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ON THE SOLUTION OF A PROBLEM OF TRUESDELL'S

BY

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1. Introduction. C. Truesdell [1 — 3] has proposed the following problem :

For an isotropic linearly elastic cylinder subject to end tractions equipollent to a torque M, define the „twist" τ in such a way that

$$M = (\mu R)\ \tau,$$

where μ is the shear modulus and R is a geometric factor determined by the cross-section of the cylinder.

As remarked by Truesdell, such a twist would generalize Saint-Venant's notion of twist so as to apply also to solutions of the torsion problem corresponding to distributions of end tractions different from the one assumed by Saint-Venant.

Truesdell's problem was solved by W. A. Day [4]. P. Podio-Guidugli [5] has solved Truesdell's problem rephrased for extension and pure bending. In [8], we have studied Truesdell's problem for flexure in the case of homogeneous and isotropic solids. In this paper we consider Saint-Venant's problem for inhomogeneous and isotropic bodies. First, we give a rational scheme of deriving solutions for extension, bending, torsion and flexure. The advantage of this method is that it does not involve artificial a priori assumptions. Then, we study Truesdell's problem rephrased for the flexure of inhomogeneous and isotropic cylinders.

2. Preliminaries. We refer to a bounded regular region B of three-dimensional Euclidean space as the body (cf. [6], Sect. 5). We let $\bar{B}$ denote the closure of B, call ∂B the boundary of B, and designate by $\mathbf{n}$ the outward unit normal of ∂B. Letters in boldface stand for tensors of an order $p \geq 1$, and if $\mathbf{v}$ has the order p, we write $v_{ij\ldots k}$ (p subscripts) for the components of $\mathbf{v}$ in the underlying rectangular Cartesian coordinate frame. We shall employ the usual summation and differentiation conventions : Greek subscripts are understood to range over the integers (1,2), whereas Latin subscripts — unless otherwise specified — are confined to the range (1, 2, 3) ; summation over repeated subscripts is implied and subscripts preceded by a comma denote partial differentiation with respect to the corresponding Cartesian coordinate.

We assume that the body occupying B is a linearly elastic material. Let $\mathbf{u}$ be a displacement field over B. Then

$$\mathbf{E}(\mathbf{u})=\operatorname{sym}\nabla\mathbf{u},$$

is the strain field associated with $\mathbf{u}$.

The stress-displacement relation may be written in the form

$$\mathbf{S}(\mathbf{u})=\mathbf{C}[\nabla\mathbf{u}].$$

Here $\mathbf{S}(\mathbf{u})$ is the stress field associated with $\mathbf{u}$, while $\mathbf{C}$ stands for the elasticity field. We assume that $\mathbf{C}$ is positive-definite, symmetric, and smooth on $\bar{B}$. For the particular case of the isotropic elastic medium the tensor field $\mathbf{C}$ admits the representation

$$C_{ijkl}=\lambda\delta_{ij}\delta_{kl}+\mu(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}),$$

where λ and μ are the Lamé moduli and δ_{ij} is the Kronecker delta.

We call a vector field $\mathbf{u}$ an equilibrium displacement field for B if and only if $\mathbf{u}\in C^1(\bar{B})\cap C^2(B)$ and

$$\operatorname{div}\mathbf{S}(\mathbf{u})=0, \tag{2.1}$$

holds on B.

Let $\mathbf{s}(\mathbf{u})$ be the surface traction at regular points of ∂B belonging to the stress field $\mathbf{S}(\mathbf{u})$ defined on $\bar{B}$, i.e.

$$\mathbf{s}(\mathbf{u})=\mathbf{S}(\mathbf{u})\,\mathbf{n}.$$

The strain energy $U(\mathbf{u})$ corresponding to a smooth displacement field $\mathbf{u}$ on $\bar{B}$ is (cf. [6], Sect. 32)

$$U(\mathbf{u})=\frac{1}{2}\int_B\nabla\mathbf{u}\cdot\mathbf{C}[\nabla\mathbf{u}]\,dv. \tag{2.2}$$

In the following, two displacement fields differing by an (infinitesimal) rigid displacement will be regarded identical.

The functional $U(\cdot)$ generates the bilinear functional

$$U(\mathbf{u},\mathbf{v})=\frac{1}{2}\int_B\nabla\mathbf{u}\cdot\mathbf{C}[\nabla\mathbf{v}]\,dv.$$

The set of smooth fields over $\bar{B}$ can be made into a real vector space with the inner product

$$\langle\mathbf{u},\mathbf{v}\rangle=2U(\mathbf{u},\mathbf{v}).$$

This inner product generates the energy norm

$$\|\mathbf{u}\|_e^2=\langle\mathbf{u},\mathbf{u}\rangle. \tag{2.3}$$

For any equilibrium displacement fields $\mathbf{u}$ and $\mathbf{v}$, one has (cf. [6] Sect. 30)

$$\langle\mathbf{u},\mathbf{v}\rangle=\int_{\partial B}\mathbf{s}(\mathbf{v})\cdot\mathbf{u}\,da, \tag{2.4}$$

which implies the well known relation

$$\int_{\partial B}\mathbf{s}(\mathbf{u})\cdot\mathbf{v}\,da=\int_{\partial B}\mathbf{s}(\mathbf{v})\cdot\mathbf{u}\,da. \tag{2.5}$$

3. Properties of the Solutions of Saint-Venant's Problem. We proceed now to Saint-Venant's problem and for this purpose stipulate that the region B from here on refers to the interior of a right cylinder of length h with the open cross-section Σ and the lateral boundary Π. The rectangular Cartesian coordinate frame is supposed to be chosen in such a way that the x_3 - axis is parallel to the generators of B and the x_1Ox_2 plane contains one of the terminal cross-sections. We denote by Σ_1 and Σ_2, respectively, the cross-section located at $x_3=0$ and $x_3=h$. We assume that the generic cross-section Σ is a simply connected regular region. Let us denote by Γ the boundary of Σ.

Saint-Venant's problem consists in the determination of an equilibrium displacement field $\mathbf{u}$ on B, subject to the conditions

$$\mathbf{s}(\mathbf{u})=0 \text{ on } \Pi,\quad \mathcal{F}(\mathbf{u})=\mathbf{F},\quad \mathcal{M}(\mathbf{u})=\mathbf{M}, \tag{3.1}$$

where $\mathbf{F}$ and $\mathbf{M}$ are prescribed vectors representing the resultant force and the resultant moment about O of the tractions acting on Σ_1. Accordingly, $\mathcal{F}(\cdot)$ and $\mathcal{M}(\cdot)$ are the vector-valued linear functionals defined by

$$\mathcal{F}(\mathbf{u})=\int_{\Sigma_1}\mathbf{s}(\mathbf{u})\,da,\quad \mathcal{M}(\mathbf{u})=\int_{\Sigma_1}\mathbf{x}\times\mathbf{s}(\mathbf{u})\,da. \tag{3.2}$$

If $\varepsilon_{\alpha\beta}$ is the two-dimensional alternator, (3.2) appears as

$$\mathcal{F}_i(\mathbf{u})=-\int_{\Sigma_1}S_{3i}(\mathbf{u})\,da, \tag{3.3}$$

$$\mathcal{M}_\alpha(\mathbf{u})=-\int_{\Sigma_1}\varepsilon_{\alpha\beta}x_\beta S_{33}(\mathbf{u})\,da,\quad \mathcal{M}_3(\mathbf{u})=-\int_{\Sigma_1}\varepsilon_{\alpha\beta}x_\alpha S_{3\beta}(\mathbf{u})\,da.$$

By a solution of Saint-Venant's problem we mean any equilibrium displacement field that satisfies (3.1).

The classification of the problem rests on various assumptions concerning the resultants $\mathbf{F}$ and $\mathbf{M}$. Throughout this paper it is convenient to use the decomposition of the problem into problems (P_1) and (P_2) characterized by

(P_1) (extension-bending-torsion) : $F_\alpha=0$,

(P_2) (flexure) : $F_3=M_i=0$.

We denote by $\mathcal{K}_I(F_3, M_1, M_2, M_3)$ the class of solutions to the problem (P_1) and by $\mathcal{K}_{II}(F_1, F_2)$ the class of solutions to the problem (P_2).

We consider that the body is occupied by an isotropic material and assume that

$$\lambda=\lambda(x_1,x_2),\quad \mu=\mu(x_1,x_2),\quad (x_1,x_2)\in\Sigma_1.$$

The functions λ and μ are supposed to belong to $C^2(\Sigma)$.

Let $\mathcal{D}$ denote the set of all equilibrium displacement fields $\mathbf{u}$ that satisfy the condition $\mathbf{s}(\mathbf{u})=0$ on the lateral boundary.

The next theorem will be of future use.

Theorem 3. 1. [8]. *If* $\mathbf{u}\in\mathcal{D}$ *and* $\mathbf{u}_{,3}\in C^1(B)\cap C^2(B)$ *then* $\mathbf{u}_{,3}\in\mathcal{D}$ *and*

$$\mathcal{F}(\mathbf{u}_{,3})=0,\qquad \mathcal{M}_\alpha(\mathbf{u}_{,3})=\varepsilon_{\alpha\beta}\mathcal{F}_\beta(\mathbf{u}),\qquad \mathcal{M}_3(\mathbf{u}_{,3})=0.$$

Theorem 3.1 has the following immediate consequence.

Theorem 3.2 [8]$_2$ *Let* $\mathbf{u}$ *be a solution of the problem* (P_1) *such that* $\mathbf{u}_{,3}\in C^1(B)\cap C^2(B)$. *Then* $\mathbf{u}_{,3}\in\mathcal{D}$, *and*

$$\mathcal{F}(\mathbf{u}_{,3})=0,\qquad \mathcal{M}(\mathbf{u}_{,3})=0.$$

This theorem allows us to establish a simple method of deriving Saint-Venant's solution to the problem (P_1). Let $\mathcal{A}$ be the class of solutions to Saint-Venant's problem corresponding to $\mathbf{F}=0$ and $\mathbf{M}=0$. The preceding theorem asserts that the partial derivative with respect to x_3 of any solution of the problem (P_1) belongs to $\mathcal{A}$. Let us note that a rigid displacement field belongs to $\mathcal{A}$. It is natural to enquire whether there exists a solution $\mathbf{v}$ of the problem (P_1) such that $\mathbf{v}_{,3}$ is a rigid displacement field. This question is settled in the next theorem.

Theorem 3.3. *Let* $\mathbf{v}\in C^1(\bar{B})\cap C^2(B)$ *be a vector field such that* $\mathbf{v}_{,3}$ *is a rigid displacement field. Then* $\mathbf{v}$ *is a solution of the problem* (P_1) *if and only if* $\mathbf{v}$ *is Saint-Venant's solution.*

Proof. Let $\mathbf{v}\in C^1(B)\cap C^2(B)$ such that

$$\mathbf{v}_{,3}=\boldsymbol{\alpha}+\boldsymbol{\beta}\times\mathbf{x},$$

where $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are constant vectors. Then it follows that

$$v_\alpha=-\frac{1}{2}a_\alpha x_3^2-a_4\varepsilon_{\alpha\beta}x_\beta x_3+w_\alpha(x_1,x_2),\tag{3.4}$$
$$v_3=(a_1x_1+a_2x_2+a_3)\,x_3+w_3(x_1,x_2),$$

except for an additive rigid displacement field. Here $\mathbf{w}$ is an arbitrary vector field independent of x_3, and we have used the notations $a_\alpha=\varepsilon_{\rho\alpha}\beta_\rho$, $a_3=\alpha_3$, $a_4=\beta_3$. Let us prove that the functions w_i and the constants $a_s (s=1,4)$ can be determined so that $\mathbf{v}\in\mathcal{K}_I(F_3, M_1, M_2, M_3)$.

The displacement field $\mathbf{v}$ given by (3.4) generates the stresses

$$S_{\alpha\beta}=\lambda(a_\rho x_\rho+a_3)\delta_{\alpha\beta}+\lambda w_{\rho,\rho}\delta_{\alpha\beta}+\mu(w_{\alpha,\beta}+w_{\beta,\alpha}),\tag{3.5}$$

$$S_{3\alpha}=\mu(w_{3,\alpha}-a_4\varepsilon_{\alpha\rho}x_\rho),\tag{3.6}$$
$$S_{33}=(\lambda+2\mu)\,(a_\rho x_\rho+a_3)+\lambda w_{\rho,\rho}.$$

The equations of equilibrium reduce to

$$S_{\alpha\beta,\beta}=0\ \text{ on }\ \Sigma,\tag{3.7}$$

$$(\mu w_{3,\alpha})_{,\alpha}=a_4\varepsilon_{\alpha\rho}(\mu x_\rho)_{,\alpha}\ \text{ on }\ \Sigma.\tag{3.8}$$

The conditions on the lateral boundary become

$$S_{\alpha\beta}n_\beta=0\ \text{ on }\ \Gamma,\tag{3.9}$$

$$\frac{\partial w_3}{\partial n}=a_4\varepsilon_{\alpha\rho}x_\rho n_\alpha\quad \text{ on }\ \Gamma.\tag{3.10}$$

First, we consider the case of homogeneous bodies. The equations (3.7) can be written in the form

$$\mu\Delta w_\alpha+(\lambda+\mu)\,w_{\rho,\rho\alpha}+\lambda a_\alpha=0,$$

from which we get $\Delta w_{\rho,\rho}=0$. From (3.5) we obtain

$$S_{\alpha\alpha}=2\lambda(a_\rho x_\rho+a_3)+2(\lambda+\mu)\,w_{\rho,\rho},$$

so that

$$\Delta S_{\alpha\alpha}=0\ \text{ on }\ \Sigma.\tag{3.11}$$

Thus, we have obtained a two-dimensional traction problem for the simply-connected region Σ, which consists in finding stresses $S_{\alpha\beta}$ that satisfy the equilibrium equations (3.7), the compatibility condition (3.11) and the boundary condition (3.9). It follows that (cf.[6], Sect. 47)

$$S_{\alpha\beta}=0\ \text{ on }\ \Sigma.\tag{3.12}$$

From (3.5) and (3.12) we obtain

$$w_{1,2}=-w_{2,1},\quad w_{1,1}=w_{2,2}=-\nu(a_\rho x_\rho+a_3)\ \text{ on }\ \Sigma,\tag{3.13}$$

where ν designates Poisson's ratio. Integration of (3.13) leads to

$$w_\alpha=\frac{1}{2}a_\alpha\nu x_\rho x_\rho-\nu a_\rho x_\rho x_\alpha-a_3\nu x_\alpha,\tag{3.14}$$

modulo a plane rigid displacement. Clearly,

$$w_3=a_4\varphi(x_1,x_2),\qquad (x_1,x_2)\in\Sigma,\tag{3.15}$$

where, in the case of homogeneous bodies, we have

$$\Delta\varphi=0\ \text{on}\ \Sigma,\quad \frac{\partial\varphi}{\partial n}=\varepsilon_{\alpha\beta}n_\alpha x_\beta\ \text{on}\ \Gamma.\tag{3.16}$$

Thus, the equilibrium equations (2.1) and the condition on the lateral boundary are satisfied if and only if w_i are given by (3.14) and (3.15), where φ is characterized by (3.16). The conditions on the terminal cross-section Σ_1 furnish the following system for the unknown constants

$$E(I_{\alpha\beta}a_\beta+A\,x_\alpha^0a_3)=\varepsilon_{\alpha\beta}M_\beta,\tag{3.17}$$
$$EA(a_\rho x_\rho^0+a_3)=-F_3,\quad \mu Da_4=-M_3,$$

where A is the area of the cross-section, x_α^0 are the coordinates of the centroid of Σ_1, E designates Young's modulus, and

$$I_{\alpha\beta}=\int_{\Sigma} x_\alpha x_\beta da, \qquad D=\int_{\Sigma}(x_\alpha x_\alpha+\varepsilon_{\alpha\beta}x_\alpha\varphi_{,\beta})\,da.$$

The relations (3.4), (3.14)—(3.17) lead to the well-known Saint-Venant's solution.

In the case of inhomogeneous and isotropic bodies, the equations of equilibrium and the condition on the lateral boundary are satisfied if and only if [7]

$$\mathbf{v}=\sum_{j=1}^{4} a_j \mathbf{v}^{(j)}, \tag{3.18}$$

where the vectors $\mathbf{v}^{(j)}(j=\overline{1,4})$ are given by

$$v_\alpha^{(\beta)}=-\frac{1}{2}x_3^2\delta_{\alpha\beta}+w_\alpha^{(\beta)}, \qquad v_3^{(\beta)}=x_\beta x_3, \tag{3.19}$$
$$v_\alpha^{(3)}=w_\alpha^{(3)}, \quad v_3^{(3)}=x_3, \quad v_\alpha^{(4)}=\varepsilon_{\beta\alpha}x_\beta x_3, \quad v_3^{(4)}=\varphi.$$

In (3.19), $w_\alpha^{(j)}(j=1,2,3)$ are the components of the displacement vectors in the following plane strain problems

$$t_{\alpha\beta}^{(j)}=\lambda e_{\rho\rho}^{(j)}\delta_{\alpha\beta}+2\mu e_{\alpha\beta}^{(j)}, \qquad 2e_{\alpha\beta}^{(j)}=w_{\alpha,\beta}^{(j)}+w_{\beta,\alpha}^{(j)},$$
$$t_{\beta\alpha,\beta}^{(\rho)}+(\lambda x_\rho)_{,\alpha}=0, \qquad t_{\beta\alpha,\beta}^{(3)}+\lambda_{,\alpha}=0 \quad \text{on} \quad \Sigma,$$
$$t_{\alpha\beta}^{(\rho)}n_\rho=-\lambda x_\rho n_\beta, \qquad t_{\alpha\beta}^{(3)}n_\beta=-\lambda n_\alpha \quad \text{on} \quad \Gamma,$$

and the function φ is characterized by

$$(\mu\varphi_{,\alpha})_{,\alpha}=\varepsilon_{\alpha\beta}(\mu x_\beta)_{,\alpha} \quad \text{on } \Sigma, \qquad \frac{\partial\varphi}{\partial n}=\varepsilon_{\alpha\beta}x_\beta n_\alpha \quad \text{on } \Gamma.$$

The conditions on the terminal cross-section Σ_1 furnish the following system for the unknown constants

$$D_{\alpha i}a_i=\varepsilon_{\alpha\beta}M_\beta, \quad D_{3i}a_i=-F_3, \quad D^{*}a_4=-M_3, \tag{3.20}$$

where

$$D_{\alpha\beta}=\int_{\Sigma} x_\alpha[(\lambda+2\mu)\,x_\beta+\lambda w_{\rho,\rho}^{(\beta)}]\,da,$$
$$D_{\alpha 3}=D_{3\alpha}=\int_{\Sigma} x_\alpha(\lambda+2\mu+\lambda w_{\rho,\rho}^{(3)})\,da, \tag{3.21}$$
$$D_{33}=\int_{\Sigma}(\lambda+2\mu+\lambda w_{\rho,\rho}^{(3)})\,da, \quad D^{*}=\int_{\Sigma}\mu(\varepsilon_{\alpha\beta}x_\alpha\varphi_{,\beta}+x_\rho x_\rho)\,da.$$

The system (3.20) uniquely determines the constants $a_j(j=\overline{1,4})$ [7]. The converse assertion from Theorem 3.3 is obtained by a simple calculation.

Let us note that the vectors $\mathbf{v}^{(j)}(j=\overline{1,4})$ depend only on the cross-section and the elasticity field. Let $\hat{a}$ be the four-dimensional vector (a_1, a_2, a_3, a_4). We will write $\mathbf{v}\{\hat{a}\}$ for the displacement vector $\mathbf{v}$ defined by (3.18), indicating thus its dependence on the constants $a_s(s=\overline{1,4})$. The next theorem is a direct consequence of the Theorem 3.1.

Theorem 3.4. *If* $\mathbf{u}\in\mathcal{X}_{II}(F_1, F_2)$, *and* $\mathbf{u}_{,3}\in C^1(\overline{B})\cap C^2(B)$ *then*

$$\mathbf{u}_{,3}\in\mathcal{X}_I(0,\ F_2,\ -F_1,\ 0).$$

This is an interesting result concerning the flexure problem.

In [7] we have derived a solution of the problem (P_2) in the case of inhomogeneous and isotropic bodies. In view of Theorems 3.3 and 3.4 it is natural to seek a solution of the problem (P_2) in the form

$$\mathbf{u}^{*}=\int_0^{x_3}\mathbf{v}\{\hat{b}\}\,dx_3+\mathbf{v}\{\hat{c}\}+\mathbf{w}^{*}, \tag{3.22}$$

where $\hat{b}=(b_1, b_2, b_3, b_4)$ and $\hat{c}=(c_1, c_2, c_3, c_4)$ are two constant four-dimensional vectors, and $\mathbf{w}^{*}$ is a vector field independent on x_3 such that $\mathbf{w}^{*}\in C^1(\Sigma)\cap C^2(\Sigma)$. We are led to the following theorem, the proof of which is analogous to that given for Theorem 3.3 (see also [7], [8]).

Theorem 3.5. *The vector field* $\mathbf{u}^{*}$ *defined by* (3.22) *is a solution of the problem* (P_2), *if and only if* $\mathbf{u}^{*}$ *is the solution given in* [7].

It is interesting to note that the determination of $\hat{b}$ from condition $\mathbf{u}^{*}\in\mathcal{X}_{II}(F_1, F_2)$ can be made in a simple way. Thus, if $\mathbf{u}^{*}\in\mathcal{X}_{II}(F_1, F_2)$, then by Theorem 3.4,

$$\mathbf{v}\{\hat{b}\}\in\mathcal{X}_I(0, F_2, -F_1, 0). \tag{3.23}$$

In view of (3.20) and (3.23) we obtain

$$D_{\alpha i}b_i=-F_\alpha, \qquad D_{3i}b_i=0, \tag{3.24}$$

and $b_4=0$.

4. On the Solution of Truesdell's Problem for Flexure. Let $\mathfrak{M}$ denote the set of all equilibrium displacement fields $\mathbf{u}$ that satisfy the conditions

$$\mathbf{u}_{,3}\in C^1(\overline{B})\cap C^2(B), \quad \mathbf{s}(\mathbf{u})=0 \text{ on } \Pi,$$
$$(S_{3\alpha}(\mathbf{u}_{,3}))\,(x_1, x_2, 0)=(S_{3\alpha}(\mathbf{u}_{,3}))\,(x_1, x_2, h), \quad (x_1, x_2)\in\Sigma_1, \tag{4.1}$$
$$\mathcal{F}_3(\mathbf{u})=0, \qquad \mathcal{M}(\mathbf{u})=0.$$

Clearly, if $\mathbf{u}\in\mathfrak{M}$ then $\mathbf{u}\in\mathcal{X}_{II}(\mathcal{F}_1(\mathbf{u}), \mathcal{F}_2(\mathbf{u}))$. In order to study Truesdell's problem, we note first that if $\mathbf{u}\in\mathfrak{M}$, then by Theorem 3.4, $\mathbf{u}_{,3}\in\mathcal{X}_I(0, \mathcal{F}_2(\mathbf{u}), -\mathcal{F}_1(\mathbf{u}), 0)$. Following [4, 5, 8] we are led to define $\hat{b}$ as the minimum of the real function

$$f(\xi_1, \xi_2, \xi_3)=\left\|\mathbf{u}_{,3}-\sum_{i=1}^{3}\xi_i\mathbf{v}^{(i)}\right\|_e^2, \tag{4.2}$$

where $\mathbf{u}\in\mathfrak{M}$ and $\mathbf{v}^{(i)}$ are the equilibrium displacement fields given by (3.19). Clearly, the function f will be a minimum at $\xi_i=\beta_i(\mathbf{u})$ if and only if $(\beta_1(\mathbf{u}), \beta_2(\mathbf{u}), \beta_3(\mathbf{u}))$ is the solution of the following system of equations

$$\sum_{j=1}^{3}<\mathbf{v}^{(i)}, \mathbf{v}^{(j)}>\beta_j(\mathbf{u})=<\mathbf{u}_{,3}, \mathbf{v}^{(i)}> \quad (i=1, 2, 3). \tag{4.3}$$

As in [4], [5], by (2.5), (3.19) and Theorem 3.1 we find

$$<\mathbf{u}_{,3},\ \mathbf{v}^{(\alpha)}> = \int_{\partial B} \mathbf{s}(\mathbf{u}_{,3}) \cdot \mathbf{v}^{(\alpha)}\, da = h\varepsilon_{\alpha 3}\mathcal{M}_3(\mathbf{u}_{,3}),$$

$$<\mathbf{u}_{,3},\ \mathbf{v}^{(3)}> = 0.$$

In view of Theorem 3.1 we arrive at

$$<\mathbf{u}_{,3},\ \mathbf{v}^{(\alpha)}> = -h\mathcal{F}_\alpha(\mathbf{u}).$$

A simple calculation yields

$$<\mathbf{v}^{(i)},\ \mathbf{v}^{(j)}> = h\ D_{ij},$$

where D_{ij} are given by (3.21).

Thus, the system (4.3) reduces to

$$D_{\alpha i}\beta_i(\mathbf{u}) = -\mathcal{F}_\alpha(\mathbf{u}), \qquad D_{3i}\beta_i(\mathbf{u}) = 0, \tag{4.4}$$

so that the formulae (3.24) apply for any $\mathbf{u} \in \mathfrak{M}$.

On the other hand, by (2.4) we obtain

$$<\mathbf{u}_{,3},\ \mathbf{v}^{(i)}> = \int_{\partial B} \mathbf{s}(\mathbf{v}^{(i)}) \cdot \mathbf{u}_{,3}\, da = E_i(\mathbf{u}), \tag{4.5}$$

where

$$E_\alpha(\mathbf{u}) = \int_{\Sigma_2} [(\lambda+2\mu)\, x_\alpha + \lambda u^{(\alpha)}_{\rho,\rho}]\, u_{3,3}\, da - \int_{\Sigma_1} [(\lambda+2\mu)\, x_\alpha + \lambda u^{(\alpha)}_{\rho,\rho}]\, u_{3,3}\, da,$$

$$E_3(\mathbf{u}) = \int_{\Sigma_2} (\lambda+2\mu+\lambda u^{(3)}_{\rho,\rho})\, u_{3,3}\, da - \int_{\Sigma_1} (\lambda+2\mu+\lambda u^{(3)}_{\rho,\rho})\, u_{3,3}\, da.$$

If we use for $<\mathbf{u}_{,3},\ \mathbf{v}^{(i)}>$ the expressions from (4.5) then the system (4.3) becomes

$$D_{\alpha i}\beta_i(\mathbf{u}) = \frac{1}{h} E_i(\mathbf{u}), \qquad D_{3i}\beta_i(\mathbf{u}) = 0.$$

The system defines $\beta_i(\mathbf{u})$ associated with any displacement field $\mathbf{u} \in \mathfrak{M}$, as global measures of strain.

Thus, Truesdell's problem as formulated in the Introduction can be solved also for the flexure of inhomogeneous and isotropic bodies.

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