(13) holds. Finally, by using (4) and taking into account that $D^\perp$ is an anti-invariant distribution, we have $g([Z, W], \xi) = g(\tilde{\nabla}_Z W - \tilde{\nabla}_W Z, \xi) = g(Z, \tilde{\nabla}_W \xi) - g(W, \tilde{\nabla}_Z \xi) = g(W, \varphi Z) - g(Z, \varphi W) = 0$, for any $Z, W \in \Gamma(D^\perp)$. Therefore (13) and (14) hold and the proof of Theorem 3 is complete.

Theorem 4. *Let M be a submanifold tangent to ξ of a Sasakian space form $\tilde{M}(c)$ with $c \neq 0$. Then $\tilde{M}$ is a semi-invariant submanifold if and only if the maximal anti-invariant subspaces $D_x^\perp \subset H_x$, $x \in M$, define a nontrivial differentiable distribution $D^\perp$ on M such that*

$$\tilde{R}(D, \varphi D ; \mu, D) = 0, \tag{15}$$

where D denotes the orthogonal complementary distribution to $D^\perp \oplus \{\xi\}$ in M, and μ denotes the distribution for which μ_x is the orthogonal complement of $\varphi D_x^\perp$ in $T_x M^\perp$.

Proof. Suppose M be a semi-invariant submanifold of $\tilde{M}(c)$. Then, by using the definition and (2), from (8) we get $\tilde{R}(X, \varphi Y ; N, Z) = 0$ for any $X, Y, Z \in \Gamma(D)$ and $N \in \Gamma(\mu)$, i.e., (15) is satisfied.

Conversely, if the maximal anti-invariant subspaces $D_x^\perp \subset H_x$ define a nontrivial differentiable distribution $D^\perp$ such that (15) holds, then by using (2) and (8) we obtain

$$\tilde{R}(X, \varphi X ; N, X) = cg(X, X)\, g(\varphi X, N) = 0 \quad \text{for any} \quad X \in \Gamma(D) \quad \text{and} \quad N \in \Gamma(\mu). \tag{16}$$

Because $c \neq 0$, from (16) it follows

$$\varphi D \perp \mu . \tag{17}$$

Also, by using (2) and taking into account that $D^\perp$ is an anti-invariant distribution, we obtain

$$\varphi D \perp D^\perp \quad \text{and} \quad \varphi D \perp \varphi D^\perp. \tag{18}$$

Since by hypothesis we have $D \perp \{\xi\}$, by means of (1) and (2) we also have

(19) $\varphi D \perp \{\xi\}$. Then (17), (18) and (19) imply $\varphi D \subset TM$ and $\varphi D = D$, i.e., D is an invariant distribution and M is a semi-invariant submanifold

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Polytechnic Institute of Iaşi
Department of Mathematics
Iaşi, R.S. România

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V-TRANSFORMATIONS OF FINSLER SPACES. II. FINSLER SPACES WHOSE ISOMETRIES ARE ALL AFFINE

BY

MAKOTO MATSUMOTO

Dedicated to Romanian Finslersh whom I met at the Romanian — Japanese Colloquium on Finsler Geometry

In Finsler geometry the theory of transformations has made remarkable progress in the 1950's by H. H i r a m a t u, G. Soós and K. T a k a n o mainly. They considered only the so-called extended point transformation, which is given by natural extension of a transformation of the underlying manifold to its tangent bundle. In the author's early works [5, 6, 7] and, in particular, the paper [9], the concept of extended point transformation was generalized to a transformation, called a V-transformation, more suitably to Finsler geometry. In the author's monograph [10] a fundamental theory of V-transformations was systematically described and several interesting results were shown. Thus Chapter VI of the monograph may be titled „V-transformations of Finsler spaces II", but the monograph is not published yet.

The main purpose of the present paper is to show Theorems 5.1 and 6.2, concerned with isometric V-rotations based on the theory in the monograph. As the contents of the monograph are not well-known, we shall sketch the theory of V-transformations in the first three sections of the present paper; the fourth section is devoted to certain generalization of the well-known theorems concerned with extended point transformations. Our main results, in particular, Theorem 5.1, give a geometrical characterization of a special class of Finsler spaces and confirm great importance of the concept of V-rotation in Finsler geometry.

§ 1. V-transformations. We consider the Finsler bundle $F(M) = (F, \pi_1, T, G(n))$ of an n-dimensional differentiable manifold M. It is the principal bundle $\pi_T^{-1}L(M)$, induced from the linear frame bundle $L(M) = (L, \pi_L, M, G(n))$ over M by the projection $\pi_T : T \to M$ of the tangent bundle $T(M) = (T, \pi_T, M, V^n)$ over M. A point $u = (y, z) \in F$ is a pair of $y \in T$ and $z \in L$ satisfying $\pi_T(y) = \pi_L(z)$. The projection $\pi_1 : F \to T$ is defined as $\pi_1(y, z) = y$ and another mapping $\pi_2 : F \to L$ given by $\pi_2(y, z) = z$ is called the induced mapping. Let β_g be the right translation of L by $g \in G(n)$ and write $\beta_g(z) = zg$ for $z \in L$. Then the right translation β_g of F is induced by $\beta_g(y, z) = (y, zg)$.

If we take account of the homogeneity with respect to $y = \dot{x}$ in a Finsler space $F^n = (M, L(x, y))$ with a fundamental function $L(x, y)$, our consideration will be restricted to nonzero vector y of $T(M)$ only.

A Finsler tensor field K of (r, s)-type on the Finsler space F^n is by definition a $(V^n)^r_s$-valued function on the total space F of $F(M)$ satisfying $K \circ \beta_g = g^{-1} K$ for any $g \in G(n)$. A Finsler tensor field ε of $(0, 1)$-type, given by $\varepsilon(y, z) = z^{-1}y$, is called the *supporting element*. If we refer to a coordinate system (x^i) of M and the canonical coordinate system (x^i, y^i, z^i_a) of F such that $y = y^i \partial/\partial x^i$ and $z = (z_a = z^i_a \partial/\partial x^i)$, we then have $z^{-1} y = z^a_i y^i e_a$ in a fixed base (e_a) of V^n, where (z^a_i) is the inverse matrix of (z^i_a). Thus the components of ε are y^i. The mapping $I = (\pi_2, \varepsilon) : (y, z) \in F \mapsto (z, z^{-1}y) \in L \times V^n$ gives a diffeomorphism of F onto the product manifold $L \times V^n$.

Let A be an element of the Lie algebra $G'(n)$ of $G(n)$ and let $\{a_t\}$ be the one-parameter group generated by A where t is the parameter. Then the one-parameter group of right translations $\{\beta_{a_t(e)}\}$ of F, e being the unit of $G(n)$, induces a so-called fundamental vector field $Z(A)$ on F. Next we consider a transformation S_v of F, corresponding to a vector v of V^n and defined by $S_v(y, z) = (y + zv, z)$. The one-parameter transformation group $\{S_{tv}\}$ of F induces a tangent vector field $Y(v)$ on F, called the *induced-fundamental vector field*. Referring to the above canonical coordinate system (x^i, y^i, z^i_a), those $Z(A)$ and $Y(v)$ are written $Z(A) = z^i_a A^a_b \partial/\partial z^i_b$ and $Y(v) = z^i_a v^a \partial/\partial y^i$, where $A = (A^a_b)$ and $v = v^a e_a$.

Let θ be the basic form on L, which is a V^n-valued differential oneform on L and given by $\theta_z = z^{-1} \circ \pi'_L$ at $z \in L$. The h-basic form θ^h on F is defined by $\theta^h = \theta \circ \pi'_2$.

Definition. *A V-transformation μ_V of $L(M)$ is a family $\{\mu_{(v)} \mid v \in V^n\}$ of transformations $\mu_{(v)}$ of L, which is parametrized by vectors of V^n and satisfies* $\beta_g \circ \mu_{(v)} = \mu_{(g^{-1}v)} \circ \beta_g$.

By the associated mapping $\alpha_v : z \in L \to zv \in T$, a V-transformation μ_V gives rise to a transformation μ of T, defined by $\mu(y) = \alpha_v \circ \mu_{(v)}(z)$ for a point $y = zv$ of T, which is called the *associated transformation* with μ_V. Next μ_V yields a transformation μ^* of the product $L \times V^n$ such that $\mu^*(z, v) = (\mu_{(v)}(z), v)$ for a point $(z, v) \in L \times V^n$. Then the diffeomorphism $I : F \to L \times V^n$ gives a transformation $\bar{\mu}$ of F by $\bar{\mu} = I^{-1} \circ \mu^* \circ I$, which is called the *lift* of μ_V to F. This $\bar{\mu}$ may be also defined by $\pi_1 \circ \bar{\mu} = \mu \circ \pi_1$ and $\pi_2 \circ \bar{\mu} = \mu_{(v)} \circ \pi_2$, where $v = \varepsilon(u)$ at $u \in F$.

Theorem 1.1. *A necessary and sufficient condition for a transformation τ of the total space F of the Finsler bundle $F(M)$ to be the lift of a V-transformation to F is that*

(i) $\beta_g \circ \tau = \tau \circ \beta_g$, (ii) $\varepsilon \circ \tau = \varepsilon$

are satisfied.

If we have a transformation π of the base manifold M, we get its differential π', a transformation of T, and its n-ple differential $\pi^{(n)}$, a transformation of L defined by $\pi^{(n)}(z) = (\pi'(z_a))$ for a point $z = (z_a)$ of L. Then the lift $\bar{\pi}^{(n)}$ of $\pi^{(n)}$ to F is obtained.

Theorem 1.2. *A necessary and sufficient condition for a transformation τ of F to be the lift to F of the n-ple differential of a transformation of the base manifold M is that*

(i) $\beta_g \circ \tau = \tau \circ \beta_g$, (ii) $\varepsilon \circ \tau = \varepsilon$,

(iii) $S_v \circ \tau = \tau \circ S_v$, (iv) $\theta^h \circ \tau' = \theta^h$

are satisfied.

Definition. *A transformation of L is called linear, if it commutes with any right-translation.*

Theorem 1.3. *A transformation τ of F is the lift to F of a linear transformation of L, if and only if the first three conditions of Theorem 1.2 are satisfied.*

We shall consider a special V-transformation μ_V such that its associated transformation μ is fibre-preserving. This μ_V is said to be *fibre-preserving*. A necessary and sufficient condition for μ_V to be fibre-preserving is that $\pi_L \circ \mu_{(v)}$ does not depend on the choice of $v \in V^n$. Therefore a fibre preserving V-transformation μ_V induces a transformation μ of the base manifold M such that $\pi_L \circ \mu_{(v)} = \mu \circ \pi_L$. Then we get the n-ple differential $\mu^{(n)}$ of μ and $g \in G(n)$ for $(z, v) \in L \times V^n$ satisfying $\mu_{(v)}(z) = \mu^{(n)}(z)\, g$. This g may be denoted by $\alpha_{(v)}(z)$, so that we obtain a mapping $\alpha_{(v)} : L \to G(n)$, depending on $v \in V^n$ and called the *deviation* of μ_V. This satisfies $\alpha_{(v)}(zg) = g^{-1} \cdot \alpha_{(gv)}(z) \cdot g$. Thus, if we consider a transformation $\zeta_{(v)}$ of L given by $\zeta_{(v)}(z) = z(\alpha_{(v)}(z))$, we get $\mu_{(v)} = \mu^{(n)} \circ \zeta_{(v)}$.

Definition. *Let α_V be a family $\{\alpha_{(v)} \mid v \in V^n\}$ of mappings $\alpha_{(v)} : L \to G(n)$ satisfying $\alpha_{(v)} \circ \beta_g = g^{-1} \cdot \alpha_{(gv)} \cdot g$. The V-transformation $\zeta_V = \{\zeta_{(v)} \mid v \in V^n\}$, defined by $\zeta_{(v)}(z) = z(\alpha_{(v)}(z))$ for a point $z \in L$, is called a V-rotation.*

The above family ζ_V is certainly a V-transformation of a kind. Then it may be said that a fibre-preserving V-transformation is a composition of an n-ple differential and a V-rotation.

§ 2. **Infinitesimal V-transformations**. Let $\{\bar{\mu}_t\}$ be the lift of a one-parameter V-transformation group to F and let X_V be the tangent vector field of F which is induced by $\{\bar{\mu}_t\}$ and called an *infinitesimal V-transformation*.

We consider the Lie differentiation $\mathcal{L}_V$ with respect to X_V. It is noted that $\mathcal{L}_V f = X_V f$ for a function f, $\mathcal{L}_V X = [X_V, X]$ for a vector field X and $\mathcal{L}_V \alpha(X) = X_V(\alpha(X)) - \alpha([X_V, X])$ for a differential oneform α. Theorem 1.1 shows

Theorem 2. 1. *A tangent vector field X_V of the total space F of the Finsler bundle $F(M)$ is an infinitesimal V-transformation, if and only if the Lie differentiation $\mathcal{L}_V$ satisfies*

(i) $\mathcal{L}_V Z(A) = 0$, (ii) $\mathcal{L}_V \varepsilon = 0$.

We shall be concerned with a Finsler connection $F\Gamma = (\Gamma^h, \Gamma^v)$, where Γ^h and Γ^v are distributions in F and called the horizontal and vertical connections of $F\Gamma$ respectively. We shall assume throughout the following that the reader is familiar with our theory of Finsler connections through the author's previous papers, in particular, the first part [9] and the monograph [10].

Let $(B^h(v), B^V(v))$ be the h- and v-basic vector fields, corresponding to $v \in V^n$. Then an infinitesimal V-transformation X_V is written in the form

$$X_V = B^h(v_V) + B^v(w_V) + Z(A_V). \tag{2.1}$$

The two conditions (i) and (ii) of Theorem 2.1 show that these v_V, $w_V \in F^n$ and $A_V \in G'(n)$ are Finsler tensor fields of (1,0)-, (1,0)- and (1,1)-type respectively and satisfy

$$A_V \,.\, \varepsilon = D(v) + w_V + C(\varepsilon, w_V), \tag{2.2}$$

where D and C are the deflection and the Cartan tensor fields of $F\Gamma$ respectively.

Next we specially consider a one-parameter transformation group $\{\pi_t\}$ of the base manifold M. Then we get the one-parameter transformation group $\{\pi_t^{(n)}\}$ of L, where $\pi_t^{(n)}$ is the n-ple differential of π_t, and the one-parameter transformation group $\{\bar{\pi}_t^{(n)}\}$ of F, where $\bar{\pi}_t^{(n)}$ is the lift of $\pi_t^{(n)}$. Thus the tangent vector field X_ε is induced by $\{\bar{\pi}_t^{(n)}\}$. This X_ε is nothing but a well-known *extended point transformation* which has been treated by almost of all authors. Theorem 1.2 shows.

Theorem 2.2. *A tangent vector field X_ε of the Finsler bundle $F(M)$ is an extended point transformation, if and only if the Lie differentiation $\mathcal{L}_\varepsilon$ with respect to X_ε satisfies*

(i) $\mathcal{L}_\varepsilon Z(A) = 0$, (ii) $\mathcal{L}_\varepsilon \varepsilon = 0$,

(iii) $\mathcal{L}_\varepsilon Y(v) = 0$, (iv) $\mathcal{L}_\varepsilon \theta^h = 0$.

If we restrict our consideration to a one-parameter linear transformation group, its lift induces a tangent vector field X_λ of F. Denoting by $\mathcal{L}_\lambda$ the Lie differentiation with respect to X_λ, Theorem 1.3 shows that $\mathcal{L}_\lambda$ satisfies the first three conditions of Theorem 2.2. In particular, the condition (iii) is noteworthy, because it may be written $[X_\lambda, Y(v)] = 0$, i.e., $\mathcal{L}_\lambda$ and ∇^0 $(=\partial)$ commute with each other for a Finsler tensor field.

On the other hand, let X_ρ be a tangent vector field of F which is induced by the lift of a one-parameter V-rotation group. The X_ρ is written in the special form

$$X_\rho = Y(A_\rho \,.\, \varepsilon) + Z(A_\rho), \tag{2.3}$$

where $A_\rho \in G'(n)$ is a Finsler tensor of (1,1)-type.

Definition. *Let a nonlinear connection N be given. An infinitesimal V-transformation X_ν is called N-natural with respect to the N, if the Lie derivative $\mathcal{L}_\nu B^h(v)$ with respect to X_ν is quasi-vertical, i.e., $\pi'_T \circ \pi_1^{-1} \mathcal{L}_\nu B^h(v) = 0$, where $B^h(v)$ is any h-basic vector field of a Finsler connection $F\Gamma = (\Gamma, N)$ having the nonlinear connection N.*

The N-naturality does not depend on the choice of a connection Γ in $F(M)$ which is paired with N. Throughout the following we shall restrict our consideration to a Finsler connection $F\Gamma = (\Gamma, N)$ having the given nonlinear connection N, whenever an N-natural transformation X_ν is concerned with.

Theorem 2.3. *Let an infinitesimal N-natural transformation X_ν be written as $X_\nu = B^h(v_\nu) + B^V(w_\nu) + Z(A_\nu)$ with respect to an $F\Gamma = (\Gamma, N)$. These v_ν, w_ν and A_ν satisfy*

(i) $w_\nu = \nabla^h v_\nu(\varepsilon) - T(v_\nu, \varepsilon) - D(v_\nu)$,

(ii) $A_\nu \,.\, v = \nabla^h v_\nu(v) - T(v_\nu, v) + C(v, w_\nu)$,

where T is the (h) h-torsion tensor of $F\Gamma$ and ∇^h is the h-covariant differentiation.

The following theorem asserts an important role of the concepts of N-naturality and V-rotation.

Theorem 2.4. *Any infinitesimal V-transformation is uniquely written as the sum $X_\nu + X_\rho$ of an infinitesimal N-natural transformation X_ν and an infinitesimal V-rotation X_ρ.*

In terms of a canonical coordinate system (x^i, y^i, z^i_a), an infinitesimal N-natural transformation X_ν is written

$$X_\nu = X^i \partial/\partial x^i + (\delta_k X^i)\,(y^k \partial/\partial y^i + z^k_a \partial/\partial z^i_a), \tag{2.4}$$

where $\delta_k = \partial_k - N^j_k \partial_j$, $N = (N^j_k)$, and $X^i(x, y)$ are components of v_ν in Theorem 2.3. Therefore, in the old-fashioned notations, X_V may be written

$$(x^i, y^i) \to (\bar{x}^i = x^i + X^i(x, y)\,dt,\ \bar{y}^i = y^i + (\delta_k X^i)\, y^k dt).$$

On the other hand, an infinitesimal V-rotation X_ρ is written

$$X_\rho = X^i_k (y^k \partial/\partial y^i + z^k_a \partial/\partial z^i_a), \tag{2.5}$$

where $X^i_k\,(x, y)$ are components of A_ρ in (2.3). Thus X_ρ may be written

$$(x^i, y^i) \to (\bar{x}^i = x^i,\ \bar{y}^i = y^i + X^i_k(x, y)\, y^k dt).$$

Proposition 2. 1. (i) *If an infinitesimal N-natural transformation is linear, it is an extended point transformation and X^i in (2.4) depend on x^i alone.*

(ii) *An infinitesimal V-rotation (2.3) is linear, if and only if $\nabla^0 A_\rho = 0$, that is, X^i_k in (2.5) depend on x^i alone.*

Finally we shall touch upon the homogeneity property of V-transformations. A V-transformation $\mu_V = \{\mu_{(v)} \mid v \in V^n\}$ is called p-homogeneous, if $\mu_{(rv)} = \mu_{(v)}$ holds for any $v \in V^n$ and any positive number r. A p-homogeneous infinitesimal V-transformation X_V is (0) p-homogeneous, and v_V, w_V, A_V in (2.1) are (0), (1), (0) p-homogeneous respectively, if the $F\Gamma$ under consideration be p-homogeneous. It is remarked [10] that "$(\alpha)p$-homogeneous" stands for "positively homogeneous of degree α".

§ 3. **Affine transformations.** We consider Lie derivatives $\mathcal{L}_V B^h(v)$ and $\mathcal{L}_V B^v(v)$ of basic vector fields $(B^h(v), B^v(v))$ of a Finsler connection $F\Gamma = (\Gamma^h, \Gamma^v)$ with respect to an infinitesimal V-transformation X_V. These are written

$$\begin{cases} \mathcal{L}_V B^h(v) = B^h(\alpha(v)) + B^v(\alpha^1(v)) + Z(\alpha^2(v)), \\ \mathcal{L}_V B^v(v) = B^h(\beta(v)) + B^v(\beta^1(v)) + Z(\beta^2(v)). \end{cases} \tag{3.1}$$

By substituting from the form (2.1) of X_V into (3.1), we get

$$\begin{cases} \alpha(v) = T(v_V, v) - \nabla^h v_V(v) - C(v, w_V) + A_V \,.\, v, \\ \alpha^1(v) = R^1(v_V, v) - P^1(v, w_V) - \nabla^h w_V(v), \\ \alpha^2(v) = R^2(v_V, v) - P^2(v, w_V) - \nabla^h A_V(v), \end{cases} \tag{3.2}$$

$$
(3.3)\quad \begin{cases} \beta(v)=C(v_V,\ v)-\nabla^v v_V(v)=-\nabla^0 v_V(v), \\ \beta^1(v)=P^1(v_V,\ v)-S^1(v,\ w_V)-\nabla^v w_V(v)+A_V\ .\ v, \\ \beta_2(v)=P^2(v_V,\ v)-S^2(v,\ w_V)-\nabla^V A_V(v), \end{cases}
$$

where ∇^h, ∇^v are h- and v-covariant differentiations, T, C, R^1, P^1, S^1 are torsion tensors and R^2, P^2, S^2 are curvature tensors of the $F\Gamma$. If we are concerned with p-homogeneous X_V and $F\Gamma$, α, α^1, α^2 of (3.2) are (0), (1), (0) p-homogeneous and β, β^1, β^2 are (-1), (0), (-1) p-homogeneous respectively.

Proposition 3.1.

(i) *For an infinitesimal N-natural transformation X_ν we have* $\alpha=0$,

(ii) *For an infinitesimal linear transformation X_λ we have* $\beta=\beta^1=0$ *and* $\beta^2=-\mathcal{L}_\lambda C$.

(iii) *For an extended point transformation X_ε we have* $\alpha=\beta=\beta^1=0$ *and* $\beta^2=-\mathcal{L}_\varepsilon C$.

Next we shall treat of an infinitesimal V-rotation X_ρ of the form (2.3). Then (3.2) and (3.3) become

$$
(3.4)\quad \begin{cases} \alpha(v)=A_\rho\ .\ v, \\ \alpha^1(v)=-P^1(v,\ A_\rho\ .\ \varepsilon)-\nabla^h(A_\rho\ .\ \varepsilon)\ v, \\ \alpha^2(v)=-P^2(v,\ A_\rho\ .\ \varepsilon)-\nabla^h(A_\rho+C(A_\rho\ .\ \varepsilon))\ v, \end{cases}
$$

$$
(3.5)\quad \begin{cases} \beta=0, \\ \beta^1(v)=C(A_\rho\ .\ \varepsilon,\ v)-\nabla^V(A_\rho\ .\ \varepsilon)\ v+A_\rho\ .\ v=-(\nabla^0 A_\rho)\ (\varepsilon,\ v)\ , \\ \beta^2(v)=-S^2(v,\ A_\rho\ .\ \varepsilon)-\nabla^V(A_\rho+C(A_\rho\ .\ \varepsilon))\ v. \end{cases}
$$

If a Finsler connection has the vanishing deflection tensor D, it is said to satisfy the *D-condition* If the Cartan tensor C satisfies $C(\varepsilon,\ v)=0$ for any $v\in V^n$, the connection is said to satisfy the *C_1-condition*.

Proposition 3.2. *If the Finsler connection, appeared in (3.1), satisfies the D- and C_1-conditions, we have* $\alpha^1(v)=\alpha^2(\varepsilon,\ v)$ *and* $\beta^1(v)=\beta^2(\varepsilon,\ v)$.

These circumstances are noteworthy in the viewpoint that it is quite analogous to the fact that $R^1(v_1,\ v_2)=R^2(\varepsilon,\ v_1,\ v_2)$ and the similar equations for P^1 and S^1 are satisfied under the D- and C_1- conditions.

Now the Jacobi identities, satisfied by X_V and $(B^h(v),\ B^v(v))$, lead to the following important equations which give the Lie derivatives of torsion and curvature tensors. (In these equations v_1 and v_2 are denoted by their subscripts and $\mathfrak{A}_{(12)}$ stands for interchange of indices 1, 2 and subtraction.)

$$
(3.6)\quad \begin{cases} \mathcal{L}_V T(1,\ 2)=-\alpha(T(1,\ 2))-\beta(R^1(1,\ 2))+ \\ \qquad \mathfrak{A}_{(12)}\{\nabla^h\alpha(2,1)-\alpha^2(1,\ 2)+T(1,\ \alpha(2))+C(1,\ \alpha^1(2))\}, \\ \mathcal{L}_V R^1(1,\ 2)=-\alpha^1(T(1,\ 2))-\beta^1(R^1(1,\ 2))+ \\ \qquad +\mathfrak{A}_{(12)}\{\Delta^h\alpha^1(2,\ 1)+R^1(1,\ \alpha(2))+P^1(1,\ \alpha^1(2))\}. \\ \mathcal{L}_V R^2(1,\ 2)=-\alpha^2(T(1,\ 2))-\beta^2(R^1(1,\ 2))+ \\ \qquad +\mathfrak{A}_{(12)}\{\nabla^h\alpha^2(2,\ 1)+R^2(1,\ \alpha(2))+P^2(1,\ \alpha^1(2))\}. \end{cases}
$$

$$
(3.7)\quad \begin{cases} \mathcal{L}_V C(1,\ 2)=\nabla^h\beta(2,\ 1)-\nabla^v\alpha(1,\ 2)-\beta^2(1,\ 2)+T(1,\ \beta(2))- \\ \qquad -\alpha(C(1,2))+C(\alpha(1),\ 2)+C(1,\ \beta^1(2))-\beta(P^1(1,2)), \\ \mathcal{L}_V P^1(1,2)=\nabla^h\beta^1(2,1)-\nabla^v\alpha^1(1,2)+\alpha^2(2,1)+R^1(1,\ \beta(2))- \\ \qquad -\alpha^1(C(1,2))-\beta^1(P^1(1,2))+P^1(\alpha(1),\ 2)+P^1(1,\ \beta^1(2))- \\ \qquad -S^1(2,\ \alpha^1(1)), \\ \mathcal{L}_V P^2(1,2)=\Delta^h\beta^2(2,1)-\nabla^v\alpha^2(1,2)-\alpha^2(C(1,2))-\beta^2(P^2(1,2))+ \\ \qquad +R^2(1,\ \beta(2))+P^2(\alpha(1),\ 2)+P^2(1,\ \beta^1(2))-S^2(2,\ \alpha^1(1)). \end{cases}
$$

$$
(3.8)\quad \begin{cases} \mathcal{L}_V S^1(1,2)=-\beta^1(S^1(1,2))+\mathfrak{A}_{(12)}\{\nabla^v\beta^1(2,1)-\beta^2(1,2)- \\ \qquad -P^1(\beta(2),\ 1)+S^1(1,\ \beta^1(2))\}, \\ \mathcal{L}_V S^2(1,\ 2)=-\beta^2(S^1(1,\ 2))+\mathfrak{A}_{(12)}\{\nabla^v\beta^2(2,1)-P^2(\beta(2),\ 1)+ \\ \qquad +S^2(1,\ \beta^1(2))\}. \end{cases}
$$

The Lie derivative of the deflection tensor D is given by

$$
(3.9)\quad \mathcal{L}_V D(v)=D(\alpha(v))+\alpha^1(v)+C(\varepsilon,\ \alpha^1(v))-\alpha^2(\varepsilon,\ v).
$$

Definition. *An infinitesimal V-transformation X_V is called an affine transformation of a Finsler connection $F\Gamma=(\Gamma^h,\ \Gamma^V)$ if $\mathcal{L}_V\Gamma^h\subset\Gamma^h$ and $\mathcal{L}_V\Gamma^v\subset\Gamma^v$ are satisfied.*

Therefore (3.1) shows that X_V is affine if and only if

$$
(3.10)\quad \alpha^1=\beta=0,\qquad \alpha^2=\beta^2=0
$$

be satisfied, and then we have

$$
(3.11)\quad \mathcal{L}_V B^h(v)=B^h(\alpha(v)),\qquad \mathcal{L}_V B^v(v)=B^V(\beta^1(v)).
$$

If the $F\Gamma$ satisfies the D- and C_1- conditions, Proposition 3.2 and (3.10) show that $\mathcal{L}_V B^v(v)=0$ for an affine X_V, so that the Lie differentiation $\mathcal{L}_V$ commutes with the v-covariant differentiation ∇^V.

Proposition 3.3. *A necessary and sufficient condition for following special infinitesimal V-transformations to be affine is that*

(i) *linear* X_λ : $\mathcal{L}_\lambda B^h(v)=B^h(\alpha(v))$ $\qquad$ $\mathcal{L}_\lambda B^v(v)=0$.

(ii) *extended* X_ε : $\mathcal{L}_\varepsilon B^h(v)=0$, $\qquad$ $\mathcal{L}_\varepsilon B^v(v)=0$.

(iii) *N-natural* X_ν : $\mathcal{L}_\nu B^h(v)=0$, $\qquad$ $\mathcal{L}_\nu B^v(v)=B^v(\beta^1(v))$.

(iv) *rotation* X_ρ : $\mathcal{L}_\rho B^h(v)=B^h(\alpha(v))$, $\qquad$ $\mathcal{L}_\rho B^v(v)=B^v(\beta^1(v))$.

This proposition asserts the conspicuous property of an affine extended X_ε: the Lie differentiation $\mathcal{L}_\varepsilon$ commutes with both of h- and v-covariant differentiations for arbitrary Finsler connection, as shown by G. Soós [13].

Theorem 3.1. *If an infinitesimal N-natural transformation is affine, it must be an extended point transformation.*

This noteworthy fact is easily proved from (3.3$_1$), (3.10) and (2.4).

Theorem 3.2. *An extended point transformation* $X_\varepsilon=B^h(v_\varepsilon)+B^v(w_\varepsilon)+$ $+Z(A_\varepsilon)$ *is affine, if and only if*

(i) $\nabla^h w_\varepsilon(v) = R^1(v_\varepsilon, v) - P^1(v, w_\varepsilon)$,

(ii) $\nabla^h A_\varepsilon(v) = R^2(v_\varepsilon, v) - P^2(v, w_\varepsilon)$,

(iii) $\nabla^h A_\varepsilon(v) = P^2(v_\varepsilon, v) - S^2(v, w_\varepsilon)$

are satisfied for any $v \in V^n$, where $\nabla^0 v_\varepsilon = 0$ and

$$w_\varepsilon = \nabla^h v_\varepsilon(\varepsilon) - T v_\varepsilon(v_\varepsilon, \varepsilon) - D(v_\varepsilon),$$

$$A_\varepsilon \cdot v = \nabla^h v_\varepsilon(v) - T(v_\varepsilon, v) + C(v, w_\varepsilon), \ \textit{for any } v \in V^n.$$

Theorem 3.3. *An infinitesimal V-rotation X_ρ, given by (2.3), is affine if and only if*

(i) $\nabla^h(A_\rho \cdot \varepsilon)\, v = -P^1(v, A_\rho \cdot \varepsilon)$,

(ii) $\nabla^h(A_\rho + C(A_\rho \cdot \varepsilon))\, v = -P^2(v, A_\rho \cdot \varepsilon)$,

(iii) $\nabla^V(A_\rho + C(A_\rho \cdot \varepsilon))\, v = -S^2(v, A_\rho \cdot \varepsilon)$,

are satisfied for any $v \in V^n$.

§ 4. Is every isometric X_ε affine? We consider an n-dimensional Finsler space $F^n = (M^n, L(x, y))$ with a fundamental function $L(x, y)$. Let $g = (g_{ij}) = \nabla^0\nabla^0 L^2/2$ be the fundamental tensor of F^n. If we are concerned with extended point transformations only, the condition $\mathcal{L}_\varepsilon g_{ij} = 0$ may be suitable to the definition of isometry in Finsler geometry, because from (ii) and (iii) of Theorem 2.2 and $g(\varepsilon, \varepsilon) = L^2$ it follows that the condition $\mathcal{L}_\varepsilon L(x, y) = 0$ is equivalent to $\mathcal{L}_\varepsilon g_{ij} = 0$ for an extended point transformation X_ε.

If we shall, however, generalize our consideration to V-transformations, it follows from (ii) of Theorem 2.1 that $\mathcal{L}_V g_{ij} = 0$ gives rise to $\mathcal{L}_V L(x, y) = 0$, but the inverse is not true. Therefore we shall define the isometry property of X_V by the stronger condition $\mathcal{L}_V g_{ij} = 0$ than $\mathcal{L}_V L(x, y) = 0$ although the latter may be more suitable in Finsler geometry.

Theorem A. *If an extended point transformation is conformal with respect to $L(x, y)$ and affine with respect to the Cartan connection $C\Gamma$ constructed from $L(x, y)$, it is homothetic with respect to $L(x, y)$.*

Theorem B. *If an extended point transformation is homothetic with respect to $L(x, y)$, it is affine with respect to both of the Cartan connection $C\Gamma$ and the Berwald connection $B\Gamma$ constructed from $L(x, y)$.*

Theorem A was proved by K. Takano [14] in 1952 and G. Soós [13] in 1956, and Theorem B by H. Hiramatu [3] in 1954. The conformity of X_ε is, of course, defined by $\mathcal{L}_\varepsilon g_{ij} = 2\tau g_{ij}$ with a scalar τ, and if τ is constant, the X_ε is called a homothety. On the other hand, denoting by $(F_j{}^i{}_k, N^i{}_j, C_j{}^i{}_k)$ the connection coefficients of a Finsler connection, they defined the affinity of X_ε by the old-fashioned form $\mathcal{L}_\varepsilon F_j{}^i{}_k = \mathcal{L}_\varepsilon C_j{}^i{}_k = 0$.

The main purpose of the present section is to show the following theorem according to our definition of affinity:

Theorem 4.1. *If an extended point transformation X_ε is an isometry of a Finsler metric $L(x, y)$, X_ε is affine with respect to both of the Cartan connection $C\Gamma$ and the Berwald connection $B\Gamma$ constructed from $L(x, y)$.*

Remark. Theorem 4.1 asserts that *every* isometric extended point transformation is affine with respect to $C\Gamma$ and $B\Gamma$. We should remember Theorem 3.1; it may be possible that there exists an N-natural transformation X_V which is an isometry but not an extended point transformation.

To prove Theorem 4.1, we shall start from a generalized Cartan connection $C\Gamma(T)$ with a given $(h)h$-torsion tensor T([2], §25 of [10]).

Definition. *A Finsler connection $C\Gamma(T)$, called a generalized Cartan connection, is uniquely determined from a Finsler metric $L(x, y)$ by the five axioms:*

(C 1) $\nabla^h g = 0$, (C 2) T be given, (C 3) $D = 0$,

(C 4) $\nabla^V g = 0$, (C 5) $S^1 = 0$.

Remark. The axioms (C 4, 5) lead to the well-known tensor $C_{ijk} = \dot\partial_k g_{ij}/2$ and the Cartan tensor $C = (C_j{}^i{}_k = g^{ir}C_{jrk})$ of $C\Gamma(T)$. But the notation C causes a confusion in case of $B\Gamma$, because the Cartan tensor C of $B\Gamma$, i.e., (h) hv-torsion tensor vanishes while $C_{ijk} = \dot\partial_k g_{ij}/2$ still survives even in case of $B\Gamma$. Therefore we shall use T. Okada's notations [12] in the following:

$$g^1 = (g_{ij} = \dot\partial_i\dot\partial_j L^2/2), \qquad g^2 = (g_{ijk} = \dot\partial_k g_{ij}/2).$$

We first consider $\mathcal{L}_\varepsilon T$, given by (3.6$_1$), for an extended point transformation X_ε. On account of (iii) of Proposition 3.1 it becomes

$$\mathcal{L}_\varepsilon T(1, 2) = \mathfrak{A}_{(12)}\{C(1, \alpha^1(2)) - \alpha^2(1, 2)\}. \tag{4.1}$$

Therefore the condition (3.10) for an affine X_ε leads to $\mathcal{L}_\varepsilon T = 0$.

Now assume that an extended point transformation X_ε be isometric with respect to a Finsler metric $L(x, y)$ and satisfy $\mathcal{L}_\varepsilon T = 0$ for the (h) h-torsion tensor T of the $C\Gamma(T)$ given by $L(x, y)$. Then (iii) of Theorem 2.2 shows $0 = [X_\varepsilon, Y(v)]\, g^1 = \mathcal{L}_\varepsilon(2g^2(v)) - \nabla^0(\mathcal{L}_\varepsilon g^1)\, v$, so that $\mathcal{L}_\varepsilon g^1 = 0$ yields $\mathcal{L}_\varepsilon g^2 = 0$. Thus $\beta^2 = 0$ for the $C\Gamma(T)$ from (iii) of Proposition 3.1.

Next (iii) of Proposition 3.1 and (3.1) give

$$\mathcal{L}_\varepsilon B^h(v) = B^v(\alpha^1(v)) + Z(\alpha^2(v)). \tag{4.2}$$

Acting this equation on g^1, we get

$$\mathcal{L}_\varepsilon \nabla^h g^1(v) - \nabla^h \mathcal{L}_\varepsilon g^1(v) = \nabla^v g^1(\alpha^1(v)) - \alpha^2(v) \cdot g^1.$$

Therefore the axioms (C 1, 4) for $C\Gamma(T)$ lead to $\alpha^2(v) \cdot g^1 = 0$, that is,

$$\alpha_{ijk} + \alpha_{jik} = 0, \tag{4.3}$$

where we put $\alpha^2 = (\alpha_j{}^i{}_k)$ and $\alpha_{jik} = g_{ir}\alpha_j{}^r{}_k$. Secondly, from $\mathcal{L}_\varepsilon T = 0$ (4.1) becomes $\mathfrak{A}_{(12)}\{\alpha^2(1,2) - g^2(1, \alpha^1(2))\} = 0$, which may be written

$$\alpha_{ijk} - \alpha_{kji} = g_{ijr}\alpha^r_k - g_{kjr}\alpha^r_i, \tag{4.4}$$

where we put $\alpha^1 = (\alpha^i_j)$. Finally, by the axiom (C 3) (3.9) becomes $\alpha^1(v) = \alpha^2(\varepsilon, v)$, that is,

$$\alpha^i_j = \alpha_0{}^i{}_j. \tag{4.5}$$

Therefore we obtain three equations (4.3), (4.4) and (4.5) for the X_ε.

Then, denoting the right-hand side of (4.4) by b_{ijk}, these equations lead to

$$\alpha_{ijk}=\alpha_{kji}+b_{ijk}=-\alpha_{jki}+b_{ijk}=\alpha_{kij}+b_{ijk}-b_{jki}=-\alpha_{ijk}+(b_{ijk}-b_{jki}+b_{kij}),$$

which is rewritten as $\alpha_{ijk}=g_{kir}\alpha^r_j-g_{kjr}\alpha^r_i$. Contracting the result by y^i and y^k we get $\alpha_{0jk}=-g_{kjr}\alpha^r_0$ and $\alpha_{0j0}=0$. These together with (4.5) yield $\alpha^1=\alpha^2=0$ at once, so that the X_ε is affine with respect to $C\Gamma(T)$. Consequently we have

Theorem 4.2. *Let an extended point transformation X_ε be isometric with respect to a Finsler metric $L(x, y)$. Then X_ε is an affine transformation of the generalized Cartan connection $C\Gamma(T)$, determined from $L(x, y)$ and a given (h) h-torsion tensor T, if and only if $\mathcal{L}_\varepsilon T$ vanishes.*

The part of Theorem 4.1 on the Cartan connection is only a corollary of Theorem 4.2. Further, if we consider a tensor field T defined by

$$T_j{}^i{}_k=c(\delta^i_j X_k-\delta^i_k X_j), \tag{4.6}$$

where $c=$ const., $X_\varepsilon=(X^i(x))$ and $X_k=g_{kr}X^r$, we have $\mathcal{L}_\varepsilon T=0$ for an isometry X_ε. Therefore Theorem 4.2 is applicable to this tensor T. It is noted that the $C\Gamma(T)$ with this T is not a Wagner connection ([2], §25 of [10]), because the covariant components X_k depend on y^i in general.

We shall turn our consideration to the Berwald connection. We shall start from a generalized one [11, 12]:

Definition. *A Finsler connection $B\Gamma(T)$, called a generalized Berwald connection, is uniquely determined from a Finsler metric $L(x, y)$ by the five axioms:*

(B 1) $\nabla^h L=0$, (B 2) T *be given*, (B 3) $D=0$,

(B 4) $P^1=0$, (B 5) $C=0$,

where the T are assumed to satisfy $\nabla^v T(1, \varepsilon, 2)=\nabla^v T(2, \varepsilon, 1)$.

If an extended point transformation X_ε is affine with respect to $B\Gamma(T)$, then we have $\mathcal{L}_\varepsilon T=0$ analogously to the case of $C\Gamma(T)$. Then assume that X_ε be isometric with respect to $L(x, y)$ and satisfy $\mathcal{L}_\varepsilon T=0$ for the (h) h-torsion tensor T of $B\Gamma(T)$ given by $L(x, y)$. $\beta^2=0$ immediately follows from (iii) of Proposition 3.1 and (B 5). Next we get (4.2), where it must be remarked that $B^r(v)=Y(v)$ from (B 5). Thus, acting (4.2) on $L(x, y)$, we get $\mathcal{L}_\varepsilon\nabla^h L(v)-\nabla^h(\mathcal{L}_\varepsilon L)\, v=\nabla^v L(\alpha^1(v))$, which yields first $\nabla^v L(\alpha^1(v))=0$ from (B 1). As the components of $\nabla^v L$ are well-known $l_i=y_i/L$ $(y_i=g_{ir}y^r)$, we get

$$\alpha^0_i(=\alpha^r_i y_r)=0. \tag{4.7}$$

Secondly, from $\mathcal{L}_\varepsilon T=0$ and (B 5) the equation (4.1) reduces to $\alpha^2(1,2)=\alpha^2(2,1)$, that is,

$$\alpha_j{}^i{}_k=\alpha_k{}^i{}_j. \tag{4.8}$$

Thirdly, from (B 4, 5) and (iii) of Proposition 3.1 the equation (3.7_2) reduces to $\nabla^v\alpha^1(1,2)=\alpha^2(2,1)$, i.e.,

$$\dot\partial_k\alpha^i_j=\alpha_k{}^i{}_j. \tag{4.9}$$

Finally (B 3, 5) and (3.9) give (4.5). Therefore we obtain four equations (4.7), (4.8), (4.9) and (4.5) for the X_ε. It should be remarked that α^1 and α^2 are (1) and (0) p-homogeneous respectively.

Differentiating (4.7) by y^j, we get $g_{jr}\alpha^r_i+y_r\alpha_i{}^r{}_j=0$ from (4.9) and (4.8). Contracting this by y^i and paying attention to the homogeneity of α^i_j, (4.5) and (4.7) give $\alpha^r_0=0$. Then $\dot\partial_j\alpha^r_0=0$ leads to $\alpha^r_j=0$ and $\alpha^2=0$ from (4.9.) Consequently we have

Theorem 4.3. *Let an extended point transformation X_ε be isometric with respect to a Finsler metric $L(x, y)$. Then X_ε is an affine transformation of the generalized Berwald connection $B\Gamma(T)$, determined from $L(x, y)$ and a given (h) h-torsion tensor T, if and only if $\mathcal{L}_\varepsilon T$ vanishes.*

The part of Theorem 4.1 on the Berwald connection is only a corollary of Theorem 4.3. Further the tensor T given by (4.6) satisfies the assumption for T mentioned in Definition, as it is easily verified from $X^i=X^i(x)$, and we have also $\mathcal{L}_\varepsilon T=0$.

§ 5. Is every isometric X_ρ affine with respect to $C\Gamma$?

We shall consider isometric rotations which are the main subject of the present paper, that is, isometric rotations which are affine with respect to a Finsler connection. As an infinitesimal V-rotation X_ρ is written in the form (2.3), X_ρ is isometric if and only if $\mathcal{L}_\rho g^1=\nabla^v g^1(A_\rho\,.\,\varepsilon)-A_\rho\,.\,g^1=0$. In terms of the notations $X^i_j(x, y)$ in (2.5) and $X_{ij}=g_{ir}X^r_j$, this equation is written as

$$X_{ij}+X_{ji}+2g_i{}^r{}_j X_{r0}=0, \tag{5.1}$$

where $g_i{}^r{}_j=g^{rs}g_{isj}$ is Cartan's torsion tensor of $C\Gamma$. Therefore, denoting $(X_{ij}-X_{ji})/2$ by Z_{ij}, we have

$$X_{ij}=Z_{ij}-g_i{}^r{}_j Z_{r0}, \qquad (Z_{ij}=-Z_{ji}). \tag{5.1'}$$

It is here noted that on account of (2.3) the Lie derivative $\mathcal{L}_\rho K$ of a tensor field K with respect to a V-rotation X_ρ is given by

$$\mathcal{L}_\rho K=\nabla^v K(A_\rho\,.\,\varepsilon)-A_\rho\,.\,K. \tag{5.2}$$

As for a general Finsler connection with the Cartan tensor C, we have $X=Y(v)=B^v(v)+Z(C(v))$ for any $v\in V^n$, so that (2.3) may be written as $X=B^v(A_\rho\,.\,\varepsilon)+Z(A_\rho+C(A_\rho\,.\,\varepsilon))$. Thus we get

$$\mathcal{L}_\rho K=\nabla^v K(A_\rho\,.\,\varepsilon)-(A_\rho+C(A_\rho\,.\,\varepsilon))\,.\,K. \tag{5.2'}$$

Now assume that an isometric X_ρ be affine with respect to the Cartan connection $C\Gamma$. For the later use we here enumerate some of Bianchi identities of $C\Gamma$ (cf. § 17 of [10]):

$$\mathfrak{S}_{(ijk)}\{R_{hijk}+g_h{}^r{}_i R_{rjk}\}=0, \tag{5.a}$$

where $\mathfrak{S}_{(ijk)}$ denotes cyclic permutation of i, j, k and summation.

$$R_{mhij}|_k+S_{mhkr}R^r{}_{ij}+\mathfrak{A}_{(ij)}\{P_{mhjk|i}+P_{mhir}P^r{}_{jk}+R_{mhir}g_j{}^r{}_k\}=0, \tag{5.b}$$

where $\mathfrak{A}_{(ij)}$ stands for interchange of i, j and subtraction.

$$P_{hijk}=\mathfrak{A}_{(hi)}\{P_{ijk|h}+P_{ikr}g_j{}^r{}_h\}, \tag{5.c}$$

which is essentially one of Bianchi identities.

$$(5.\mathrm{d}) \qquad \mathfrak{S}_{(ijk)} \{S_{mhi|j|k}\} = 0.$$

From (5.1′) we get $A_\rho \,.\, \varepsilon = Z(\varepsilon) = (Z^i_0)$ and $A_\rho + C(A_\rho \,.\, \varepsilon) = (Z^i_j)$, where $Z^i_j = g^{ir} Z_{rj}$. Therefore (5.2′) may be written as

$$(5.2'') \qquad \mathcal{L}_\rho K = \nabla^v K(Z(\varepsilon)) - Z \,.\, K.$$

We first treat of the v-curvature tensor S^2: On account of (3.10) and Proposition 3.2, $\mathcal{L}_\rho S^2$ for this X_ρ, given by (3.8), reduces to so that (5.2″) yields

$$(5.3) \quad \nabla^v S^2(1, 2, Z(\varepsilon)) - Z(S^2(1, 2)) + S^2(Z, 1, 2) + S^2(Z(1), 2) + S^2(1, Z(2)) = 0.$$

In terms of components this may be written

$$S_{hijk} |_r Z^r_0 - Z^i_r S_h{}^r{}_{jk} + S_r{}^i{}_{jk} Z^r_h + S_h{}^i{}_{rk} Z^r_j + S_h{}^i{}_{jr} Z^r_k = 0,$$

or, in terms of contravariant components $Z^{ij} = Z^{ir} g_{rj}$, we have

$$(5.3') \qquad Z^{rs}(S_{hijk} |_r y_s - S_{hsjk} g_{ir} + S_{rijk} g_{sh} + S_{hirk} g_{sj} - S_{hirj} g_{sk}) = 0.$$

In the last section we showed that every isometric extended point transformation is also affine with respect to $C\Gamma$ as well as $B\Gamma$. We now assume that *every* isometric V-rotation X_ρ be also affine with respect to $C\Gamma$. Then (5.3′) must be satisfied by arbitrary skew-symmetric Z^{rs}, so that we have

$$(5.4) \qquad \mathfrak{A}_{(rs)} \{S_{hijk} |_r y_s - S_{hsjk} g_{ir} + S_{rijk} g_{sh} + S_{hirk} g_{sj} - S_{hirj} g_{sk}\} = 0.$$

If we introduce the so-called v-Ricci tensor $S_{ij} = S_i{}^r{}_{jr}$, (5.d) shows $S_{hijk} |_s g^{hs} = S_{ik}|_j - S_{ij}|_k$. Then contraction of (5.4) by g^{hs} gives

$$(5.5) \qquad (n-2)\, S_{rijk} + y_r(S_{ij} |_k - S_{ik} |_j) - g_{jr} S_{ik} + g_{kr} S_{ij} = 0.$$

Further contraction by y^r yields

$$(5.6) \qquad L^2(S_{ij} |_k - S_{ik} |_j) = y_j S_{ik} - y_k S_{ij},$$

so that (5.5) is rewritten as

$$(5.5') \qquad (n-2)\, S_{rijk} = h_{jr} S_{ik} - h_{kr} S_{ij},$$

where $h_{ij} = g_{ij} - l_i l_j$ is the so-called angular metric tensor. (5.5′) immediately yields $S_{rj} = \{S^i{}_i/(n-1)\}\, h_{rj}$, and (5.5′) becomes

$$(5.5'') \qquad L^2 S_{hijk} = S(h_{hj} h_{ik} - h_{hk} h_{ij}),$$

where $S = L^2 S^i{}_i/(n-1)(n-2)$, in case of $n > 2$. (5.5″) gives $L^2 S_{ik} = S(n-2) h_{ik}$ and (5.6) leads to $S = S(x)$.

Remark. It is well-known [8], [10] that in case of $n=3$, S_{hijk} is always of the form (5.5″), so that the v-curvature $S = S(x)$ is a condition for the space under consideration. On the other hand, a Finsler space of dimension $n > 3$ is called $S3$-like if S_{hijk} is of the form (5.5″), and then the v-curvature S is necessarily a function of position alone. In case of $n=2$ the equation (5.4) does not give any condition because of $S_{hijk} = 0$ identically.

Secondly we treat of the hv-curvature tensor P^2. It follows from (3.10) and Proposition 3.2 that $\mathcal{L}_\rho P^2$ for the X_ρ, given by (3.7), reduces to

$$(5.7) \qquad \mathcal{L}_\rho P^2(1, 2) = P^2(\alpha(1), 2).$$

The left-hand side is written in the form similar to that of (5.3), and (3.4) gives $\alpha(v) = Z(v) - g^2(Z(\varepsilon), v)$. Therefore (5.7) is written

$$(5.7') \qquad \begin{aligned} &\nabla^v P^2(1, 2, Z(\varepsilon)) - Z(P^2(1, 2)) + P^2(Z, 1, 2) \\ &+ P^2(1, Z(2)) + P^2(g^2(Z(\varepsilon), 1), 2) = 0. \end{aligned}$$

Owing to $P^1(1, 2) = P^2(\varepsilon, 1, 2)$, (5.7′) yields

$$\nabla^v P^1(1, 2, Z) - Z(P^1(1, 2)) + P^1(1, Z(2)) + P^1(g^2(Z(\varepsilon), 1), 2) = 0.$$

In terms of components this may be written as

$$Z^{rs}(P_{ijk} |_r y_s - P_{sjk} g_{ir} + P_{ijr} g_{sk} + P_{itk} g_r{}^t{}_j y_s) = 0.$$

Then the arbitrariness of $Z^{rs} (= -Z^{sr})$ as above mentioned leads to

$$\mathfrak{A}_{(rs)} \{P_{ijk} |_r y_s - P_{sjk} g_{ir} + P_{ijr} g_{sk} + P_{itk} g_r{}^t{}_j y_s\} = 0.$$

From this the symmetry property $P_{ijk} = P_{ikj}$ of $P_{ijk} = g_{ijk} |_0$ yields

$$\mathfrak{A}_{(rs)} \{P_{ijr} g_{sk} - P_{ikr} g_{sj} + (P_{itk} g_r{}^t{}_j - P_{itj} g_r{}^t{}_k)\, y_s\} = 0.$$

The contraction by $y^k y^s$ immediately gives $P_{ijr} = 0$, and then (5.c) shows $P_{hijk} = 0$. Thus (5.7′) becomes trivial. It is noted that we get $P_{hijk} = 0$ even in case of $n=2$.

Thirdly we treat of the h-curvature tensor R^2. Analogously to the above, (3.6₃) reduces to

$$(5.8) \qquad \mathcal{L}_\rho R^2(1, 2) = R^2(1, \alpha(2)) - R^2(2, \alpha(1)),$$

and this is written

$$(5.8') \qquad \begin{aligned} &\nabla^v R^2(1, 2, Z(\varepsilon)) - Z(R^2(1, 2)) + R^2(Z, 1, 2) + \\ &+ R^2(g^2(Z(\varepsilon), 1), 2) - R^2(g^2(Z(\varepsilon), 2), 1) = 0. \end{aligned}$$

In terms of components this may be written

$$Z^{rs}\{(R_{hijk} |_r + R_{hijt} g_r{}^t{}_k - R_{hikt} g_r{}^t{}_j)\, y_s - R_{hsjk} g_{ir} + R_{rijk} g_{sh}\} = 0.$$

Then the arbitrariness of $Z^{rs} (= -Z^{sr})$ leads to

$$(5.9) \quad \mathfrak{A}_{(rs)}\{(R_{hijk} |_r + R_{hijt} g_r{}^t{}_k - R_{hikt} g_r{}^t{}_j)\, y_s - R_{hsjk} g_{ir} + R_{rijk} g_{sh}\} = 0.$$

Contraction by y^h yields

$$(5.10) \qquad \mathfrak{A}_{(rs)} \{(R_{ijk} |_r + R_{ijt} g_r{}^t{}_k - R_{ikt} g_r{}^t{}_j)\, y_s - R_{sjk} g_{ir}\} = 0,$$

where $R_{ijk} = R_{0ijk}$ are components of the $(v)h$-torsion tensor R^1. Moreover contraction by y^s gives

$$L^2(R_{ijk} |_r + R_{ijt} g_r{}^t{}_k - R_{ikt} g_r{}^t{}_j) = R_{ijk} y_r - R_{rjk} y_i.$$

Then (5.10) becomes $R_{rjk} h_{is} = R_{sj} h_{ir}$, so that we get $(n-2)\, R_{rjk} = 0$. Consequently R_{rjk} vanishes in case of $n > 2$.

From $R_{ijk}=0$ it follows that (5.a) becomes $R_{hijk}+R_{hjki}+R_{hkij}=0$, so that we get $R_{hijk}=R_{jkhi}$ analogously to the case of Riemannian geometry (cf. (22.7) of [10]). Then we have $R_{hijk}\mid_r y^j=R_{jkhi}\mid_r y^j=-R_{hkhi}$ and $R_{hij0}=0$. Therefore, contracting (5.9) by $y^j y^s$, we get finally $R_{rkhi}=0$.

Conversely we consider a Finsler space F^n $(n>2)$ with $R^2=P^2=0$ and S^2 of the $S3$-like form (5.5″) where $S=S(x)$. The differential equations (ii) and (iii) in Theorem 3.3 are now written in the form

$$(5.11)\qquad \nabla^h Z(v)=0. \qquad\qquad (5.12)\qquad \nabla^v Z(v)=S^2(Z(\varepsilon),\ v),$$

for any $v\in V^n$, and the equation (i) is solely a consequence of (5.11). The first of integrability conditions of the system (5.11) and (5.12) is

$$\nabla^h\nabla^h Z(1,2)-\nabla^h\nabla^h Z(2,1)=R^2(Z,\ 1,2)-Z(R^2(1,2))-\nabla^v Z(R^1(1,2)),$$

which is only trivial for the F^n. The second of them is

$$\nabla^v\nabla^h Z(1,2)-\nabla^h\nabla^v Z(2,1)=P^2(Z,\ 1,2)-Z(P^2(1,2))-$$
$$-\nabla^h Z(g^2(1,2))-\nabla^v Z(P^1(1,2)).$$

All the terms of this equation vanish except $\nabla^h\nabla^v Z(2,1))\ \{=\nabla^h(S^2(Z(\varepsilon),v_2))\,v_1\}$, which, in virtue of (5.5″), is written as

$$S_{hirj\mid k}Z^r_0=S\mid_k(h_{hr}h_{ij}-h_{hj}h_{ir})\,Z^{rs}y\,/L^2.$$

Then the arbitrariness of Z^{rs} as above leads to $S_{\mid k}\mathfrak{A}_{(rs)}\{(h_{hr}h_{ij}-h_{hj}h_{ir})\,y_s\}=0$. Contracting by y^s, this yields $S_{\mid k}=0$ for $n>2$, so that S becomes constant.

The third of them is

$$(5.13)\qquad \nabla^v\nabla^v Z(1,2)-\nabla^v\nabla^v Z(2,1)=S^2(Z,\ 1,2)-Z(S^2(1,2)).$$

From (5.12) we get $\nabla^v Z(\varepsilon,v)=S^2(\varepsilon,Z(\varepsilon),v)=0$, so that the left-hand side of (5.13) is rewritten $\mathfrak{A}_{(12)}\{\nabla^v S^2(Z(\varepsilon),\ 1,2)-S^2(Z(1),\ 2)\}$. Then, on account of (5.d) it is observed that (5.13) is equivalent to (5.3) which we have concerned with.

Consequently the system (5.11) and (5.12) is completely integrable in the F^n if and only if the v-curvature S is constant.

We shall turn our consideration to (5.9) in the special case of $n=2$. We already have $P^2=0$. Referring to the Berwald frame $(l_i,\ m_i)$ (§28 of [10]), we have $R_{hijk}=R(l_hm_i-l_im_h)(l_jm_k-l_km_j)$ where R is the h-scalar curvature, so that we observe

$$(5.14)\qquad \mathfrak{A}_{(rs)}\{R_{hsjk}g_{ir}-R_{rijk}g_{sh}\}=0.$$

Thus (5.9) becomes

$$\mathfrak{A}_{(rs)}\{(R_{hijk}\mid_r+R_{hijt}g_r{}^t{}_k-R_{hikt}g_r{}^t{}_j)\,y_s\}=0.$$

This is trivial because of (5.b) which now reduces to $R_{mhij}\mid_k+R_{mhir}g_j{}^r{}_k-R_{mhjr}g_i{}^r{}_k=0$.

Conversely we consider a two-dimensional Finsler space F^2 with $P^2=0$. The system (5.11) and (5.12) is now $\nabla^h Z=\nabla^v Z=0$, and the second and third of integrability conditions are trivial. The first of them is written as $R^2(Z,1,2)-Z(R^2(1,2))=0$, that is, $Z^{rs}(R_{rijk}g_{sh}-R_{hsjk}g_{ir})=0$, which is only a consequence of (5.14).

Summarizing up all the above results, we have

Theorem 5.1. *A necessary and sufficient condition for an n-dimensional Finsler space F^n to be such that every isometric V-rotation be affine with respect to the Cartan connection is that*

(i) $n>3$: *F^n is an S3-like space with vanishing h- and hv-curvature tensors and constant v-curvature.*

(ii) $n=3$: *The h- and hv-curvature tensors of F^3 vanish and its v-curvature is constant.*

(iii) $n=2$: *The hv-curvature tensor of F^2 vanishes, that is, F^2 is a Landsberg space*

Remark 1 : We notice that the Bianchi identity (5.b) plays an important role in the above proof in case of $n=2$. In fact (5.b) is a single surviving Bianchi identity in case of $n=2$. (See (7.8) of Berwald's paper [1]). In the present author's notations it is written

$$(5.15)\qquad R_{;2}+RI+I_{,1,1}=0,$$

where R is the h-scalar curvature as above appeared and I is the main scalar. Cf. (28.14) of [10].

Remark 2. Finsler spaces having $R^2=P^2=0$ and S^2 of $S3$-like form where S is constant have been studied by S. Hojo [4]. According to his Theorem 3.3 the fundamental functions of such spaces are known, provided that $S\geqq 0$ and $S\leqq -1$. Theorem 5.1 gives an geometrical characterization to such spaces in the viewpoint of the theory of transformations.

§ 6. Is every isometric V_ρ affine with respect to $B\Gamma$?

We consider the problem, similar to the subject of §5, with respect to the Berwald connection $B\Gamma$. First we must pay attention to the equation (iii) of Theorem 3.3 ; it reduces to $\nabla^v A_\rho=0$ for $B\Gamma$ because of vanishing of the Cartan tensor C, so that (ii) of Proposition 2.1 shows the following analogous to Theorem 3.1 :

Theorem 6.1. *If an infinitesimal V-rotation X_ρ is affine with respect to the Berwald connection, X_ρ must be linear, that is, $\nabla^v A_\rho=0$ in (2.3), or $X^i_j=X^i_j(x)$ in (2.5), and we have $\beta^1=0$ in (3.1).*

Now assume that an isometric linear V-rotation X_ρ be affine with respect to $B\Gamma$. From $C=0$ it follows that the right-hand sides of (5.2) and (5.2′) are of the same form and the equation (3.8) for $\mathcal{L}_\rho S^2$ becomes trivial.

It should be remarked that the connection coefficients $G_j{}^i{}_k$ of $B\Gamma$ are given by $\dot\partial_j\dot\partial_k G^i$ from the quantities G^i which appear in the differential equations of geodesic curve. The hv-curvature tensor P^2 of $B\Gamma$, usually denoted by $G=(G_h{}^i{}_{jk})$, is given by $G_h{}^i{}_{jk}=\dot\partial_h G_j{}^i{}_k$. Consequently $G_h{}^i{}_{jk}$ are completely symmetric in subscripts and $G_0{}^i{}_{jk}=0$. Next, the h-curvature tensor R^2, usually denoted by $H=(H_h{}^i{}_{jk})$, is given by

$$(6.a)\qquad H_h{}^i{}_{jk}=\dot\partial_h R^i{}_{jk},$$

from the $(v)h$-torsion tensor $R^1=(R^i{}_{jk})$ which is the same with R^1 of $C\Gamma$. Further we have one of Bianchi identities, just corresponding to (5.b),

$$(6.b)\qquad \dot\partial_k H_m{}^h{}_{ij}+\mathfrak{A}_{(ij)}\{G_m{}^h{}_{jk;i}\}=0,$$

where (;) and $\dot\partial_k$ stand for ∇^h and ∇^v with respect to $B\Gamma$ respectively.

We shall treat of $\mathcal{L}_\rho P^2$, i.e., $\mathcal{L}_\rho G$, given by (3.7). From (3.10) and Proposition 3.2 and paying attention to (3.4 ${}_1$), the equation (3.7) becomes

$$\mathcal{L}_\rho G(1,2)=G(X(1),\ 2), \tag{6.1}$$

where $X=(X^i_j(x))$ appears in (2.5). From (5.2) we have

$$\mathcal{L}_\rho G(1,2)=\nabla^v G(1,2,\ X(\varepsilon))-X(G(1,2))+G(X,\ 1,2)+G(X(1),\ 2)+G(1,\ X(2)),$$

analogous to the left-hand side of (5.3), so that (6.1) is written

$$(\dot\partial_r G_h{}^i{}_{jk})X^r_0-X^i_r G_h{}^r{}_{jk}+G_r{}^i{}_{jk}X^r_h+G_h{}^i{}_{jr}X^r_k=0. \tag{6.1'}$$

Owing to $G_h{}^i{}_{jk}=G_h{}^i{}_{kj}$, (6.1') yields $G_h{}^i{}_{jr}X^r_k=G_h{}^i{}_{kr}X^r_j$, so that we get $G_h{}^i{}_{jr}X^r_0=0$. Substituting into this from (5.1'), we get $G_h{}^i{}_{jr}Z^{rs}y_s=0$. Then the arbitrariness of $Z^{rs}(=-Z^{sr})$ leads to $G_h{}^i{}_{jr}y_s=G_h{}^i{}_{js}y_r$, which yields $G_h{}^i{}_{jr}=0$ at once. Therefore the $B\Gamma$ is linear, that is, $G_j{}^i{}_r=G_j{}^i{}_r(x)$, or the space is a so-called Berwald space (affinely connected space). Cf. Theorem 25.1 of [10].

We next treat of $\mathcal{L}_\rho R^2$, i.e., $\mathcal{L}_\rho H$, given by (3.6). It is first noted that $\dot\partial_k H_m{}^h{}_{ij}=0$ from (6.b) for our F^n. Therefore (3.6), which is now written as

$$\mathcal{L}_\rho H(1,\ 2)=H(1,\ X(2))-H(2,\ X(1)), \tag{6.2}$$

reduces solely to $H(X,\ 1,2)-X(H(1,2))=0$, that is,

$$H_r{}^i{}_{jk}X^r_h-X^i_r H_h{}^r{}_{jk}=0. \tag{6.2'}$$

Contracting by y^h, (6.2') gives $(\dot\partial_r R^i{}_{jk})\,X^r_0-X^i_r R^r{}_{jk}=0$ on account of (6.a). Then, substituting from (5.1'), we have

$$Z^{rs}\{(\dot\partial_r R^i{}_{jk}+g_t{}^i{}_r R^t{}_{jk})\,y_s-R_{sjk}\delta^i_r\}=0.$$

Then the arbitrariness of $Z^{rs}(=-Z^{sr})$ yields

$$\mathfrak{A}_{(rs)}\{(\dot\partial_r R^i{}_{jk}+g_t{}^i{}_r R^t{}_{jk})\,y_s-R_{sjk}\delta^i_r\}=0. \tag{6.3}$$

Contracting by y^s, this gives $L^2(\dot\partial_r R^i{}_{jk}+g_t{}^i{}_r R^t{}_{jk})=R^i{}_{jk}y_r-R_{rjk}y^i$. Therefore (6.3) can be rewritten as $R_{rjk}h_{is}=R_{sjk}h_{ir}$, which immediately yields $(n-2)R_{rjk}=0$. Thus, for our space F^n of $n>2$ we have $R_{rjk}=0$, so that (6.a) shows $H_k{}^h{}_{ij}=0$. Consequently the $F^n(n>2)$ is of $H=G=0$, that is, it is locally Minkowski. Cf. Theorem 24.4 of [10].

In this case the differential equations (ii) and (iii) in Theorem 3.3 are written as $\nabla^h X=\nabla^v X=0$ and their integrability conditions are obviously satisfied.

We shall turn our consideration to (6.2') in the special case of the two-dimensional space F^2. Then, referring to the Berwald frame $(l_i,\ m_i)$ again we have the special expression of H_{hijk}:

$$H_{hijk}=\{R(l_h m_i-l_i m_h)+R_{;2}\,m_h m_i\}\,(l_j m_k-l_k m_j), \tag{6.4}$$

where $R_{;2}=\partial R/\partial\theta$ is the derivative of the h-scalar curvature R, appearing in R_{hijk} of $C\Gamma$, by the Landsberg angle θ. Cf. (28.17) of [10]. Next the equation $Lg_{hij}=Im_h m_i m_j$ is well-known, where I is the main scalar. Therefore (5.1') is written

$$X_{ij}=Z_{ij}+IZ_{12}m_i m_j, \tag{6.5}$$

where we put $Z_{12}=Z_{ij}l^i m^j$. Substituting these expressions, it is easily shown that (6.2') becomes

$$(R_{;2}+RI)\,Z_{12}=0. \tag{6.6}$$

Now the F^2 under consideration is a Berwald space as it was shown above. In §8 of the paper [1] Berwald showed that a Berwald space F^2 is locally Minkowski or has the constant main scalar I. Cf. Theorem 28.2 of [10]. As to the locally Minkowski F^2 the condition (6.2') is obviously satisfied by $H_{hijk}=0$. On the other hand, if I of F^2 be constant, (6.6) is solely a consequence of the Bianchi identity (5.15). Therefore (6.2') holds in every case.

As for our F^2 the differential equations (ii) and (iii) of Theorem 3.3 become $\nabla^h X=\nabla^v X=0$ and a non-trivial integrability condition is only $\nabla^h\nabla^h X(1,2)-\nabla^h\nabla^h X(2,1)=0=H(X,\ 1,2)-X(H(1,2))$, which is nothing but (6.2').

Summarizing up all the above results, we have

Theorem 6.2. *A necessary and sufficient condition for an n-dimensional Finsler space F^n to be such that every isometric linear V-rotation be affine with respect to the Berwald connection is that*

(i) $n\geqq 3$: *F^n is a locally Minkowski space.*
(ii) $n=2$: *F^2 is a Berwald space.*

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Shotoku Academy
Gifu College of Education Japan

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