Analele ştiinţifice ale Universităţii „Al. I. Cuza" din Iaşi
Tomul XXXII, s. I a, Matematică, 1986, f. 1

RELATIONS BETWEEN BINARY AND TERNARY OPERATIONS IN A SET

BY

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In previous notes [2], [3], I have shown how ternary operations in a set M can be generated by means of binary operations. If β is a binary operation on M, $\beta : M\times M\to M$, and τ is a ternary operation on M, $\tau : M\times\times M\times M\to M$, I have denoted for any $x, y, z\in M$ by:

(1) $$\beta[(x, y)]=(x, y)_\beta ,$$

(2) $$\tau(x, y, z)=(x, y, z)_\tau .$$

If β_1, β_2 are two binary operations on the set M, we define on M ternary operations τ^1_{ij}, τ^2_{ij} with $i, j\in\{1, 2\}$ as follows:

(3) $$(x, y, z)_{\tau^1_{ij}}=((x, y)_{\beta_i}, z)_{\beta_j}$$

(4) $$(x, y, z)_{\tau^2_{ij}}=(x, (y, z)_{\beta_i})_{\beta_j}$$

for any $x, y, z\in M$. The ternary operations τ^1_{ii}, τ^2_{ii} are thus defined by means of a unique binary operation β_i. $\tau^1_{ii}=\tau^2_{ii}$, if and only if β_i is an associative ternary operation.

In general, ternary operations τ^k_{ij}, $i, j, k\in\{1, 2\}$ are distinct. If $\tau^1_{ij}=\tau^2_{ij}$ we say that the binary operation β_1 is associated with the binary operation β_j. The relation of association between binary operations is not symmetric. If β_i is associated with β_j and β_j with β_i we say that β_i and β_j are mutually associated.

If $\tau^k_{ij}=\tau^k_{ji}$, we say that the binary operation β_i is permutable with the binary operation β_j in the ternary operation τ^k_{ij}. In general one of the casses does not imply other one. If for any $x, y, z\in M$:

(5) $$((x, y)_{\beta_i}, z)_{\beta_j}=((x, z)_{\beta_j}, (y, z)_{\beta_j})_{\beta_i},$$

we say that β_j is left distributive with respect to β_i and if:

(6) $$(x, (y, z)_{\beta_i})_{\beta_j}=((x, y)_{\beta_j}, (x, z)_{\beta_j})_{\beta_i},$$

we say that β is right distributive with respect to β_i. If β_j is left (right) distributive with respect to β_i it does not follow that generally β_i is left (right) distributive with respect to β_j. If this condition is verified, we say that the operations β_i and β_j are mutually left (right) distributive. If the set M has n elements, then n^{n^2} binary operations and n^{n^3} ternary operations can be defined on M. The number of the ternary operations of τ_{ij}^k-type is $2\,n^{2n^2}$ at most. So, for $n \geq 3$, there are ternary operations which are not of τ_{ij}^k-type.

In [2] the following theorem has been proved:

Theorem 1. *Let M be a set endowed with two distinct binary operations β_1 and β_2 which verify the conditions:*

i) β_1 and β_2 are associative and mutually associated;

ii) β_1 and β_2 are permutable in the ternary operations τ_{12}^1 and τ_{12}^2.

Then the operations τ_{ij}^k, satisfy the ternary associativity condition:

$$(x, y, z)_{\tau_{ij}^k}, u, v)_{\tau_{ij}^k} = (x, (y, z, u)_{\tau_{ij}^k}, v)_{\tau_{ij}^k} = (x, y, (z, u, v)_{\tau_{ij}^k})_{\tau_{ij}^k}. \tag{7}$$

In this note we shall consider some methods to define binary operations starting from a certain ternary operation on the set M. Let the operation τ be given on the set M and let $a \in M$. We can define on M the binary operation α by:

$$(x, y)_\alpha = (a, x, y)_\tau, \tag{8}$$

The operation α is associative if:

$$(a, (a, x, y)_\tau, z)_\tau = (a, x, (a, y, z)_\tau)_\tau, \tag{9}$$

and has a identity element if:

$$(\exists) c \in M \text{ so that } (\forall) x \in M, (a, c, x)_\tau = (a, x, c)_\tau = x, \tag{10}$$

Theorem 2. *If τ is a ternary operation on M, $a \in M$ and if τ verifies conditions (9), (10) and:*

$$(\forall) x \in M, (\exists) x' \in M \text{ so that } (a, x', x) = c, \tag{11}$$

then the binary operation α defined on M by (8) endows M with a group structure. If in addition:

$$(\forall) x, y \in M, \ (a, x, y)_\tau = (a, y, x)_\tau, \tag{12}$$

then the resulting group is commutative.

Remark. If τ is a ternary and associative operation on M and if α is a group, then if e is the identity element for α, $(x, y, z)_\tau = ((a, e, x)_\tau, y, z)_\tau = (a, (e, x, y)_\tau, z)_\tau$, and so: $(y, z)_\tau = ((e, a, y)_\tau, z)_\tau$.

It results that $y = (e, a, y)_\tau$, $(\forall) y \in M$ which shows that the couple (e, a) is a unitary couple in the left ternary operation.

Given the ternary operation τ we can form two binary operation of the previous type:

$$(x, y)_\alpha = (a, x, y)_\tau \quad \text{and} \quad (x, y)_\gamma = x, y, c)_\tau, \tag{13}$$

where $a, c \in M$ are fixed.

A new ternary operation τ' on M can be obtained by:

$$(x, y, z)_\tau = ((x, y)_\alpha, z)_\gamma = ((a, x, y)_\tau, z, c)_{\tau'}. \tag{14}$$

If M is a group with respect to α, it is a semigroup with respect to γ and if γ is distributive with respect to α then τ determines a ring structure on M.

Theorem 3. *If M is endowed with two binary operations β_1, β_2 with the properties:*

i) $\beta_i (i = 1, 2)$ have left identity elements.

ii) β_1, β_2 are mutually left distributive, then every element of M is idempotent with respect to β_i $(i = 1, 2)$.

Proof: Let us denote by e_i the left identity element for the β_i $(i = 1, 2)$. So $(e_i, x)_{\beta_i} = x$, $(\forall) x \in M$ $(i = 1, 2)$ and:

$$((e_1, e_1)_{\beta_2}, e_2)_{\beta_1} = ((e_1, e_2)_{\beta_1}, (e_1, e_2)_{\beta_1})_{\beta_2} = (e_2, e_2)_{\beta_1} = e_2. \tag{15}$$

Then:

$$x = (e_2, x)_{\beta_2} = (((e_1, e_1)_{\beta_2}, e_2)_{\beta_1}, x)_{\beta_2} = (((e_1, e_1)_{\beta_2}, x)_{\beta_2}, (e_2, x)_{\beta_2})_{\beta_1} =$$
$$= (((e_1, e_1)_{\beta_2}, x)_{\beta_2} \ x)_{\beta_1}.$$

For $x = e_2$ it results:

$$e_2 = (((e_1, e_1)_{\beta_2}, e_2)_{\beta_2}, e_2)_{\beta_1} = (((e_1, e_1)_{\beta_2}, e_2)_{\beta_1}, (e_2, e_2)_{\beta_1})_{\beta_2} =$$
$$= (((e_1, e_2)_{\beta_1}, (e_1, e_2)_{\beta_1})_{\beta_2}, (e_2, e_2)_{\beta_1})_{\beta_2} = ((e_2, e_2)_{\beta_1}, (e_2, e_2)_{\beta_1})_{\beta_2} = (e_2, e_2)_{\beta_1}.$$

So $(e_2, e_2)_{\beta_1} = e_2$. Then:

$$x = (e_2, x)_{\beta_2} = ((e_2, e_2)_{\beta_1}, x)_{\beta_2} = ((e_2, x)_{\beta_2}, (e_2, x)_{\beta_2})_{\beta_1} = (x, x)_{\beta_1}.$$

It follows that every element of M is idempotent with respect to the operation β_1. In the same manner we can prove that: $x = (x, x)_{\beta_2}$.

Remarks. i) If in the previous theorem the left sense is replaced with the right sense we also see that the elements of M are idempotent with respect to β_i, $i = 1, 2$.

ii) If $\beta_i (i = 1, 2)$ are Boole algebras they satisfy the conditions from theorem 3.

iii) Let be $M = \{0, 1\}$ and the binary operations

(16)

β_1	0	1
0	0	0
1	0	1

β_2	0	1
0	0	1
1	0	1

β_3	0	1
0	0	1
1	1	1

The set M, structured with the binary operations β_1, β_2 or β_2, β_3, satisfies conditions from theorem 3 and is not a Boole algebra.

iv) Let β_1, β_2 be two binary operations on the set M which verifies conditions from theorem 3 and let the ternary operations τ_{ij}^k, $i, j, k \in \{1, 2\}$ be defined by β_1, β_2 (after (3) and (4)). Then:

$$(x, y, z)_{\tau_{12}^1} = ((x, z)_{\beta_2}, (y, z)_{\beta_2})_{\beta_1}; \quad (x, y, z)_{\tau_{21}^1} = ((x, z)_{\beta_1}, (y, z)_{\beta_1})_{\beta_2} \tag{17}$$

$$(x, y, z)_{\tau_{12}^2} = ((x, y)_{\beta_2}, (x, z)_{\beta_2})_{\beta_1}; \quad (x, y, z)_{\tau_{21}^2} = ((x, y)_{\beta_1}, (x, z)_{\beta_1})_{\beta_2}, \tag{18}$$

$$(x, x, y)_{\tau_{12}^1} = (x, y)_{\beta_2}; \ (x, x, y)_{\tau_{21}^1} = (x, y)_{\beta_1}, \tag{19}$$

$$(x, y, y)_{\tau_{12}^1} = (x, y)_{\beta_2}, \ (x, y, y)_{\tau_{21}^2} = (x, y)_{\beta_1}, \tag{20}$$

$$(x, y, y)_{\tau_{12}^1} = (x, y, y)_{\tau_{21}^1}. \tag{21}$$

So a) e_1 and e_2 are left identity elements for the ternary operation τ^1_{12}: $(e_i, x, x)_{\tau^1_1} = x$; $(\forall)x \in M$.

b) The couple (e_1, e_2) is a left unitary couple for the operation τ^1_{12}; i.e., $(e_1, e_2, x)_{\tau^1_{12}} = x$, $(\forall)x \in M$.

c) τ^1_{12} is a ternary idempotent operation i.e., $(x, x, x)_{\tau^1_{12}} = x$, $(\forall)x \in M$.

d) e_2 is a left biunity for τ^1_{12}, so: $(e_2, e_2, x)_{\tau^1_{12}} = x$, $(\forall)x \in M$.

e) e_1 is a middle unity for τ^1_{21} so: $(x, e_1, x)_{\tau^1_{21}} = x$, $(\forall)x \in M$.

For the ternary product τ^1_{21}:

a) e_1, e_2 are left identity elements
b) (e_2, e_1) is a unitary couple (right);
c) τ^1_{21} is idempotent ternary operation;
d) e_1 is a left biunity
e) e_1 is a middle unity.

If in the ternary product τ^1_{12}, we have left distributivity denoting by f_1, respectively f_2, the right identity elements in the binary operations β_1, respectively β_2, we have:

a) f_1, f_2 are right unities $(x, x, f_i)_{\tau^2_{12}} = x$, $(\forall)x \in M$, $(i = 1, 2)$;
b) (f_2, f_1) is a unitary couple for τ^2_{12} (to the right): $(x, f_2, f_1)\ \tau^2_{12} = x$;
c) τ^2_{12} is ternary idempotent operation: $(x, x, x)_{\tau^2_{12}} = x$, $(\forall)x \in M$;
d) f_2 is right biunity in τ^1_{12}: $(x, f_2, f_2)\tau^2_{21} = x$;
e) f_2 is middle unity in τ^2_{12}: $(x, f_2, x)\tau^2_{12} = x$.

For τ^2_{21} we get;

a) f_1, f_2 are right unities in τ^2_{21};
b) (f_1, f_2) is unitary couple in τ^2_{21};
c) τ^2_{21} is ternary idempotent operation;
d) f_1 is right biunity in τ^2_{21};
e) f_1 is middle unity in τ^2_{31}.

In (13) I have defined two binary operations α, γ obtained from the ternary operation τ; we consider the ternary operation τ' related to τ by:

$$(a, x, y)_\tau = (x, y)_\alpha,\ (x, y, c)_\tau = (x, y)_\gamma,\quad (x, y, z)_{\tau'} = ((x, y)_\alpha, z)_\gamma \tag{14'}$$
$$(a, c \in M) \text{ so}: (x, y, z)_{\tau'} = ((a, x, y)_\tau, z, c)_\tau.$$

If we denote $\alpha = \beta_1$ and $\gamma = \beta_2$, it results that τ' is an operation of τ^1_{12}-type: $((x, y)_{\beta_1}, z)_{\beta_2} = ((x, y)_\alpha, z)_\gamma = (x, y, z)_{\tau'}$.

Conditions in order that α, γ be mutually left distributive are expressed by: $((x, y)_\alpha, z)_\gamma = ((x, z)_\gamma, (y, z)_\gamma)_\alpha$ that is:

$$((a, x, y)_\tau, z, c)_\tau = (a, (x, z, c)_\tau, (y, z, c)_\tau)_\tau, \tag{22}$$

and $((x, y)_\gamma, z)_\alpha = ((x, z)_\alpha, (y, z)_\alpha)_\gamma$ that is:

$$(a, (x, y, c)_\tau, z)_\tau = ((a, x, z)_\tau, (a, y, z)_\tau, c)_\tau \tag{23}$$

Conditions that the binary operations α, γ have left identity elements are expressed by:

$$(\forall)x \in M,\ (\exists)e_1 \in M \text{ so that } (a, e_1, x)_\tau = x \tag{24}$$

and

$$(\exists)e_2 \in M \text{ so that } (e_2, x, c)_\tau = x. \tag{25}$$

If (22), (23), (24), (25) are fulfilled we get:

$$(x, x)_\alpha = (a, x, x)_\tau = x,\qquad (x, x)_\gamma = (x, x, c)_\tau = x, \tag{26}$$

according to theorem 3; this shows that τ has an element a as left ternary unity and c as right ternary unity element.

Theorem 4. *If (22)—(25) are satisfied and in addition both α and γ are commutative* i.e.:

$$(a, x, y)_\tau = (a, y, x)_\tau,\ (x, y, c)_\tau = (y, x, c)_\tau, \tag{27}$$

and also associative i.e.;

$$(a, (a, x, y)_\tau, z)_\tau = (a, x, (a, y, z)_\tau)_\tau, \tag{28}$$
$$((x, y, c)_\tau, z, c)_\tau = (x, (y, z, c)_\tau, c)_\tau,$$

and, there exists a bijection $i: M \to M$ defined by $i(x) = \overline{x} (x, \overline{x} \in M)$ so that

$$(a, x, \overline{x})_\tau = e_1 \text{ and } (x, \overline{x}, c)_\tau = e_2, \tag{29}$$

the set M becomes a Boole algebra with respect to the operations α, γ.

Remark 1. Given the ternary operation τ in M, the ternary operations: $(a, x, y)_\tau$, $(x, b, y)_\tau$, $(x, y, c)_\tau$ can be considered pair-operations, where a, b, c are parameters that cover independently the set M, but are fixed for every binary operation.

Example: Let a and $a' \in M$ be fixed; we consider

$$(a, x, y)_\tau = (x, y)_\alpha;\ (a', x, y)_\tau = (x, y)_{\alpha'}; \tag{30}$$

and

$$(x, y, z)_{\tau^1_{12}} = ((x, y)_\alpha, z)_{\alpha'} = (a', (a, x, y)_\tau, z)_\tau. \tag{31}$$

Conditions that α and α' be mutually distributive are, in this case:

$$(a', (a, x, y)_\tau, z)_\tau = (a, (a', x, z)_\tau, (a', y, z)_\tau)_\tau \tag{32}$$
$$(a, (a', x, y)_\tau, z)_\tau = (a', (a, x, z)_\tau, (a, y, z)_\tau)_\tau.$$

Conditions for the existence of the identity elements for the binary operations α and α' are: $(a, e_1, x) = x$ and $(a', e_2, x) = x$ $(\forall)x \in M$. Conditions (26) become: $(a, x, x)_\tau = x$, $(a', x, x)_\tau = x$.

Then the conditions of commutativity, associativity and the existence of the complementary element are.

$$(a, x, y)_\tau = (a, y, x)_\tau;\ (a', x, y)_\tau = (a', y, x)_\tau \tag{33}$$

$$(a, (a, x, y)_\tau, z)_\tau = (a, x, (a, y, z)_\tau)_\tau;\ (a', (a', x, y)_\tau, z)_\tau = (a', x, (a', y, z)_\tau)_\tau, \tag{34}$$

$$(a, x, \overline{x})_\tau = e_1,\quad (a', x, \overline{x})_\tau = e_2, \tag{35}$$

for any corresponding pair $x, \overline{x}, \in M$.

Remark 2. It is known that in every distributive lattice $\mathscr{L}$ with elements 0,1 so that $(\forall)x\in\mathscr{L}$ we have $0\vee x=x$ and $1\wedge x=x$, a ternary operation can be defined by :

$$(x,y,z)_\tau=(x\wedge y)\vee(y\wedge z)\vee(z\wedge x). \tag{36}$$

Puting $x=0$ and replacing (y,z) by (x,y), and puting $x=1$ and replacing (y,z) with (x,y) we get :

$$(0,x,y)_\tau=x\wedge y\,;\;(1,x,y)_\tau=x\vee y. \tag{37}$$

Conversely, in a ternary structure M, with τ commutative operation with elements (0, 1) forming a unitary couple for the ternary product and with the property that every element of M is a biunity for τ and in which the ternary product satisfies the ternary associativity :

$$((x,y,z)_\tau,u,v)_\tau=((x,u,v)_\tau,y,(z,u,v)_\tau)_\tau, \tag{38}$$

by fixing in turn two of the elements of the ternary product $(x,y,z)_\tau$ with elements 0 and 1 (for instance) :

$$(x,0,y)_\tau=x\wedge y\,;\;(x,1,y)_\tau=x\vee y, \tag{39}$$

the ternary structure of M becomes a distributive lattice with elements 0,1. [1], [4], [5].

A ternary operation τ on a set M can generate binary operations β by various methods.

The method of fixing an element in each tern of τ has already been used.

Another method is the equalization of two of the three factors in each triple of τ. So three binary operations are obtained :

$$(x,x,y)_\tau=(x,y)_{\beta_1}\,;\quad(x,y,y)_\tau=(x,y)_{\beta_2}\,;\quad(x,y,x)_\tau=(x,y)_{\beta_3} \tag{40}$$

By pairs of the binary operations $\beta_i(i=1,2,3)$ we can get six ternary operations, generally different one from another and different from the initial ternary operation τ :

$$\begin{aligned}
&(x,y,z)_{\tau^1_{12}}=((x,y)_{\beta_1},z)_{\beta_2}=((x,x,y)_\tau,z,z)_\tau,\\
&(x,y,z)_{\tau^1_{21}}=((x,y)_{\beta_2},z)_{\beta_1}=((x,y,y)_\tau,(x,y,y)_\tau,z)_\tau,\\
&(x,y,z)_{\tau^1_{23}}=((x,y)_{\beta_2},z)_{\beta_3}=((x,y,y)_\tau,z,(x,y,y)_\tau)_\tau,\\
&(x,y,z)_{\tau^1_{32}}=((x,y)_{\beta_3},z)_{\beta_2}=((x,y,x)_\tau,z,z)_\tau,\\
&(x,y,z)_{\tau^1_{31}}=((x,y)_{\beta_3},z)_{\beta_1}=((x,y,x)_\tau,(x,y,x)_\tau,z)_\tau,\\
&(x,y,z)_{\tau^1_{13}}=((x,y)_{\beta_1},z)_{\beta_3}=((x,x,y)_\tau,z,(x,x,y)_\tau)_\tau,
\end{aligned} \tag{41}$$

and six more of τ^2_{ij}-type.

Remark 1. If the ternary operation τ on the set M is ternary two-sided associative, then τ^1_{12} from (41) written as joint operation by β_1, β_2 defined in (40) is equal to τ^2_{21} (the operations β_1 and β_2 are associated in this order).

Indeed, by definition, τ is two-sided associative when :

$$((x,y,z)_\tau,u,v)_\tau=(x,y,(z,u,v)_\tau)_\tau,\;(\forall)x,y,z,u,v\in M.$$

Considering $x=y$, $u=v$ and replacing z by y and u by z, we get :

$$((x,x,y)_\tau,z,z)_\tau=(x,x,(y,z,z)_\tau)_\tau,\;(\forall)x,y,z\in M,$$

which means :

$$((x,y)_{\beta_1},z)_{\beta_2}=(x,(y,z)_{\beta_2})_{\beta_1}.$$

Remark 2. If τ is left associative, i. e. :

$$((x,y,z)_\tau,u,v)_\tau=(x,(y,z,u)_\tau,v)_\tau,$$

considering $y=z$, $x=u=v$ we get :

$$((x,y)_{\beta_2},x)_{\beta_2}=(x,(y,x)_{\beta_1})_{\beta_1}.$$

Also, if τ satisfies the ternary associativity :

$$(x,(y,z,u)_\tau,v)_\tau=(x,y,(z,u,v)_\tau)_\tau,$$

then we get :

$$(x,(x,y)_{\beta_2})_{\beta_1}=(x,(y,x)_{\beta_1})_{\beta_1}.$$

Remark 3. According to what has been analysed in the previous paragraphs we conclude that :

Considering the ternary operation τ on the set M from which the binary operations $(x,x,y)_\tau=(x,y)_{\beta_1}$ and $(x,y,y)_\tau=(x,y)_{\beta_2}$ are detached the conditions that have to be imposed to τ in order that M has a Boole algebra structure with respect to the binary operations β_1 and β_2, are :

(I) $(\forall)x\in M,\,(\exists)e_1,e_2\in M$ such that $(e_1,e_1,x)_\tau=x,\,(e_2,x,x)_\tau=x,$

(II) $(\forall)x,y,z\in M,\;((x,x,y)_\tau,z,z)_\tau=((x,z,z)_\tau,(x,z,z)_\tau,(y,z,z)_\tau)_\tau,$

(III) $(\forall)x,y;z\in M;\,((x,y,y)_\tau,(x,y,y)_\tau,z)_\tau=$
$=((x,x,z)_\tau,(y,y,z)_\tau,(y,z,z)_\tau)_\tau$

(IV) $(\forall)x,y\in M,\,(x,x,y)_\tau=(y,y,x)_\tau;\;(x,y,y)_\tau=(y,x,x)_\tau,$

(V) $(\forall)x,y,z\in M,\,((x,x,y)_\tau,(x,x,y)_\tau,z)_\tau=(x,x,(y,y,z)_\tau)_\tau,$

VI) $(\forall)x,y,z\in M,\,((x,y,y)_\tau,z,z)_\tau=(x,(y,y,z)_\tau,(y,y,z)_\tau)_\tau,$

(VII) $(\forall)x\in M,\;(x,x,e_2)_\tau=e_2\,;\;(x,e_1,e_1)_\tau=e_1,$

(VIII) $(\forall)x\in M,\;(x,x,\overline{x})_\tau=e_2,\;(x,\overline{x},\overline{x})_\tau=e_1.$

Also, in order that the operations β_1 and β_3, defined in (40), give a Boole algebra structure to M, it is required :

(I') $(\forall)x\in M,\,(\exists)e_1,e_3\in M,\,(e_1,e_1,x)_\tau=x,\,(e_3,x,e_3)_\tau=x,$

(II') $(\forall)x,y,z\in M,\,((x,x,y)_\tau,z,(x,x,y)_\tau)=$
$=((x,z,x)_\tau,(x,z,x)_\tau,(y,z,y)_\tau)_\tau.$

$$\text{(III}')\quad (\forall)x, y, z \in M,\ ((x, y, x)_\tau,\ (x, y, x)_\tau, z)_\tau = \\ = ((x, x, z)_\tau,\ (y, y, z)_\tau,\ (x, x, z)_\tau)_\tau,$$

$$\text{(IV}')\quad (\forall)x, y \in M,\ (x, x, y)_\tau = (y, y, x)_\tau;\ (x, y, x)_\tau = (y, x, y)_\tau,$$

$$\text{(V}')\quad (\forall)x, y, z, \in M,\ ((x, x, y)_\tau, (x, x, y)_\tau, z)_\tau = (x, x, (y, y, z)_\tau)_\tau,$$

$$\text{(VI}')\quad (\forall)x, y, z \in M,\ ((x, y, z)_\tau, z, (x, y, x)_\tau)_\tau = (x, (y, z, y)_\tau, x)_\tau,$$

$$\text{(VII}')\quad (\forall)x \in M,\ (x, x, e_3)_\tau = e_3;\ (x, e_1, e_1)_\tau = e_1,$$

$$\text{(VIII}')\quad (\forall)x \in M,\ (\exists)\bar{x} \in M,\ (x, x, \bar{x})_\tau = e_3,\ (x, \bar{x}, x)_\tau = e_1.$$

Conditions that the operations β_2, β_3 lead to a Boole algebra in M are obtained in the same way.

Remark 4. Considering the ternary operation τ which fulfils the associativity condition :

$$(\forall)x, y, z, u, v \in M,\ ((x, y, z), u, v)_\tau = ((x, z, v)_\tau, y, u)_\tau,$$

for $y=z$, $u=v$ and by applying again the associativity law we get :

$$((x, y, y)_\tau, u, u)_\tau = ((x, u, u)_\tau, y, y)_\tau,$$

or

$$((x, y)_{\beta_2}, u)_{\beta_2} = ((x, u)_{\beta_2}, y)_{\beta_2},\ (\forall)x, y, u \in M.$$

If β_2 has a left identity element e_2, then β_2 is a binary commutative operation, for, considering $x=e_2$, we get :

$$(\forall)y, u \in M,\ (y, u)_{\beta_2} = (u, y)_{\beta_2}.$$

Provided that β_1 too admits a trivial left element e_1 so that : $(e_1, x)_{\beta_1} = (e_1, e_1, x)_\tau = x$, $(\forall)x \in M$ and moreover

$$((x, x, y)_\tau, z, z)_\tau = ((x, z, z)_\tau, (x, z, z)_\tau, (y, z, z)_\tau)$$

$$((x, y, y)_\tau,\ (x, y, y)_\tau, z)_\tau = ((x, x, z)_\tau, (y, z, z)_\tau, (y, y, z)_\tau)$$

then :

$$(x, x)_{\beta_1} = (x, x, x)_\tau = x = (x, x)_{\beta_2}.$$

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Tomul XXXII, s. I a, Matematică, 1986, f. 1

A NOTE ON ADMISSIBLE NULL CONTROLLABILITY AND ON VARIATIONAL PERTURBATIONS OF THE MINIMUM TIME PROBLEM

BY

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1. Introduction. We consider in this paper the abstract control system

$$x'(t) = Ax(t) + Bu(t),\ t \geqslant 0, \tag{1.1}$$

where the state $x(t)$, $t \geqslant 0$, takes values in a reflexive Banach space X and the control $u(t)$, $t \geqslant 0$, is in another reflexive Banach space U. Assume the operator A generates a strongly continuous semigroup $S(t)$ on X for $t \geqslant 0$, and B is a bounded linear operator from U into X.

For $x(0) = x_0 \in X$ and $u \in L^1_{loc}(R^+; U)$, the solution of the equation (1.1) is understood in the mild sense, that is,

$$x(t) = S(t)\, x_0 + \int_0^t S(t-s)\ Bu(s)\, ds,$$

where the integral is in the sense of Bochner.

We take the set of admissible controls as

$$U_{ad} = \{u \in L^\infty_{loc}(R^+;\ U)\ ;\ \| u(t) \| \leqslant 1 \text{ a.e.}\}. \tag{1.2}$$

In [10], the author gives a sufficient condition of admissible null controllability for (1.1). More precisely, it is proved there that if $S(t)$ is a contraction semigroup and if the system (1.1) is null controllable on $[0, T]$ for some $T > 0$, then it is admissible null controllable in finite time. Here we prove the above result with a weaker condition on $S(t)$ (Corollary 1). The approach is different and allows to obtain additional informations about the transferable sets (Theorem 2). Unfortunately, we are forced to impose the reflexivity to the spaces X and U while in [10] they are general Banach spaces. This is done in Section 2.

In Section 3 we are concerned with variational perturbations of the problem of reaching the origin from an initial point in minimum time. This problem, which is very important from the numerical point of view has been considered, by a number of authors (see, for example, [6], [11] when dim X is finite). In the infinite dimensional case, S a s a i and