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BOUNDARY VALUE PROBLEMS FOR NONLINEAR SECOND ORDER INTEGRODIFFERENTIAL EQUATIONS

BY

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1. **Introduction.** In a recent paper [2], J. Morchalo has studied the nonlinear integrodifferential equation of the form

$$y''(t)=g\left(t, y(t), \int_0^t h[t, s, y(s)]ds\right), \tag{1}$$

with the given boundary conditions

$$y(0)=y(b)=0, \tag{2}$$

where $h\in C[J\times J\times R, R]$, $g\in C[J\times R\times R, R]$, $J=[0, b]$ and R denotes the set of real numbers. In [2] this problem has been shown to have a unique solution when $g(t, y, u)$ and $h(t, s, y)$ have bounded partial derivatives of first order with respect to variables y and u such that $0\le g_y\le N_1$, $0\le g_u\le \le N_2$, $0\le h_y\le N_3$ and b satisfies the inequality $(1+b)b^2N<1$, where $N= =\max\{N_1, N_2, N_3\}$. Further, by assuming the existence of a unique solution of boundary value problem (BVP for short) (1)—(2) the upper and lower bounds on the solutions are also obtained. All the results in [2] are obtained by using the general idea of Chaplygin. Formulation of the existence and uniqueness results for BVP (1)—(2) or its more general form under weaker conditions on the functions motivates our present work.

Consider the nonlinear integrodifferential equation of the form

$$x''(t)=F\left(t, x(t), x'(t), \int_a^t K(t, s, x(s), x'(s))ds\right), \tag{3}$$

with the given boundary conditions

$$x(a)=A,\quad x(b)=B, \tag{4}$$

where $K\in C[I\times I\times R^n\times R^n, R^n]$ $F\in C[I\times R^n\times R^n\times R^n, R^n]$, $I=[a, b]$, R^n denotes the n dimensional Euclidean space with norm $\|\cdot\|$, A and B are real constants. Here we are interested in obtaining the existence and uniqueness of solutions of BVP (3)—(4) without any conditions on the derivatives of the functions F and K involved in (3). Using Lipschitz conditions of F and K we obtain the existence and uniqueness of solutions of BVP (3)—(4). We also establish the existence of solutions of BVP (3)—(4) when F and K are independent of x', by using the method of upper and lower solutions.

Thus here our approach to the problem is somewhat different from that in [2] which permits us to avoid the cumbersome conditions on the derivatives of the functions used in [2].

2. Existence and uniqueness of solutions. In this section we establish our results on the existence and uniqueness of solutions of BVP (3)—(4) by using the well known Banach contraction mapping principle.

Theorem 1. *Let the functions K and F satisfy the conditions*

$$\|K(t, s, x_1, y_1)-K(t, s, x_2, y_2)\| \leq P\|x_1-x_2\|+Q\|y_1-y_2\|, \tag{5}$$

$$\|F(t, x_1, y_1, z_1)-F(t, x_2, y_2, z_2)\| \leq L\|x_1-x_2\|+M\|y_1-y_2\|+N\|z_1-z_2\|, \tag{6}$$

for all $(t, s, x_i, y_i) \in I\times I\times R^n\times R^n$, $(t, x_i, y_i, z_i) \in I\times R^n\times R^n\times R^n$, $i=1, 2$, *where* P, Q, L, M, N *are nonnegative constants such that*

$$\max\left\{[1+N(b-a)]\left[L\frac{(b-a)^2}{8}+M\frac{(b-a)}{2}\right],\ [1+N(b-a)]\left[P\frac{(b-a)^2}{8}+Q\frac{(b-a)}{2}\right]\right\}<1. \tag{7}$$

Then the BVP (3)—(4) *has a unique solution.*

Proof. Let S be the Banach space of functions $u \in C^{(1)}[I, R^n]$ with the norm

$$\|u\|_1=\max\{\max_I[L\|u(t)\|+M\|u'(t)\|],\ \max_I[P\|u(t)\|+Q\|u'(t)\|]\}.$$

Define a mapping $T: S\to S$ by setting for each $x\in S$

$$Tx(t)=\Phi(t)+\int_a^b G(t, s)F(s, x(s), x'(s), \int_a^s K(s, \tau, x(\tau), x'(\tau))d\tau)ds,$$

where $G(t, s)$ is the Green's function associated with the BVP $x''=0$, $x(a)=0$, $y(b)=0$ and Φ satisfies $\Phi''=0$ and the boundary conditions (4). Then if $x, y\in S$, we have by (5) and (6)

$$\begin{aligned}\|Tx(t)-Ty(t)\| \leq \int_a^b |G(t, s)|[L\|x(s)-y(s)\|+M\|x'(s)-y'(s)\|+\\ +N\int_a^s (P\|x(\tau)-y(\tau)\|+Q\|x'(\tau)-y'(\tau)\|)d\tau]ds \leq [1+\\ +N(b-a)][(b-a)^2/8]\|x-y\|_1\end{aligned} \tag{8}$$

and similarly, we have

$$\|Tx'(t)-Ty'(t)\| \leq [1+N(b-a)][(b-a)/2]\|x-y\|_1. \tag{9}$$

From (8) and (9) it follows that

$$\begin{aligned}\|Tx-Ty\|_1 \leq \max\{[1+N(b-a)][L[(b-a)^2/8]+M(b-a)/2],\\ [1+N(b-a)][P[(b-a)^2/8]+Q(b-a)/2]\}\|x-y\|_1\end{aligned}$$

Now in view of assumption (7), we apply the Banach contraction mapping principle to conclude that T has a unique fixed point in S. The fixed point

of T is however a solution of the BVP (3)—(4). This completes the proof of the theorem.

As an immediate consequence of our Theorem 1, we establish the following result on the existence and uniqueness of the solutions of the following BVP

$$x''(t)=f(t, x(t), \int_a^t k(t, s, x(s))ds), \tag{10}$$

$$x(a)=A, \quad x(b)=B, \tag{11}$$

studied by M o r c h a l o [2] (when $A=B=0$ and $R^n=R$), where $k\in C[I\times I\times R^n, R^n]$, $f\in C[I\times R^n\times R^n, R^n]$ and A and B are real constants. Here, we are able to show that the main result in [2, Theorem 1] remains valid without assuming the major hypotheses on the derivatives of the functions f and k involved in (10).

Theorem 2. *Let the functions k and f satisfy the conditions*

$$\|k(t, s, x_1)-k(t, s, x_2)\| \leq P\|x_1-x_2\|, \tag{12}$$

$$\|f(t, x_1, y_1)-f(t, x_2, y_2)\| \leq L\|x_1-x_2\|+M\|y_1-y_2\|, \tag{13}$$

for all $(t, s, x_i) \in I\times I\times R^n$, $(t, x_i, y_i) \in I\times R^n\times R^n$, $i=1, 2$, *where* P, L, M *are nonnegative constants such that*

$$\max\{[1+N(b-a)]L[(b-a)^2/8],\ [1+N(b-a)]P[(b-a)^2/8]\}<1. \tag{14}$$

Then the BVP (10)—(11) *has a unique solution.*

Remark 1. We note that in the special case when the function F in (3) is independent of the integral term, then our Theorem 1 reduces to Theorem 1.1.1 given in [1, p. 5].

3. Existence and bounds on the solutions. In [2] the author has obtained upper and lower bounds on the solutions of BVP (1)—(2) by assuming the existence of a unique solution. In this section we apply the method of upper and lower solutions to obtain the existence of solutions of BVP (10)—(11). An important consequence of this approach is that in the course of proving the existence of a solution of the BVP (10)—(11), one obtains simultaneously bounds on the solution in terms of the solutions of the integrodifferential inequalities.

We list the following assumptions for convenience.

(H_1) There exist functions α, $\beta \in C^2(I)$ with $\alpha(t) \leq \beta(t)$ on I such that

$$\alpha''(t) \geq f(t, \alpha(t), \int_a^t k(t, s, \alpha(s))ds), \tag{15}$$ $$\alpha(a) \leq A, \quad \alpha(b) \leq B, \tag{16}$$

and

$$\beta''(t) \leq f(t, \beta(t), \int_a^t k(t, s, \beta(s))ds), \tag{17}$$ $$\beta(a) \geq A, \quad \beta(b) \geq B. \tag{18}$$

(H_2) The function $k(t, s, x)$ is nondecreasing in x for fixed t, s in I and the function $f(t, x, u)$ is nonincreasing in x and u for fixed t in I.

Remark 2. We note that the functions α and β satisfying the inequalities (15)—(16) and (17)—(18) are called the lower and upper solutions respectively of the BVP (10)—(11).

Theorem 3. *Assume that the hypothes* (H_1) *and* (H_2) *hold. Then there exists a solution* $x(t)$ *of* BVP (10)–(11) *such that* $\alpha(t) \le x(t) \le \beta(t)$, *which may be computed by the method of successive approximations.*

Proof. The proof is similar to that of Theorem 8 in [5] but for the sake of completeness we shall outline the main steps It is easy to observe that the Grren's function $G(t, s)$ is nonpositive on $I \times I$. Define the operator T by

$$Tx(t)=\Phi(t)+\int_a^b G(t,s)f\left(s, x(s), \int_a^s k(s, \tau, x(\tau))d\tau\right)ds.$$

If $x(t) \le z(t)$, then we obtain $Tx(t) \le Tz(t)$ by hypothesis (H_2). Furthermore, since

$$T\beta''(t)=f\left(t, \beta(t), \int_a^t k(t, s, \beta(s)ds\right),$$

and since $T\beta(t)$ satisfies (11), the function $T\beta(t)-\beta(t)$ is a convex function which satisfies (16) with $A=B=0$. From this we conclude that $T\beta(t) \le \beta(t)$. Similarly we show that $\alpha(t) \le T\alpha(t)$. Further we have $T\alpha(t) \le T\beta(t)$ since T is monotone operator. We now define two sequences $\{x_n\}$ and $\{z_n\}$ by setting $x_n = Tx_{n-1}$, $z_n = Tz_{n-1}$, with $x_0=\alpha$ and $z_0=\beta$. It is then easy to see that $\alpha \le x_1 \le \ldots \le x_n \le z_n \le \ldots \le z_1 \le \beta$.
Thus the sequences $\{x_n\}$ and $\{z_n\}$ are both bounded and monotone, and it is easy to verify the existence of the limits

$$\lim_{n\to\infty} x_n(t)=x(t)\le z(t)=\lim_{n\to\infty} z_n(t),$$

and further that $Tx(t)=x(t)$, $Tz(t)=z(t)$. This completes the proof of the theorem.

Remark 3. In the case when $x(t)=z(t)$, then there exists exactly one solution whose graph stays between the graphs of α and β. We also note that existence and bounds on the solutions of BVP (3)–(4) by reducing it to two systems of more general forms of first order integrodifferential equations is established by the present author in [4] by using monotone method under slightly different conditions on the functions involved therein (see, also [3]).

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ON NORMAL SEMI-INVARIANT SUBMANIFOLDS IN SASAKIAN SPACE FORMS

BY

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1. **Introduction.** The geometry of classes of submanifolds in Sasakian manifolds is one of the most interesting topic of the differential geometry. Very recently, A. Bejancu and the present author introduced and studied in [2], [3] the semi-invariant submanifolds of a Sasakian manifold.

The purpose of the present paper is to obtain results on geometry of normal semi-invariant submanifolds in Sasakian space forms. In § 2 we give definitions and fundamental formulas for semi-invariant submanifolds of Sasakian manifolds. In § 3 there are given fundamental results on normal semi-invariant submanifolds of a Sasakian manifold. The main results of the paper are given in § 4 and they are concerned with normal semi-invariant submanifolds in Sasakian space forms (see Theorems 4.1–4.5).

2. **Preliminaries.** Let $\tilde{M}$ be a $(2n+1)$ — dimensional Sasakian manifold with the Sasakian structure (φ, ξ, η, g), where φ is a tensor field of type (1.1.) on $\tilde{M}$, ξ is a vector field, η is a 1-form and g is the Riemannian metric on $\tilde{M}$. These tensor fields are related by

(2.1) $$\varphi^2 X = -X + \eta(X)\xi\ ;\ \varphi\xi = 0\ ;\ \eta(\varphi X) = 0\ ;\ \eta(\xi) = 1.$$

(2.2) $$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y) \text{ and}$$

(2.3) $$(\tilde{\nabla}_X \varphi)Y = g(X, Y)\xi - \eta(Y)X$$

for any vector fields X, Y tangent to $\tilde{M}$, where $\tilde{\nabla}$ denotes the Riemannian connection on $\tilde{M}$. From (2.3) we also have

(2.4) $$\tilde{\nabla}_X \xi = -\varphi X \text{ for any vector field } X \text{ tangent to } \tilde{M}.$$

Let M be an m-dimensional Riemannian manifold isometrically immersed in $\tilde{M}$ and suppose that the structure vector field ξ of $\tilde{M}$ is tangent to M. We denote by TM and $TM^{\perp}$ the tangent bundle to M and respectively the normal bundle to M. The submanifold M of $\tilde{M}$ is called a *semi-invariant submanifold* if it is endowed with the pair of distributions $(D, D^{\perp})$ satisfying conditions:

(i) $TM = D \oplus D^{\perp} \oplus \{\xi\}$, where $\{\xi\}$ denotes the distribution spanned by ξ on M,

(ii) the distribution D is invariant by φ, that is, $\varphi(D_x) = D_x$ for each $x \in M$,