$$\text{(III')}\quad (\forall)x, y, z\in M,\ ((x, y, x)_\tau,\ (x, y, x)_\tau, z)_\tau = ((x, x, z)_\tau,\ (y, y, z)_\tau,\ (x, x, z)_\tau)_\tau,$$

$$\text{(IV')}\quad (\forall)x, y\in M,\ (x, x, y)_\tau=(y, y, x)_\tau;\ (x, y, x)_\tau=(y, x, y)_\tau,$$

$$\text{(V')}\quad (\forall)x, y, z, \in M,\ ((x, x, y)_\tau, (x, x, y)_\tau, z)_\tau=(x, x, (y, y, z)_\tau)_\tau,$$

$$\text{(VI')}\quad (\forall)x, y, z\in M,\ ((x, y, z)_\tau, z, (x, y, x)_\tau)_\tau=(x, (y, z, y)_\tau, x)_\tau,$$

$$\text{(VII')}\quad (\forall)x\in M,\ (x, x, e_3)_\tau=e_3;\ (x, e_1, e_1)_\tau=e_1,$$

$$\text{(VIII')}\quad (\forall)x\in M,\ (\exists)\overline{x}\in M,\ (x, x, \overline{x})_\tau=e_3,\ (x, \overline{x}, x)_\tau=e_1.$$

Conditions that the operations β_2, β_3 lead to a Boole algebra in M are obtained in the same way.

Remark 4. Considering the ternary operation τ which fulfils the associativity condition :

$$(\forall)x, y, z, u, v\in M,\ ((x, y, z), u, v)_\tau=((x, z, v)_\tau, y, u)_\tau,$$

for $y=z$, $u=v$ and by applying again the associativity law we get :

$$((x, y, y)_\tau, u, u)_\tau=((x, u, u)_\tau, y, y)_\tau,$$

or

$$((x, y)_{\beta_2}, u)_{\beta_2}=((x, u)_{\beta_2}, y)_{\beta_2},\ (\forall)x, y, u\in M.$$

If β_2 has a left identity element e_2, then β_2 is a binary commutative operation, for, considering $x=e_2$, we get :

$$(\forall)y, u\in M,\ (y, u)_{\beta_2}=(u, y)'_{\beta_2}.$$

Provided that β_1 too admits a trivial left element e_1 so that : $(e_1, x)_{\beta_1}= =(e_1, e_1, x)_\tau=x$, $(\forall)x\in M$ and moreover

$$((x, x, y)_\tau, z, z)_\tau=((x, z, z)_\tau, (x, z, z)_\tau, (y, z, z)_\tau)$$

$$((x, y, y)_\tau,\ (x, y, y)_\tau, z)_\tau=((x, x, z)_\tau, (y, z, z)_\tau, (y, y, z)_\tau)$$

then :

$$(x, x)_{\beta_1}=(x, x, x)_\tau=x=(x, x)_{\beta_2}.$$

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A NOTE ON ADMISSIBLE NULL CONTROLLABILITY AND ON VARIATIONAL PERTURBATIONS OF THE MINIMUM TIME PROBLEM

BY

OVIDIU CÂRJĂ

1. Introduction. We consider in this paper the abstract control system

$$x'(t)=Ax(t)+Bu(t),\ t\geq 0, \tag{1.1}$$

where the state $x(t)$, $t\geq 0$, takes values in a reflexive Banach space X and the control $u(t)$, $t\geq 0$, is in another reflexive Banach space U. Assume the operator A generates a strongly continuous semigroup $S(t)$ on X for $t\geq 0$, and B is a bounded linear operator from U into X.

For $x(0)=x_0\in X$ and $u\in L^1_{loc}(R^+; U)$, the solution of the equation (1.1) is understood in the mild sense, that is,

$$x(t)=S(t)\,x_0+\int_0^t S(t-s)\,Bu(s)\,ds,$$

where the integral is in the sense of Bochner.

We take the set of admissible controls as

$$U_{ad}=\{u\in L^\infty_{loc}(R^+; U);\ \|u(t)\|\leq 1\ \text{a.e.}\}. \tag{1.2}$$

In [10], the author gives a sufficient condition of admissible null controllability for (1.1). More precisely, it is proved there that if $S(t)$ is a contraction semigroup and if the system (1.1) is null controllable on $[0, T]$ for some $T>0$, then it is admissible null controllable in finite time. Here we prove the above result with a weaker condition on $S(t)$ (Corollary 1). The approach is different and allows to obtain additional informations about the transferable sets (Theorem 2). Unfortunately, we are forced to impose the reflexivity to the spaces X and U while in [10] they are general Banach spaces. This is done in Section 2.

In Section 3 we are concerned with variational perturbations of the problem of reaching the origin from an initial point in minimum time. This problem, which is very important from the numerical point of view has been considered, by a number of authors (see, for example, [6], [11] when dim X is finite). In the infinite dimensional case, Sasai and

Shimemura [12] consider $S(t)$ an analytic semigroup and $B=I$, Knowles [8] and Lasieska [9] study a Ritz-Galerkin approximation when the control acts on the boundary and the target is a ball. In [2] we have studied this problem under the assumption that the original system together with the perturbed systems are exactly controllable on $[0, t]$ for each $t>0$ and furthermore the perturbed systems are exactly controllable on $[0, t]$ uniformly with respect to n. Here we relax that assumption by replacing the words „exactly controllable" by „null controllable". The result can be applied to systems of type (1.1) with $B=I$ (the identity operator in X).

Apparently, the two objectives of this paper are quite different. However they are unified by the main tool used for the proof of the results. Namely, it is given by the following theorem on majorisation and comajorisation of operators in Banach spaces.

Theorem 1. [7]. *Let X, Y, Z be Banach spaces and let $E\in L(X, Y)$, $F\in L(X, Z)$. The following three conditions are equivalent* :

(i) range $E^*\subset$ range F^*,

(ii) $\gamma\|Ex\|\leqslant\|Fx\|$, *for all $x\in X$ and for some $\gamma>0$,*

(iii) $E^*B(0, \delta)\subset F^*B(0, 1)$, *for some $\delta>0$.*

Moreover, in the equivalence (ii)⇔(iii) one can take $\gamma=\delta$.

We denote by $L(X, Y)$ the space of all linear and bounded operators from X into Y, while $B(0, \delta)$ stands for the closed ball centered at zero with radius δ.

2. Admissible null controllability. Let us introduce the operator

$$V(T): L^\infty(0, T; U)\to X \text{ by } V(T)\,u=\int_0^T S(T-s)\,Bu(s)\,ds.$$

The transferable set at time T is defined by

$$R(T)=\{x\in X\ ;\ S(T)\,x=V(T)\,u \text{ for some } u\in U_{ad}\}.$$

In fact, $R(T)$ is the set of all initial points which can be transferred to zero during $[0, T]$ with admissible controls.

The transferable set in free time is defined by

$$R=\bigcup_{T>0} R(T).$$

Definition 1. *The control system is called null controllable on $[0, T]$ if all points of X can be transferred to zero during $[0, T]$ with controls in $L^\infty(0, T; U)$. This is equivalent to*

$$\text{range } S(T)\subset\text{range } V(T).$$

Definition 2. *The control system (1.1) is called admissible null controllable if all points of X can be transferred to zero in finite time with controls in U_{ad}. This means $R=X$.*

Our aim here is to search remarkable subsets of R, $R(T)$ and also to give sufficient conditions to assure that the system (1.1) is admissible null controllable.

Define $G(T): X^*\to L^1(0, T; U^*)$ by $G(T)\,x^*=B^*S^*(T-.)\,x^*$. So, the adjoint of $G(T)$, $G^*(T): L^\infty(0, T; U)\to X$ is exactly $V(T)$ defined above.

Remark 1. By Theorem 1 we can conclude that the system (1.1) is null controllable on $[0, T]$ if and only if

$$\int_0^T\|B^*S^*(s)\,x^*\|\,ds\geqslant C(T)\,\|S^*(T)\,x^*\| \quad\text{for some } C(T)>0 \tag{2.1}$$

and for all $x^*\in X^*$. Moreover, (2.1) holds if and only if all points from the ball $B(0, C(T))$ can be transferred to zero during $[0, T]$ with admissible controls.

The main result of this section is

Theorem 2. *Let the system (1.1) be null controllable on $[0, T]$ for some $T>0$ and more exactly let (2.1) hold. Then for integers $n\geqslant 1$, all points from the ball $B(0, C(T)\sum_{k=0}^{n-1} 1/\|S(kT)\|)$ can be transferred to zero during $[0, nT]$ with admissible controls. This means*

$$B(0, C(T)\sum_{k=0}^{n-1} 1/\|S(kT)\|)\subset R(nT). \tag{2.2}$$

Also we have

$$\text{int } B(0, C(T)\sum_{k=0}^{\infty} 1/\|S(kT)\|)\subset R. \tag{2.3}$$

Proof. If (2.1) holds, we have

$$\int_0^{nT}\|B^*S^*(s)\,x^*\|\,ds=\sum_{k=0}^{n-1}\int_{kT}^{(k+1)T}\|B^*S^*(s)\,x^*\|\,ds=$$

$$=\sum_{k=0}^{n-1}\int_0^T\|B^*S^*(kT+s)\,x^*\|\,ds=\sum_{k=0}^{n-1}\int_0^T\|B^*S^*(s)\,S^*(kT)\,x^*\|\,ds\geqslant$$

$$\geqslant C(T)\sum_{k=0}^{n-1}\|S^*(T)\,S^*(kT)\,x^*\|\geqslant C(T)\sum_{k=0}^{n-1}(1/\|S^*(kT)\|)\,\|S^*(nT)\,x^*\|.$$

By Theorem 1 and Remark 1, the assertion of Theorem 2 follows easily.

Corollary 1. *Under the assumption of Theorem 2, if the series $\sum_{k=0}^{\infty} 1/\|S(kT)\|$ is divergent, then (1.1) is admissible null controllable, that is,*

$$R=X. \tag{2.4}$$

In particular, if

$$\|S(t)\|\leqslant Mt \text{ for } t\to\infty, \tag{2.5}$$

then (2.4) holds.

The assumption that the control system (1.1) is null controllable on $[0, T]$ for some $T>0$ is in fact a necessary condition for admissible null controllability. By a standard argument using the Baire Category Theorem, one can prove

Theorem 3. *If the control system* (1.1) *is admissible null controllable then there exists $T>0$ such that it is null controllable on* $[0, T]$.

It would be interesting to obtain a characterization for the admissible null controllability. In the finite dimensional case, it is proved in [4] that the admissible null controllability condition is equivalent to condition (a) below.

$$(a) \qquad \int_0^\infty \| B^*S^*(-t)\, x^* \| \, dt = +\infty \quad \text{for all} \quad x^* \in X^*, \ x^* \neq 0.$$

(see also [13] for other results in the finite dimensional case).

Remark 2. Results of the same kind as in Theorem 2 can be obtained if instead of (1.2) we take

$$U_{ad} = \bigcup_{T>0} \{u \in L\,(0, T ; U) : \| u \|_p \leqslant 1\}, \quad p>1.$$

In this case, instead of (2.2) and (2.3) we shall have

$$B(0, C(T) \sum_{k=0}^{n-1} 1/\| S(kT) \|^q) \subset R(nT) \ ; \ \operatorname{int} B(0, C(T) \sum_{k=0}^{\infty} 1/\| S(kT) \|^q) \subset R,$$

where $1/p + 1/q = 1$.

Remark 3. Similar results can be obtained for boundary-distributed control systems, using the semigroup representation for solution (see [1]). More precisely, consider the control system

$$(2.6) \qquad x'(t) = Ax(t) + \Lambda u(t), \quad y(t) = \Pi x(t), \quad y(0) = y_0,$$

where A is as above, $\Pi = A - \lambda_0 I$, $\lambda_0 \in \rho(A)$ (the rezolvent of A), $\Lambda \in L(U, X)$ Assume the following condition: for each $t>0$ and $u \in U$, $S(t)\Lambda u \in \mathcal{D}(A)$ (the domain of A) and there exists $\gamma \in L^q_{loc}\,(R^+)$ such that

$$\| \Pi S(t)\Lambda \| \leqslant \gamma(t), \quad a.e. \ t>0.$$

Under this assumption, if $u(.) \in L^p_{loc}\,(R^+ ; U)$, we can define the mild solution of (2.6) by

$$y(t) = S(t)\, y_0 + \int_0^t \Pi S(t-s)\Lambda u(s)\, ds$$

and moreover $y(.)$ is a continuous function. As results from [1], the principal parabolic boundary control systems can be written in the form (2.6). The system (2.6) is null controllable on $[0, T]$ (with controls in $L\,(0, T ; U)$ if and only if

$$\int_0^T \| (\Pi S(s)\Lambda)^* x^* \|^q \, ds \geqslant C(T)\, \|S^*(T)\, x^* \|^q, \text{ for all } x^* \in X^*.$$

The proof of Theorem 2 for the system (2.6) is similar to that for (1.1) using the commutativity of Π with $S(t)$ on $\mathcal{D}(A)$ and

$$(\Pi S(t+s)\Lambda)^* = (\Pi S(t)\Lambda)^* S^*(s).$$

We conclude this section with the special case $B=I$ in (1.1). So we have the equation

$$(2.7) \qquad x'(t) = Ax(t) + u(t), \quad t \geqslant 0.$$

Simple calculations give

$$(2.8) \qquad \int_0^t \| S^*(s)\, x^* \| \, ds \geqslant [\exp(\omega t) - 1)/\omega M \exp(\omega t)] \, \| S^*(t)\, x^* \|$$

for all $x^* \in X^*$, where $M \geqslant 1$ and $\omega \in R^1$ (the real line) are such that $\| S(t) \| \leqslant \leqslant M \exp(\omega t)$, $t \geqslant 0$.

Corollary 2. *For the control system* (2.7) *we have*

(i) $B(0, [(\exp(\omega t)-1)/\omega M \exp(\omega t)] \sum_{k=0}^{n-1} 1/\| S(kt) \|) \subset R(nt)$ *for integers* $n \geqslant 1$ *and* $t>0$.

(ii) $\operatorname{int} B(0, [(\exp(\omega t)-1)/\omega M \exp(\omega t)] \sum_{k=0}^{\infty} 1/\|S(kt)\|) \subset R$. *for* $t>0$.

(iii) $\operatorname{int} B(0, 1/\omega M) \subset R$ *for* $\omega>0$ *and* $R = X$ *for* $\omega \leqslant 0$.

(iv) *If for some $t>0$ the series* $\sum_{k=0}^{\infty} 1/\| S(kt) \|$ *is divergent, then* $R = X$.

3. **Variational perturbations of the minimum time problem.** Along with the control system (1.1), let us consider the following sequence of perturbed control systems

$$(3.1) \qquad x'(t) = A_n x(t) + B_n u(t), \quad t \geqslant 0,$$

for $n = 1, 2, \ldots$, in the space X. For $n \geqslant 1$. A_n are assumed to be the infinitesimal generators of the strongly continuous semigroups of bounded linear operators $S_n(t)$ on X and $B_n \in L(U, X)$.

Our aim is to give conditions for the convergence of the sequence of minimum times of the approximate systems (3.1) to the minimum time of the original system (1.1).

We now list the assumptions which will be in effect throughout this section.

(A.1) *There exist* $M>0$ *and* $\omega \in R^1$ *such that*

$$\| S_n(t) \| \leqslant M \exp(\omega t), \ \textit{for} \ n = 1, 2, \ldots \ \textit{and} \ t \geqslant 0.$$

Here M and ω are independent of n.

(A.2) *$S_n(t)\, x$ converges to $S(t)\, x$ for any $x \in X$, uniformly for t in* ***bounded*** *intervals.*

(A.3) *$B_n u$ converges to Bu for any $u \in U$.*

(A.4) *$B_n^* S_n^*(t)\, x^*$ converges to $B^* S^*(t)\, x^*$ for any $x^* \in X^*$, uniformly for t in bounded intervals.*

(A.5) *There exists a function $c : (0, \infty) \to (0, \infty)$ such that the following condition holds : for each $t>0$ there exists a positive integer $n(t)$ such* ***that*** *for $n \geqslant n(t)$,*

$$\int_0^t \| B_n^* S_n^*(s)\, x^* \| \, ds \geqslant c(t)\, \| S_n^*(t)\, x^* \|, \ \textit{for any} \ x^* \in X^*.$$

Remark 4. Condition (A.5) says that the original system together with the perturbed systems are null controllable on $[0, t]$ for each $t>0$

uniformly with respect to n. As results from [5], the null controllability is fragil in the presence of perturbations. However, under Assumption (A.1), in the case $B_n = B = I$ the condition (A.5) is satisfied, because of (2.8).

Let us denote by $R_n(t)$ and R_n, $n = 1, 2, \ldots$, the transferable set and the transferable set in free time associated with the system (3.1). Define also the functions

$$T_n(x) = \inf\{t\ ;\ x \in R_n(t)\},\quad n = 1, 2, \ldots, x \in R_n$$

and

$$T(x) = \inf\{t\ ;\ x \in R(t)\},\quad x \in R,$$

which are called the minimum time functions associated with the systems (1.1) and (3.1), respectively.

The main result of this section is

Theorem 4. *Assume* (A.1)—(A.5). *If* $x \in R$ *and* x_n *converges to* x_0 *then* $x_n \in R_n$ *for* n *sufficiently large and* $T_n(x_n)$ *converges to* $T(x)$.

For the proof of Theorem 4 we need two lemmas.

Lemma 1. *Assume* (A.1)—(A.4). *If* $x_n \in R_n(t_n)$, t_n *converges to* T *and* x_n *converges to* x, *then* $x \in R(T)$.

For the proof see [2].

Lemma 2. *Assume* (A.1), (A.3) *and* (A.5). *If* x_n *converges to* x *and* $x \in R(t)$, *then for each* $\varepsilon > 0$ *there exists* $n(\varepsilon) > 0$ *such that for* $n \geq n(\varepsilon)$ *we have* $x_n \in R_n(t + \varepsilon)$.

Proof. Let $\varepsilon > 0$ be fixed. Since $x \in R(t)$, there exists $u \in U_{ad}$ such that

$$S(t)\, x = \int_0^t S(t-s)\, Bu(s)\, ds. \tag{3.2}$$

By (A.5) and Remark 1 there exists $n(\varepsilon)$ such that for any $n \geq n(\varepsilon)$ and any $x \in X$ with $\|x\| \leq c(\varepsilon)$ there exists $u_1^n \in U_{ad}$ such that

$$S_n(\varepsilon)\, x = \int_0^\varepsilon S_n(\varepsilon - s)\, B_n u_1^n(s)\, ds.$$

On the other hand, by (A.1), (A.2) and (A.4) we have

$$S_n(t)\, x_n \text{ converges to } S(t)\, x\ ;$$

$$\int_0^t S_n(t-s)\, B_n u(s)\, ds \text{ converges to } \int_0^t S(t-s)\, Bu(s)\, ds.$$

These facts imply the existence of a positive integer n_1 such that for $n \geq n_1$ we have

$$\left\| S_n(t)\, x_n - \int_0^t S_n(t-s)\, B_n u(s)\, ds \right\| \leq c(\varepsilon).$$

Thus, for $n \geq \max\,(n_1, n(\varepsilon))$, there exists $u_n \in U_{ad}$ such that

$$S_n(\varepsilon)\, S_n(t)\, x_n - S_n(\varepsilon) \int_0^t S_n(t-s)\, B_n u(s)\, ds = \int_0^\varepsilon S_n(\varepsilon - s)\, B_n u_n(s)\, ds.$$

This means

$$S_n(t+\varepsilon)\, x_n = \int_0^{t+\varepsilon} S_n(t+\varepsilon-s)\, B_n v_n(s)\, ds$$

where

$$v_n(s) = \begin{cases} u(s) & \text{on } [0, t] \\ u_n(s-t) & \text{on } [t, t+\varepsilon]. \end{cases}$$

Clearly $v_n \in U_{ad}$. Hence $x_n \in R_n(t+\varepsilon)$ and the proof is complete.

Now, the proof of Theorem 4 is the same as that of Theorem 2.1 in [2]. However for the reader's convenience we shall give it here.

Proof of Theorem 4. From Lemma 2 we deduce that the sequence $T_n(x_n)$ is bounded. Thus, there exists a subsequence of $T_n(x_n)$ denoted $T_{n'}$, which converges, say to T'. Taking $t = T(x)$ in Lemma 1 we have: for every $\varepsilon > 0$ there exists $n(\varepsilon)$ such that for $n' > n(\varepsilon)$, we have $T_{n'} \leq T(x) + \varepsilon$. Thus $T' \leq T(x)$. On the other hand, by Lemma 1, we deduce $x_0 \in R(T')$ and so $T(x) \leq T'$. Thus $T' = T(x)$. Since the last equality can be obtained for all convergent subsequences, the proof is complete.

Remark 5. In particular, Theorem 4 says that if the system (1.1) is null controllable on $[0, t]$ for each $t > 0$, then the minimum time function, is continuous. This is the main result of [3].

Remark 6. In infinite dimensional case, for practical applications, it is important that the spaces X and U be also approximated by a sequence of finite dimensional spaces. This is the case, for example, in [2] where the spaces X, U and the operators A and B are approximated in the sense of Trotter. Theorem 4 can be proved in the setting of [2] but we have adopted the above setting for only simplicity.

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