Proof. If possible, let $y(t)$ be a bounded oscillatory or Z-type solution of (5). Let $|y(t)|\leqslant M$. Choose $t_0>0$ such that $y(t_0)=0$ and $y'(t_0)\leqslant 0$. Integrating (5) from t_0 to t twice yields

$$y(t)\leq M^{\gamma}\int_{t_0}^{t} 1/r(s)\Big(\int_{t_0}^{t} |p(\theta)|d\theta\Big)ds+\int_{t_0}^{t} 1/r(s)\Big(\int_{t_0}^{s} f(\theta)d\theta\Big)ds.$$

So $\liminf\limits_{t\to\infty} y(t)=-\infty$, a contradiction. Hence the theorem.

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ON SOME FUNDAMENTAL MATRIX INTEGRAL INEQUALITIES AND THEIR APPLICATIONS

BY

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§1. Introduction. In the study of differential, partial and integral equations, Bellman-Gronwall's inequality and its various linear and non-linear generalizations have played significant roles in proving uniqueness, boundedness, comparison, continuous dependence, perturbation, and stability results [1, 2, 4, 6, 7, 8, 10]. One version of the celebrated Bellman-Gronwall's Lemma is that in which the exact solution of an operator equation of the form $\omega=a+T\omega$ for a monotone operator T provides an upper bound on all solutions of the inequality $u\leqslant a+Tu$. In this context we mention the survey paper of Chandra and Fleishman [5] and the references given there, in addition to Beckenbach and Bellman [1], Bellman [3], Beesack [2], Losonczi [9] and Walter [11]. Our objective in this paper is to establish a number of new matrix integral inequalities which are ready tools in obtaining pointwise estimates on solutions of certain matrix differential inequalities.

§2. Main Results. Let $\mathcal{A}$ denote the linear space of real $n\times n$ symmetric matrices and $\mathcal{K}$ be a cone of nonnegative definite matrices in $\mathcal{A}$. By means of $\mathcal{K}$, we introduce a partial ordering in $\mathcal{A}$ as follows: For

$$X, Y\in\mathcal{A},\ X\leqslant Y \ \textit{if and only if}\ Y-X\in\mathcal{K}.$$

A function $f:\mathcal{A}\to\mathcal{A}$ is said to be monotonic nondecreasing on $\mathcal{A}$ if $X, Y\in\mathcal{A}\Rightarrow f(X)\leqslant f(Y)$. So f is monotone nondecreasing if and only if $f(Y)-f(X)$ is a non-negative definite matrix.

Let $R^+=[0, \infty)$ and assume that the following conditions hold:

(A_1) $G: R^+\times R^+\times\mathcal{A}\to\mathcal{A}$ is continuous and has the following properties:

(i) $G(t, s, .)$ is non-negative definite and uniformly continuous on $0\leqslant s\leqslant t\leqslant T<\infty$, and for each s, it is nondecreasing in t,

(ii) $G(t, ., v)$ is nondecreasing in v for each $t\in[0, T]$.

(A_2) (i) $W: R^+\times\mathcal{A}\to\mathcal{A}$ is uniformly Lipschitz continuous, non-negative definite, and monotonic nondecreasing in the variables separately.

(ii) There exists a real constant $n\times n$ symmetric matrix $Y>0$ such that for $H(t)\in\mathcal{A}$, $\int_0^T G(T, s, H(s)+W(s, Y))ds\leqslant Y$.

A useful generalization of a matrix inequality due to Chandra and Fleishman [5] is embodied in the following result.

Theorem 1. *Let $X(t)$ and $H(t)$ be real $n\times n$ symmetric matrices, and suppose G and W satisfy (A_1) and (A_2). If for $0\leq t\leq T<\infty$*

$$X(t)\leq H(t)+W\left(t, \int_0^t G(t, s, X(s))ds\right), \tag{1}$$

then, $X(t)\leq H(t)+W(t, R(t))$ on the common interval of existence, where $R(t)$ is the unique maximal solution of the matrix integral equation

$$R(t)=\int_0^t G(t, s, H(s)+W(s, R(s)))ds. \tag{2}$$

Proof. Define

$$\Phi(t)=\int_0^t G(t, s, X(s))ds$$

then $\Phi(t)\in A$ and so

$$X(t)\leq H(t)+W(t, \Phi(t)) \tag{3}$$

By the monotonicity property of G, in the last variable and (3),

$$\Phi(t)\leq \int_0^t G(t, s, H(s)+W(s, \Phi(s))ds \tag{4}$$

Define for $0\leq t\leq T<\infty$, $R_0(t)=\int_0^t G(t, s, H(s)+W(s, Y))ds$, and for $n=1, 2, 3,\ldots$

$$R_n(t)=\int_0^t G(t, s, H(s)+W(s, R_{n-1}(s)))ds.$$

Then clearly, $\{R_n(t)\}\subset \mathcal{A}$ for $n=0, 1, 2, \ldots$ and

$$R_1(t)=\int_0^t G(t, s, H(s)+W(s, R_0(s)))ds=\int_0^t G(t, s, H(s)+W(s, \int_0^s G(s, r,$$

$$H(r)+W(r, Y))dr))ds\leq \int_0^t G(t, s, H(s)+W(s, Y))ds=R_0(t).$$

Monotonicity of G and W implies

$$R_2(t)=\int_0^t G(t, s, H(s)+W(s, R_1(s)))ds\leq R_1(t),$$

and hence by introduction, $R_{n+1}(t)\leq R_n(t)\leq \ldots\leq R_0(t)\leq Y$. Moreover, for any n and $t_1, t_2\in[0, T]$,

$$\|R_n(t_1)-R_n(t_2)\|=\left\|\int_0^{t_1} G(t_1, s, H(s)+W(s, R_{n-1}(s)))ds-\int_0^{t_2} G(t_2, s, H(s)+\right.$$

$$\left.+W(s, R_{n-1}(s)))ds\right\|\leq \int_0^{t_1}\| G(t_1, s, H(s)+W(s, R_{n-1}(s)))-G(t_2, s, H(s)+$$

$$+W(s, R_{n-1}(s)))\|ds+\int_{t_1}^{t_2}\| G(t_2, s, H(s)+W(s, Y))\|ds.$$

Since $G(t, s, .)$ is uniformly continuous on any closed interval and (A_2) holds, we see that we can make the right hand side of the last inequality as small as we please. Hence $\{R_n(t)\}$ is equicontinuous and also equibounded so that Ascoli-Arzela's theorem implies $\{R_n(t)\}$ is precompact. Moreover since it is monotonic it converges uniformly to $R(t)$ say and continuity of G and W, imply that $R(t)$ is a solution of the matrix integral equation

$$R(t)=\int_0^t G(t, s, H(s)+W(s, R(s)))ds\ldots\ldots \tag{5}$$

and $R(t)\leq Y$. Let $\tilde{R}(t)$ be any other solution of (5) such that $\tilde{R}(t)\leq Y$, then

$$R_0(t)-\tilde{R}(t)=\int_0^t G(t, s, H(s)+W(s, Y))ds-\int_0^t G(t, s, H(s)+W(s, \tilde{R}(t))ds\geq 0.$$

Similarly $R_1(t)-\tilde{R}(t)\geq 0$ and by induction, it easily follows that $R_n(t)-\tilde{R}(t)\geq 0$ for all n. Hence taking limits we have $R(t)\geq \tilde{R}(t)$, so that $R(t)$ is the maximal solution of (5).

Suppose $\Phi(t)\in S_Y=\{\Phi(t)\in\mathcal{A}: \Phi(t)\leq Y\}$. Define $Z_0(t)=\Phi(t)$ and for $n=1, 2, \ldots$ set

$$Z_n(t)=\int_0^t G(t, s, H(s)+W(s, z_{n-1}(s)))ds.$$

Clearly by induction and (4), $\Phi(t)\leq Z_1(t)\leq \ldots\leq Z_n(t)\leq Z_{n+1}(t)\ldots\ldots$ However, $\{Z_n(t)\}$ converges to a solution $Z(t)$ of (5) in S_Y. The maximality of $R(t)$ as a solution of (5) implies, $Z(t)\leq R(t)$. Hence $\Phi(t)\leq Z(t)\leq R(t)$, for all t in the common interval of existence. Monotonicity of W implies

$$X(t)\leq H(t)+W(t, \Phi(t))\leq H(t)+W(t, R(t))$$

and the proof is complete.

The following is another variation of Theorem 1.

Theorem 2. *Let $X(t)$, $H(t)$, G and W be as in Theorem 1. If $E(t)$ is a real symmetric $n\times n$ matrix, such that $E(t)\geq I$ is the identity matrix, then $X(t)\leq H(t)+W(t, \int_0^t G(t, s, E(s)X(s))ds)$, implies $X(t)\leq H(t)+W(t, R(t))$ on their common interval of existence in $[0, T]$ where $R(t)$ is the unique maximal solution of the matrix integral equation,*

$$R(t)=\int_0^t G(t, s, E(s)[H(s)+W(s, R(s))])ds.$$

Proof. Define

$$\Phi(t)=\int_0^t G(t, s, E(s)X(s))ds$$

and proceed as in the last theorem.

Remark. If in Theorem 1 we set $W(t, u)=u$, and $G(t, s, \omega)=G(\omega)$ for all $t, s\in[0, T]$, then we obtain Theorem 4 of [5] as a special case of our result. If $E(t)\equiv I$, then Theorem 2 reduces to Theorem 1 and so Theorem 2 is a generalization of Theorem 1.

We now derive a new useful matrix integral inequality of the Bellman-Gronwall type. In what follows we take a real symmetric $n\times n$ matrix $X(t)$ to be positive if $X(t)\in\mathcal{K}$ with $X(0)\neq 0$. We assume that the following conditions hold:

(A_3) $H(t)$ is a positive, continuous, nondecreasing real symmetric $n\times n$ matrix defined on $[0, T]$,

(A_4) $\Phi(t)$, $X(t)$, $C(t)\in\mathcal{A}$ are continuous and non-negative definite matrices on $[0, T]$.

(A_5) $Q(t)\in\mathcal{A}$ is continuous on $[0, T]$ and $Q(t)\geqslant I$ where I is the identity matrix.

Theorem 3. *Let $H(t)$, $\Phi(t)$, $X(t)$, $C(t)$ be as defined in (A_3)–(A_5) and G, W be as defined in (A_1) and (A_2). If*

$$\Phi(t)\leqslant H(t)+Q(t)\Big[\int_0^t X(s)\,\Phi(s)ds+\int_0^t X(s)Q(s)\Big(\int_0^s C(u)\Phi(u)du\Big)ds\Big]+ W\Big(t, \int_0^t G(t, s, \Phi(s))ds\Big) \tag{6}$$

and $\exp\big(-\int_0^t (X(\tau)+C(\tau))Q(\tau)d\tau\big)$ commutes with $[X(t)+C(t)]Q(t)$, then

$$\Phi(t)\leqslant A(t)\{H(t)+W(t, R(t))\} \tag{7}$$

on their common interval of existence in $[0, T]$, where $A(t)$ is the real symmetric $n\times n$ matrix defined by

$$A(t)=Q(t)\Big[I+\int_0^t X(s)Q(s)\exp\Big\{\int_0^s (X(u)+C(u))Q(u)du\Big\}ds\Big] \tag{8}$$

and $R(t)$ is the unique maximal solution of the matrix integral equation

$$R(t)=\int_0^t G(t, s, A(s)[H(s)+W(s, R(s))])ds. \tag{9}$$

Proof. Define

$$M(t)=H(t)+W\Big(t, \int_0^t G(t, s, \Phi(s))ds\Big)\ ; \tag{10}$$

then $M(t)\in\mathcal{A}$ and is positive, so that (6) becomes

$$\Phi(t)\leqslant M(t)+Q(t)\Big[\int_0^t X(s)\Phi(s)ds+\int_0^t X(s)Q(s)\Big(\int_0^s C(u)\Phi(u)ds\Big)ds\Big] \tag{11}$$

$M(t)\in\mathcal{A}$ is positive and nondecreasing so $M^{-1}(t)$ exists on $[0, T]$ and since $Q(t)\geqslant I$, post multiplying equation (11) by $M^{-1}(t)$, we have

$$\Phi(t)M^{-1}(t)\leqslant I+Q(t)\Big[\int_0^t X(s)\Phi(s)ds+\int_0^t X(s)Q(s)\Big(\int_0^s C(u)\Phi(u)du\Big)ds\Big]M^{-1}(t). \tag{12}$$

Now $0<s<t$ and $0<u<s\leqslant t\Rightarrow M(u)\leqslant M(s)\leqslant M(t)\Rightarrow M^{-1}(t)\leqslant M^{-1}(s)\leqslant M^{-1}(u)$, so from (12)

$$\Phi(t)M^{-1}(t)\leqslant I+Q(t)\Big[\int_0^t X(s)\Phi(s)M^{-1}(s)ds+ \int_0^t X(s)Q(s)\Big(\int_0^s C(u)\Phi(u)M^{-1}(u)du\Big)ds\Big]. \tag{13}$$

Define $J(t)=\Phi(t)M^{-1}(t)$, then (13) becomes

$$J(t)\leqslant Q(t)\Big[I+\int_0^t X(s)J(s)ds+\int_0^t X(s)Q(s)\Big(\int_0^s C(u)J(u)du\Big)ds\Big]. \tag{14}$$

Define another matrix $S(t)$ as follows:

$$S(t)=I+\int_0^t X(s)J(s)ds+\int_0^t X(s)Q(s)\Big(\int_0^s C(u)J(u)du\Big)ds \tag{15}$$

then $S(t)\in\mathcal{A}$, $S(0)=I$ and (14) becomes, $J(t)\leqslant Q(t)S(t)$ and differentiating (15), we have

$$\frac{dS}{dt}=X(t)J(t)+X(t)Q(t)\Big(\int_0^t C(u)J(u)du\Big)\leqslant \leqslant X(t)Q(t)\Big[S(t)+\int_0^t C(u)Q(u)S(u)du\Big]. \tag{16}$$

Let

$$P(t)=S(t)+\int_0^t C(u)Q(u)S(u)du\ ; \tag{17}$$

then, $P(0)=I$, $P(t)\in\mathcal{A}$ and (16) becomes

$$\frac{dS}{dt}\leqslant X(t)Q(t)P(t). \tag{18}$$

Now $C(u)$ and $S(u)$ are both non-negative definite matrices and $Q(u)\geqslant I$ so $P(t)-S(t)=\int_0^t C(u)Q(u)S(u)du\in\mathcal{K}$, since $P(0)-S(0)=0$. Hence $S(t)\leqslant P(t)$ and differentiating (17) and in view of (18),

$$\frac{dP}{dt}\leqslant [X(t)+C(t)]Q(t)P(t), \tag{19}$$

$$\text{i.e.}\quad \frac{dP}{dt}-[X(t)+C(t)]Q(t)P(t)\leqslant 0. \tag{20}$$

Multiply (20) by $\exp\big(-\int_0^t [X(\tau)+C(\tau)]Q(\tau)d\tau\big)$ on the left; we have

$$\frac{d}{dt}\Big\{\exp\Big(-\int_0^t [X(\tau)+C(\tau)]Q(\tau)d\tau\Big)P(t)\Big\}\leqslant 0. \tag{21}$$

Integrating (21) from 0 to t, we have,

$$\exp\Big(-\int_0^t [X(\tau)+C(\tau)]Q(\tau)d\tau\Big)P(t)-P(0)\leqslant 0. \tag{22}$$

$\exp(\int_0^t [X(\tau)+C(\tau)]Q(\tau)d\tau)$ is positive, so since (22) implies

$$P(0)-\exp(-\int_0^t [X(\tau)+C(\tau)]Q(\tau)d\tau)P(t)\leqslant \mathcal{K},$$

then,

$$\exp(\int_0^t [X(\tau)+C(\tau)]Q(\tau)d\tau).[P(0)-\exp(-\int_0^t [X(\tau)+C(\tau)]Q(\tau)d\tau)P(t)]\in\mathcal{K}.$$

Hence, $P(t)\leqslant\exp(\int_0^t [X(\tau)+C(\tau)]Q(\tau)d\tau$. Moreover, (18) implies

$$X(t)Q(t)P(t)-\frac{dS}{dt}\in\mathcal{K},\text{ and}$$

$$X(t)Q(t)\exp(\int_0^t[X(\tau)+C(\tau)]Q(\tau)d\tau)-X(t)Q(t)P(t)\in\mathcal{K}.$$

Hence,

$$\frac{dS}{dt}\leqslant X(t)Q(t)\exp(\int_0^t [X(\tau)+C(\tau)]Q(\tau)d\tau). \tag{23}$$

Integrating (23) from 0 to t, we have

$$S(t)\leqslant I+\int_0^t X(s)Q(s)\exp(\int_0^t [X(u)+C(u)]Q(u)du)ds \tag{24}$$

It can be shown easily, using (24), and $J(t)\leqslant Q(t)S(t)$, that

$$J(t)\leqslant Q(t)[I+\int_0^t X(s)Q(s)\exp(\int_0^t [X(u)+C(u)]Q(u)du)ds]=A(t)\ldots \tag{25}$$

where $A(t)$ is as defined in (8). It follows that by post multiplying (25) by matrix $M(t)$, we have

$$\Phi(t)\leqslant A(t)M(t). \tag{26}$$

By the monotonicity property of W in the last veriable, (10) and (26) imply

$$M(t)\leqslant H(t)+W(t,\int_0^t G(t,s,A(s)M(s))ds). \tag{27}$$

Applying Theorem 2 to (27), we have

$$M(t)\leqslant H(t)+W(t,R(t)) \tag{28}$$

on their common interval of existence on $[0, T]$ and $R(t)$ is the unique maximal solution of the matrix integral equation

$$R(t)=\int_0^t G(t,s,A(s)[H(s)+W(s,R(s))])ds.$$

(26) and (28) imply (7) with $A(t)$ and $R(t)$ as defined in (8) and (9).

Remark. If we use the fact that $S(t)\leqslant P(t)$ in the proof of the last theorem we can obtain the following sharper estimate in place of (7):

$$\Phi(t)\leqslant Q(t)\exp(\int_0^t [X(\tau)+C(\tau)]Q(\tau)d\tau)\{H(t)+W(t,R(t))\},$$

where $R(t)$ is as defined in (9).

Corollary I. *In Theorem 3, if $C(t)=0$, then*

$$\Phi(t)\leqslant H(t)+Q(t)\int_0^t X(s)\Phi(s)ds+W(t,\int_0^t G(t,s,\Phi(s))ds)$$

implies

$$\Phi(t)\leqslant A_1(t)\{H(t)+W(t,R(t))\}$$

on their common interval of existence in $[0, T]$ where

$$A_1(t)=Q(t)[I+\int_0^t X(s)Q(s)\exp(\int_0^t X(u)Q(u))du)ds]$$

and $R(t)$ is the unique solution of the matrix integral equation

$$R(t)=\int_0^t G(t,s,A_1(s)[H(s)+W(s,R(s))])ds.$$

§3. Some applications. In this section we derive explicit pointwise bounds on solution of the matrix integrodifferential inequality

$$\frac{d}{dt}\Phi(t)\leqslant F(t,\Phi(t))+G(t,\Phi(t),\int_0^t k(t,s,\Phi(s))ds) \tag{29}$$

satisfying the initial condition

$$\Phi(0)=H \tag{30}$$

where H is a real constant symmetric $n\times n$ matrix.

We shall assume that F and G satisfy the following conditions:

(A_6) $F: R^+\times\mathcal{A}\to\mathcal{A}$ is a continuous function, uniformly Lipschitz in the second variable, and monotone nondecreasing, in the second variable.

(A_7) $G: R\times\mathcal{A}\times\mathcal{A}\to\mathcal{A}$ is a continuous function non-negative definite and there exists a continuous non-negative definite matrix $X(t)$, such that

$$X(t)[U(t)+V(t)]-G(t,U(t),V(t))\in\mathcal{K},\text{ for } U(t),\ V(t)\in\mathcal{A}.$$

(A_8) $k: R\times R\times\mathcal{A}\to\mathcal{A}$ is a continuous function and such that there exist matrices $C(t)$, $\omega(t)$ which are non-negative definite and for which $C(s)\omega(s)-k(t,s,\omega(s))\in\mathcal{K}$.

Theorem 4. *Let F, G, satisfy (29) and (30) such that $(A_6)-(A_8)$ hold. Then*

$$\Phi(t)\leqslant A(t)\{H+R(t)\} \tag{31}$$

on their common interval of existence where

$$A(t)=I+\int_0^t X(s)\exp(\int_0^s \{X(u)+C(u)\}du)ds$$

and $R(t)$ is the unique solution of the equation

$$R(t)=\int_0^t F(s, A(s)[H+R(s)])ds. \tag{32}$$

Proof. Integrating (29) from 0 to t, and using (A_6), (A_7), (A_9),

$$\Phi(t)\leq H+\int_0^t F(s, \Phi(s))ds+\int_0^t G(s, \Phi(s), \int_0^s (s, u, \Phi(u)du)ds\leq H+$$

$$+\int_0^t F(s, \Phi(s))ds+\int_0^t X(s)\Phi(s)ds+\int_0^t X(s)(\int_0^s C(u)\Phi(u)du)ds.$$

If we apply Theorem 3 with $Q(t)=I$ and $W(t, z)=z$, we have the required result.

Corollary 2. *Assume that the conditions of Theorem 4 hold. Then*

$$\Phi(t)\leq A(t)\omega(t) \tag{33}$$

on their common interval of existence, where $\omega(t)$ is the unique solution of the non-linear matrix differential equation

$$\frac{d\omega(t)}{dt}=F(t, A(t)\omega(t)),\ \omega(0)=H \tag{34}$$

when

$$A(t)=I+\int_0^t X(s)\exp(\int_0^s (X(u)+C(u))du)ds. \tag{35}$$

Remark. Let $F(t, \Psi)=J(t)+D(\Psi)$, and $G(t, u, v)=0$ then (29) and (30) become

$$\frac{d\Phi}{dt}\leq J(t)+D(\Phi),\ \Phi(0)=H,$$

and the Corollary implies, $A(t)=I$, $\Phi(t)\leq\omega(t)$ where $\omega(t)$ is the unique solution of

$$\frac{d\omega(t)}{dt}=F(t, \omega(t))=J(t)+D(\omega)$$

which is Corollary 1 of [5].

The following can be derived from our results.

Theorem 5. *If all solutions of (34) are bounded and*

$$\int_0^\infty X(s)\exp(\int_0^s (X(u)+C(u)du)ds<\infty$$

then the solutions of (29) and (30) with F and G satisfying $(A_6)-(A_8)$ are bounded.

Theorem 6. *Let $F(t, 0)=0$ and $G(t, 0, 0)=0$ in addition to the properties $(A_6)-(A_8)$. Then the stability properties of the zero solution of (34) implies the corresponding stability properties of the zero solution of (29) and (30) provided*

$$\int_0^\infty X(s)\exp\ (\int_0^s (X(u)+C(u))du)ds<\infty.$$

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