$$(4.27) \qquad \nabla_X Z \in \Gamma(\tilde{D} \oplus \{\xi\}), \qquad \forall X \in \Gamma(D \oplus D^{\perp}).\ Z \in \Gamma(\tilde{D}).$$

By using the remark 3.6 in the lemma 3.3, we get either $D \oplus D^{\perp} = \{0\}$, or $\tilde{D} = \{0\}$. Thus the proof is complete.

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Faculty of Mathematics
University of Iaşi
Iaşi 6600, R.S. Romania

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SOME THEOREMS ON SEMI-INVARIANT SUBMANIFOLDS OF A SASAKIAN MANIFOLD

BY

NECULAI PAPAGHIUC

Let $\tilde{M}$ be a $(2n+1)$-dimensional Sasakian manifold with the Sasakian structure (φ, ξ, η, g), where φ is a tensor field of type (1, 1) on $\tilde{M}$, ξ is a vector field, η is a 1-form and g is the Riemannian metric on $\tilde{M}$, these tensor fields being related by

$$(1) \qquad \varphi^2 X = -X + \eta(X)\,\xi\,;\ \varphi\xi = 0\,;\ \eta(\varphi X) = 0\,;\ \eta(\xi) = 1,$$

$$(2) \qquad g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\,\eta(Y) \qquad \text{and}$$

$$(3) \qquad (\tilde{\nabla}_X \varphi)\,Y = g(X, Y)\,\xi - \eta(Y)\,X$$

for any vector fields X, Y tangent to $\tilde{M}$, where $\tilde{\nabla}$ denotes the Riemannian connection on $\tilde{M}$. From (3) we also have

$$(4) \qquad \tilde{\nabla}_X \xi = -\varphi X \text{ for any vector field } X \text{ tangent to } \tilde{M}.$$

Let M be an m-dimensional Riemannian manifold isometrically immersed in $\tilde{M}$ and suppose that the structure vector field ξ of $\tilde{M}$ is tangent to M. We denote by TM and $TM^{\perp}$ the tangent bundle to M and respectively the normal bundle to M. Then, the submanifold M of $\tilde{M}$ is called a *semi-invariant submanifold* if it is endowed with a pair of distributions $(D, D^{\perp})$ satisfying the conditions:

(i) $TM = D \oplus D^{\perp} \oplus \{\xi\}$, where $\{\xi\}$ denotes the distribution spanned by ξ,
(ii) the distribution D is invariant by φ, that is, $\varphi(D_x) = D_x$, for each $x \in M$.
(iii) the distribution $D^{\perp}$ is anti-invariant by φ, that is, $\varphi(D^{\perp}_x) \subset T_x M^{\perp}$, for each $x \in M$.

Remark. From the definition, it follows that the distributions D, $D^{\perp}$ and $\{\xi\}$ are mutually orthogonal to each other.

The semi-invariant submanifolds of a Sasakian manifold have been introduced and studied by A. Bejancu and the present author in [2], [3], [6]. They are corresponding to CR-submanifolds of Kaehlerian manifolds introduced by A. Bejancu in [1] and studied then by many authors. In this note, we obtain some results concerned with the maximal invariant subspaces D_x of $T_x M$ and respectively with the maximal anti-invariant subspaces $D^{\perp}_x$ of $T_x M$, where M is a submanifold of a Sasakian manifold $\tilde{M}$

and to obtain some characterizations in order that a submanifold M of $\tilde{M}$ be a semi-invariant submanifold in the particular case when $\tilde{M}$ is a Sasakian space form of constant φ-sectional curvature $\tilde{M}(c)$. The corresponding results for a CR-submanifold of a Kaehler manifold have been obtained by D. E. Blair and B. Y. Chen in [4] and Chen-Jung Hsu in [5].

In the sequel, if M is a submanifold of a Sasakian manifold $\tilde{M}$ such that the structure vector field ξ of $\tilde{M}$ is tangent to M, we denote by H the orthogonal complementary distribution of $\{\xi\}$ in M, i.e. we have $T_xM = H_x \oplus \xi_x$ for each $x \in M$.

Proposition 1. *Let M be a submanifold tangent to ξ of a Sasakian manifold $\tilde{M}$ and D_x the maximal invariant subspace of H_x. Suppose* $\dim D_x =$ *constant. Then the invariant distribution D is never integrable.*

Proof. By using (2) and (3) we obtain

$$g([X, Y], \xi) = g(\tilde{\nabla}_X Y - \tilde{\nabla}_Y X, \xi) = g(X, \tilde{\nabla}_Y \xi) - g(Y, \tilde{\nabla}_X \xi) = 2g(\varphi X, Y) \tag{5}$$

for any $X, Y \in \Gamma(D)$, where $\Gamma(S)$ denotes the module of all differentiable sections of a vector bundle S on M. Then, if we take a unit vector field $X \in \Gamma(D)$ and $Y = \varphi X$, from (5) we get $g([X, Y], \xi) = 2$, i.e. the invariant distribution D is never integrable.

Theorem 1. *Let M be a submanifold tangent to ξ isometrically immersed in a Sasakian manifold $\tilde{M}$ and D_x the maximal invariant subspace of H_x. Suppose* $\dim D_x =$ *constant. Then the distribution $D \oplus \{\xi\}$ is integrable if and only if the second fundamental form h of M satisfies $h(X, \varphi Y) = h(\varphi X, Y)$ for any $X, Y \in \Gamma(D)$.*

Proof. Let $X, Y \in \Gamma(D)$. Then, by using (3) and the Gauss and Weingarten formulas, we obtain

$$h(X, \varphi Y) = fh(X, Y) + \varphi Q \nabla_X Y \tag{6}$$

where $fh(X, Y)$ denotes the normal component of $\varphi h(X, Y)$, ∇ (resp. $\nabla^{\perp}$) denotes the Riemannian connection on M (resp. the linear connection induced by $\tilde{\nabla}$ on $TM^{\perp}$) and $Q\nabla_X Y$ denotes the component of $\nabla_X Y$ from the orthogonal complementary distribution of $D \oplus \{\xi\}$ in M. Since h is a symmetrical morphism of vector bundles, from (6) it follows that $h(X, \varphi Y) - h(\varphi X, Y) = \varphi Q([X, Y])$. In this way, we get that $[X, Y] \in \Gamma(D \oplus \{\xi\})$ if and only if $\varphi Q([X, Y]) = 0$, i.e. $h(X, \varphi Y) = h(\varphi X, Y)$. On the other hand, by using (2), (3), (4) and the Gauss and Weingarten formulas, we obtain $[X, \xi] \in \Gamma(D \oplus \{\xi\})$ for each vector field $X \in \Gamma(D)$. Thus, the proof of Theorem 1 is complete.

Theorem 2. *Let M be a submanifold tangent to ξ of a Sasakian space form $\tilde{M}(c)$ with $c \neq 1$. Then M is a semi-invariant submanifold if and only if the maximal invariant subspaces $D_x = H_x \cap \varphi(H_x)$, $x \in M$ define a nontrivial differentiable distribution D on M such that*

$$\tilde{R}(D, D; D^{\perp}, D^{\perp}) = 0, \tag{7}$$

where $D^{\perp}$ denotes the orthogonal complementary distribution to $D \oplus \{\xi\}$ in M and $\tilde{R}$ denotes the curvature tensor of the Sasakian space form $\tilde{M}(c)$.

Proof. It is well known that the curvature tensor of a Sasakian space form $\tilde{M}(c)$ is given by

$$\begin{aligned} \tilde{R}(X, Y)Z = {} & \frac{c+3}{4}\{g(Y, Z)X - g(X, Z)Y\} + \frac{c-1}{4}\{\eta(X)\eta(Z)Y - \\ & - \eta(Y)\eta(Z)X + g(X, Z)\eta(Y)\xi - g(Y, Z)\eta(X)\xi + g(Z, \varphi Y)\varphi X - \\ & - g(Z, \varphi X)\varphi Y + 2g(X, \varphi Y)\varphi Z\} \end{aligned} \tag{8}$$

for any vector fields X, Y, Z tangent to $\tilde{M}(c)$. Then, if M is a semi-invariant submanifold of $\tilde{M}(c)$, by direct computation, from (8) we obtain $\tilde{R}(X, Y; Z, W) = g(\tilde{R}(X, Y)Z, W) = 0$ for any $X, Y \in \Gamma(D)$ and $Z, W \in \Gamma(D^{\perp})$, i.e. (7) holds.

Conversely, suppose the maximal invariant subspaces D_x of H_x define a nontrivial differentiable distribution such that (7) holds. Then (7) and (8) imply $\tilde{R}(X, \varphi X; Z, W) = -[(c-1)/2]\, g(X, X)\, g(\varphi Z, W) = 0$ for all $X \in \Gamma(D)$ and $Z, W \in \Gamma(D^{\perp})$, from which because $c \neq 1$, we get $g(\varphi Z, W) = 0$ for any $Z, W \in \Gamma(D^{\perp})$, i.e., we have

$$\varphi D^{\perp} \perp D^{\perp}. \tag{9}$$

Since D is an invariant distribution, by using (2) we have $g(X, \varphi Z) = -g(\varphi X, Z) = 0$ for any $X \in \Gamma(D)$ and $Z \in \Gamma(D^{\perp})$, i.e. we get

$$\varphi D^{\perp} \perp D. \tag{10}$$

Finally, $g(\xi, \varphi Z) = 0$ for any $Z \in \Gamma(D^{\perp})$ and taking account of (9) and (10) we obtain $\varphi D^{\perp} \subset TM^{\perp}$, i.e., $D^{\perp}$ is an anti-invariant distribution. Thus M is a semi-invariant submanifold.

Theorem 3. *Let M be a submanifold tangent to ξ of a Sasakian manifold $\tilde{M}$ and $D_x^{\perp}$ the maximal anti-invariant subspace of H_x. Suppose* $\dim D_x^{\perp} =$ *constant. Then the anti-invariant distribution $D^{\perp}$ is always integrable.*

Proof. By using (2), (3) and the Gauss and Weingarten formulas we get

$$-A_{\varphi Z}W + \nabla^{\perp}_W \varphi Z = \varphi(\nabla_W Z + h(Z, W)) + g(Z, W)\xi \text{ for any } Z, W \in \Gamma(D^{\perp}). \tag{11}$$

Then (11) implies

$$(A_{\varphi Z}W - A_{\varphi W}Z) + (\nabla^{\perp}_Z \varphi W - \nabla^{\perp}_W \varphi Z) = \varphi([Z, W]). \tag{12}$$

Since $A_{\varphi Z}W - A_{\varphi W}Z$ is tangential to M and $\nabla^{\perp}_Z \varphi W - \nabla^{\perp}_W \varphi Z$ is normal to M, from (12) it follows that $[Z, W] \in \Gamma(D^{\perp})$ for any $Z, W \in \Gamma(D^{\perp})$ if and only if we have

$$A_{\varphi Z}W = A_{\varphi W}Z \text{ for any } Z, W \in \Gamma(D^{\perp}) \text{ and} \tag{13}$$

$$g([Z, W], \xi) = 0 \text{ for any } Z, W \in \Gamma(D^{\perp}). \tag{14}$$

On the other hand, by using (2), (3), (4) and Gauss's and Weingarten's formulas we obtain $g(A_{\varphi Z}W, X) = g(h(W, X), \varphi Z) = g(\tilde{\nabla}_X W, \varphi Z) = -g(\varphi\tilde{\nabla}_X W, Z) = -g(\tilde{\nabla}_X \varphi W, Z) = g(\varphi W, \tilde{\nabla}_X Z) = g(h(X, Z), \varphi W) = g(A_{\varphi W}Z, X)$ for any $Z, W \in \Gamma(D^{\perp})$ and $X \in \Gamma(TM)$, i.e.,

(13) holds. Finally, by using (4) and taking into account that $D^\perp$ is an anti-invariant distribution, we have $g([Z, W], \xi) = g(\tilde\nabla_Z W - \tilde\nabla_W Z, \xi) = g(Z, \tilde\nabla_W \xi) - g(W, \tilde\nabla_Z \xi) = g(W, \varphi Z) - g(Z, \varphi W) = 0$, for any $Z, W \in \Gamma(D^\perp)$. Therefore (13) and (14) hold and the proof of Theorem 3 is complete.

Theorem 4. *Let M be a submanifold tangent to ξ of a Sasakian space form $\tilde M(c)$ with $c \neq 0$. Then $\tilde M$ is a semi-invariant submanifold if and only if the maximal anti-invariant subspaces $D_x^\perp \subset H_x$, $x \in M$, define a nontrivial differentiable distribution $D^\perp$ on M such that*

$$\tilde R(D, \varphi D ; \mu, D) = 0, \tag{15}$$

where D denotes the orthogonal complementary distribution to $D^\perp \oplus \{\xi\}$ in M, and μ denotes the distribution for which μ_x is the orthogonal complement of $\varphi D_x^\perp$ in $T_x M^\perp$.

Proof. Suppose M be a semi-invariant submanifold of $\tilde M(c)$. Then, by using the definition and (2), from (8) we get $\tilde R(X, \varphi Y ; N, Z) = 0$ for any $X, Y, Z \in \Gamma(D)$ and $N \in \Gamma(\mu)$, i.e., (15) is satisfied.

Conversely, if the maximal anti-invariant subspaces $D_x^\perp \subset H_x$ define a nontrivial differentiable distribution $D^\perp$ such that (15) holds, then by using (2) and (8) we obtain

$$\tilde R(X, \varphi X ; N, X) = cg(X, X)\, g(\varphi X, N) = 0 \quad \text{for any } X \in \Gamma(D) \text{ and } N \in \Gamma(\mu). \tag{16}$$

Because $c \neq 0$, from (16) it follows

$$\varphi D \perp \mu . \tag{17}$$

Also, by using (2) and taking into account that $D^\perp$ is an anti-invariant distribution, we obtain

$$\varphi D \perp D^\perp \quad \text{and} \quad \varphi D \perp \varphi D^\perp. \tag{18}$$

Since by hypothesis we have $D \perp \{\xi\}$, by means of (1) and (2) we also have

(19) $\varphi D \perp \{\xi\}$. Then (17), (18) and (19) imply $\varphi D \subset TM$ and $\varphi D = D$, i.e., D is an invariant distribution and M is a semi-invariant submanifold

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Polytechnic Institute of Iaşi
Department of Mathematics
Iaşi, R.S. România

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V-TRANSFORMATIONS OF FINSLER SPACES. II. FINSLER SPACES WHOSE ISOMETRIES ARE ALL AFFINE

BY

MAKOTO MATSUMOTO

Dedicated to Romanian Finslerish whom I met at the Romanian – Japanese Colloquium on Finsler Geometry

In Finsler geometry the theory of transformations has made remarkable progress in the 1950's by H. H i r a m a t u, G. Soós and K. T a k a n o mainly. They considered only the so-called extended point transformation, which is given by natural extension of a transformation of the underlying manifold to its tangent bundle. In the author's early works [5, 6, 7] and, in particular, the paper [9], the concept of extended point transformation was generalized to a transformation, called a V-transformation, more suitably to Finsler geometry. In the author's monograph [10] a fundamental theory of V-transformations was systematically described and several interesting results were shown. Thus Chapter VI of the monograph may be titled „V-transformations of Finsler spaces II", but the monograph is not published yet.

The main purpose of the present paper is to show Theorems 5.1 and 6.2, concerned with isometric V-rotations based on the theory in the monograph. As the contents of the monograph are not well-known, we shall sketch the theory of V-transformations in the first three sections of the present paper ; the fourth section is devoted to certain generalization of the well-known theorems concerned with extended point transformations. Our main results, in particular, Theorem 5.1, give a geometrical characterization of a special class of Finsler spaces and confirm great importance of the concept of V-rotation in Finsler geometry.

§ 1. V-transformations. We consider the Finsler bundle $F(M) = (F, \pi_1, T, G(n))$ of an n-dimensional differentiable manifold M. It is the principal bundle $\pi_T^{-1}L(M)$, induced from the linear frame bundle $L(M) = (L, \pi_L, M, G(n))$ over M by the projection $\pi_T : T \to M$ of the tangent bundle $T(M) = (T, \pi_T, M, V^n)$ over M. A point $u = (y, z) \in F$ is a pair of $y \in T$ and $z \in L$ satisfying $\pi_T(y) = \pi_L(z)$. The projection $\pi_1 : F \to T$ is defined as $\pi_1(y, z) = y$ and another mapping $\pi_2 : F \to L$ given by $\pi_2(y, z) = z$ is called the induced mapping. Let β_g be the right translation of L by $g \in G(n)$ and write $\beta_g(z) = zg$ for $z \in L$. Then the right translation β_g of F is induced by $\beta_g(y, z) = (y, z, g)$.

If we take account of the homogeneity with respect to $y = x$ in a Finsler space $F^n = (M, L(x, y))$ with a fundamental function $L(x, y)$, our consideration will be restricted to nonzero vector y of $T(M)$ only.