, a')$ such that $y''(t)<0$ for $t\in(c,d)$. Since $r(t)$ is differentiable, (2) can be written as

$$r(t)y''(t)+r'(t)y'(t)=f(t)+p(t)g(y(t))+q(t)(y'(t))^\alpha.$$

So $r(b)y''(b)=f(b)\geq 0$, a contradiction. This completes the proof of the theorem.

3. In this section, we allow $p(t)$ and $f(t)$ to change sign. We consider

$$(r(t)y')'-q(t)(y')^\alpha-p(t)|y|^\gamma \operatorname{sgn} y=f(t), \tag{5}$$

where p, q, r and $f\in C([0, \infty), R)$ such that $r(t)>0$, and $\alpha>0$ is a quotient of odd integers and $\gamma>0$.

Theorem 7. *If $q(t)$ does not change sign for large t, $f(t)>0$ and $\lim_{t\to\infty} f(t)/|p(t)|=\infty$, then all bounded solutions of (5) are nonoscillatory.*

Proof. Let $q(t)\geq 0$ for $t\geq t_0$. Let $y(t)$ be a bounded solution of (5) such that $|y(t)|\leq M$. From the given condition it follows that there exists a $t_1\geq t_0$ such that $f(t)\geq M^\gamma|p(t)|$ for $t\geq t_1$. If possible, let $y(t)$ be oscillatory or Z-type. So there exist t_2 and $t_3(t_1<t_2<t_3)$ such that $y'(t_2)=0=y'(t_3)$ and $y'(t)\geq 0$ for $t\in(t_2, t_3)$. Now integrating (5) from t_2 to t_3, we obtain

$$0\geq\int_{t_2}^{t_3} f(t)dt+\int_{t_2}^{t_3} p(t)|y(t)|^\gamma \operatorname{sgn} y(t)dt\geq\int_{t_2}^{t_3}[f(t)-M^\gamma|p(t)]dt>0,$$

a contradiction. So $y(t)$ is nonoscillatory.

If $q(t)\leq 0$ for $t\geq t_0$, then we can choose t_2 and $t_3(t_1<t_2<t_3)$ such that $y'(t_2)=0=y'(t_3)$ and $y'(t)\leq 0$ for $t\in(t_2, t_3)$ to get a contradiction.

This completes the proof of the theorem.

Theorem 8. *Let $q(t)=0$. If*

$$\lim_{t\to\infty}\int_0^t 1/r(s)\left(\int_0^s |p(\theta)|d\theta\right)ds<\infty$$

and

$$\lim_{t\to\infty}\inf\int_0^t 1/r(s)\left(\int_0^s f(\theta)d\theta\right)ds=-\infty,$$

then all bounded solutions of (5) are nonoscillatory.

Proof. If possible, let $y(t)$ be a bounded oscillatory or Z-type solution of (5). Let $|y(t)| \leq M$. Choose $t_0 > 0$ such that $y(t_0) = 0$ and $y'(t_0) \leq 0$. Integrating (5) from t_0 to t twice yields

$$y(t) \leq M + \int_{t_0}^{t} 1/r(s)\left(\int_{t_0}^{t} |p(\theta)|d\theta\right)ds + \int_{t_0}^{t} 1/r(s)\left(\int_{t_0}^{s} f(\theta)d\theta\right)ds.$$

So $\liminf\limits_{t\to\infty} y(t) = -\infty$, a contradiction. Hence the theorem.

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ON SOME FUNDAMENTAL MATRIX INTEGRAL INEQUALITIES AND THEIR APPLICATIONS

BY

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§1. Introduction. In the study of differential, partial and integral equations, Bellman-Gronwall's inequality and its various linear and non-linear generalizations have played significant roles in proving uniqueness, boundedness, comparison, continuous dependence, perturbation, and stability results [1, 2, 4, 6, 7, 8, 10]. One version of the celebrated Bellman-Gronwall's Lemma is that in which the exact solution of an operator equation of the form $\omega = a + T\omega$ for a monotone operator T provides an upper bound on all solutions of the inequality $u \leq a + Tu$. In this context we mention the survey paper of Chandra and Fleishman [5] and the references given there, in addition to Beckenbach and Bellman [1], Bellman [3], Beesack [2], Losonczi [9] and Walter [11]. Our objective in this paper is to establish a number of new matrix integral inequalities which are ready tools in obtaining pointwise estimates on solutions of certain matrix differential inequalities.

§2. Main Results. Let $\mathcal{A}$ denote the linear space of real $n \times n$ symmetric matrices and $\mathcal{K}$ be a cone of nonnegative definite matrices in $\mathcal{A}$. By means of $\mathcal{K}$, we introduce a partial ordering in $\mathcal{A}$ as follows: For

$$X, Y \in \mathcal{A}, \quad X \leq Y \quad \textit{if and only if} \quad Y - X \in \mathcal{K}.$$

A function $f : \mathcal{A} \to \mathcal{A}$ is said to be monotonic nondecreasing on $\mathcal{A}$ if $X, Y \in \mathcal{A} \Rightarrow f(X) \leq f(Y)$. So f is monotone nondecreasing if and only if $f(Y) - f(X)$ is a non-negative definite matrix.

Let $R^+ = [0, \infty)$ and assume that the following conditions hold:

(A_1) $G : R^+ \times R^+ \times \mathcal{A} \to \mathcal{A}$ is continuous and has the following properties:

(i) $G(t, s, .)$ is non-negative definite and uniformly continuous on $0 \leq s \leq t \leq T < \infty$, and for each s, it is nondecreasing in t,

(ii) $G(t, ., v)$ is nondecreasing in v for each $t \in [0, T]$.

(A_2) (i) $W : R^+ \times \mathcal{A} \to \mathcal{A}$ is uniformly Lipschitz continuous, non-negative definite, and monotonic nondecreasing in the variables separately.

(ii) There exists a real constant $n \times n$ symmetric matrix $Y > 0$ such that for $H(t) \in \mathcal{A}$, $\int_0^T G(T, s, H(s) + W(s, Y))ds \leq Y$.

A useful generalization of a matrix inequality due to Chandra and Fleishman [5] is embodied in the following result.