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ON CATEGORIES OF H-CONES AND OF HYPERHARMONIC CONES

BY

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§0. In [2] Eriksson proposes a definition for morphisms between hyperharmonic cones. Results concerning the connection with the definition proposed in [4] for the morphisms of H-cones, are given in [3]. The present paper begins with an alternative description of the hyperharmonic cone associated with an H-cone, which is eassier to handle in some situations. In § 2, the different cones associated to an H-cone, are compared. § 3 contains the main result (Theorem 4), which asserts an adjunction situation between a category of H-cones and a category of hyperharmonic cones.

The notions and notations about H-cones are as in [1]. For questions about categories, see [5]. The notion of hyperharmonic cone was introduced in [2].

§ 1. Let S be an H-cone. We denote by $H(S)$ the hyperharmonic cone constructed in [3]. Let us recall that the elements of $H(S)$ are the classes of non-void, increasing subsets of S, for the equivalence relation associated with the following preorder relation :

$$A \leqslant B \text{ iff } \bigvee s \wedge A \leqslant \bigvee s \wedge B, \quad \forall s \in S.$$

We prove next, that there exists a canonical, complete system of representatives. This fact will shorten notably certain proofs. Notice for instance, that the order relation will reduce simply to the inclusion.

Let us denote by $\tilde{S}$ the set of all subsets $A \subseteq S$, which are non-void, solid and conditionally upper complete. $\tilde{S}$ will be endowed with the order relation given by the inclusion.

Proposition 1. *The map $\varphi : \tilde{S} \to H(S)$, defined by : $\varphi(A)$ is the equivalence class of A, is a bimonotone bijection.*

Proof. Since any $A \in \tilde{S}$ is non-void and increasing, φ is well defined. If $A \subseteq B$, then clearly $\varphi(A) \leqslant \varphi(B)$, hence φ is monotone. Conversely, let A, $B \in \tilde{S}$ be such that $\varphi(A) \leqslant \varphi(B)$. For any $s \in A$, we have : $s = \bigvee s \wedge A \leqslant \bigvee s \wedge B \leqslant s$. Let us denote : $s_b = s \wedge b$, $\forall b \in B$. $(s_b)_{b \in B}$ is an increasing and bounded family from B. Hence : $s = \bigvee\limits_{b \in B} s_b \in B$, and φ results bimonotone. In order to define the inverse map, we need :

Lemma 2. *Let $A \subseteq S$ be non-void and increasing. Then :*
$A' = \{s \in S \mid \exists F \subseteq S, F$ *increasing*, $s = \bigvee F$, $\forall t \in F\ \exists u \in A$ *such that* $t \leqslant u\}$ *is*

the smallest subset of S, which contains A, and is solid and conditionally upper complete.

Proof. It is clear that there exists the smallest subset of S, which contains A and is solid, conditionally upper complete. Let $\bar{A}$ be this set. We prove first that A' is solid and conditionally upper complete. If $s \in A'$ and $t \leqslant s$, then : $t = \bigvee t \wedge F$. Let $s, t \in A'$ be such that : $s = \bigvee F$ and $t = \bigvee F'$. Define F'' as the set of all the elements of the form : $(u \vee v) \wedge (s \vee t)$, where $u \in F$, $v \in F'$. F'' is increasing and : $s \vee t = \bigvee F''$. Since A is increasing, it follows that $s \vee t \in A'$. Finally, let $F \subseteq A$ be increasing and bounded in S. Let us denote $s = \bigvee F$ and : $A_s = \{u \in S \mid \exists t \in F$ such that $u \leqslant t$, $\exists v \in A$ such that $u \leqslant v\}$ A is increasing and dominated by s. Since for any $t \in F$ we have : $t \leqslant \bigvee A_s$, it follows that : $s = \bigvee A_s \in A'$. Since obviously $\bar{A} \subseteq A'$ we get $A \subseteq A'$. Conversely, let $s \in A'$, and let F be as in the definition. Let, for each $t \in F$, $u \in A$ be such that $t \leqslant u$. Hence $u \in \bar{A}$, so that $t \in \bar{A}$ and finally $s = \bigvee F \in A$.

We continue the proof of the theorem. Let A, B be non-void, increasing, such that $[A] \leqslant [B]$. For any $s \in A'$ we have: $s = \bigvee F = \bigvee s \wedge F \leqslant \bigvee s \wedge A$ hence: $s \leqslant \bigvee s \wedge B \leqslant s$. Let us denote : $s_b = s \wedge b$, $\forall b \in B$. $(s_b)_{b \in B}$ is an increasing family from B, and : $s = \bigvee_{b \in B} s_b$, hence $s \in B'$. It results that $A' \subseteq B'$. From these facts, we get a well defined, monotone map : $\psi : H(S) \to \tilde{S}$, $\psi([A]) = A'$. Directly from the definition we have : $(\psi \circ \varphi)(A) = \psi([A]) = A' = A$, $\forall A \in \tilde{S}$. Finally, let A be non-void and increasing. For any $s \in S$ and $t \in A'$, we have : $s \wedge t = s \wedge \bigvee F = \bigvee s \wedge F \leqslant \bigvee s \wedge A$ hence : $\bigvee s \wedge A' \leqslant \bigvee s \wedge A$. The converse inequality being obvious it follows that A and A' are equivalent (in the sense of [3]), hence :

$$(\varphi \circ \psi)([A]) = \varphi(A') = [A'] = [A].$$

Remark. Defining cone operations on $\tilde{S}$ by : $(A, B) \to A + B$; $(\alpha, A) \to \alpha . A$, we obtain that $\tilde{S}$ is isomorphic to $H(S)$ as an ordered convex cone and a complete lattice.

§2. Let S be an H-cone and $\hat{S}$ be the H-cone of all cancellable elements from $H(S)$ ([2]). We are going to establish some connections of $\hat{S}$ with other cones.

Proposition 3. *Let S be an H-cone of functions on a set X. There exists a bimonotone, additive bijection between $\hat{S}$ and $\bar{S}$* [1, th. 3.5]

Proof. We define $\psi : \bar{S} \to \hat{S}$ as follows : if $f \in \bar{S}$, then : $\psi(f) = \{s \in S \mid s \leqslant f\}$ Since S is solid in $\bar{S}$, it follows that $\psi(f) \in H(S)$. Moreover, $\psi(f)$ is cancellable in $H(S)$: if $s \in \alpha \cdot \psi(f)$, $\forall \alpha > 0$, then : $\alpha^{-1} . s \leqslant f$, $\forall \alpha > 0$, hence $[s = 0]$ is a finely dense set. We get thus $s = 0$, and ψ is correctly defined. Clearly ψ is monotone. If $\psi(f) \leqslant \psi(g)$ then $f \leqslant g$, since S is increasingly dense in $\bar{S}$. Finally, ψ is additive from the Riesz decomposition property. We define next $\varphi : \hat{S} \to \bar{S}$: for $A \in \hat{S}$, $\varphi(A) : X \to \bar{R}_+$ is the map defined by : $\varphi(A)(x) = \sup_{s \in A} s(x)$ φ is correctly defined : if $[\varphi(A) = +\infty]$ would have non-void fine interior, there exists u, $v \in S$ such that $u \leqslant v$ on X and : $\varnothing \neq [u < v] \subseteq [\varphi(A) = +\infty]$. For each $t \in S$ we have : $\sup_{s \in A} (\min (t(x), u(x) + s(x))) \geqslant \sup_{s \in A} (\min (t(x), v(x) + s(x)))$, $\forall x \in X$. Hence : $u + A \geqslant v + A$. Since A is cancellable, it follows $u \geqslant v$, contradiction. Hence φ is correctly defined. Now :

$$(\varphi \circ \psi)(f)(x) = \varphi[s \leqslant f](x) = \sup_{s \in f} s(x) = f(x).$$

Conversely, let $A \in \hat{S}$. Then :

$$(\psi \circ \varphi)(A) = \{s \in S \mid s(x) \leqslant \sup_{t \in A} t(x)\}.$$

Let A' denote the set in the second member. Clearly $A \subseteq A'$. It remains to prove that :

$$\bigvee u \wedge A' \leqslant \bigvee u \wedge A, \quad \forall u \in S$$

which comes to :

$$\sup_{s \in A'} (\min (u(x), s(x))) \leqslant \sup_{t \in A} (\min (u(x), t(x))) = \min (u(x), \sup_{t \in A} t(x))$$

and this is true from the definition of A'.

Remarks. a) If S is a standard H-cone of functions on the saturated space X, then $\bar{S}$ and S^{**} are isomorphic [1, cor. 4.4.b)]. In general, there exists the following connection between $\hat{S}$ and S^{**}. If we define $\varphi : S^{**} \to \hat{S}$ by : $\varphi(\alpha) = \{s \in S \mid \tilde{s} \leqslant \alpha\}$, then φ is correctly defined. φ is bimonotone (or only injective) if and only if S is dense in S^{**}. Indeed, if $\tilde{s} \leqslant \alpha \Rightarrow \tilde{s} \leqslant \beta$ and S is dense in S^{**}, it follows that $\alpha \leqslant \beta$. Conversely, if S is not dense in S^{**}, let $\alpha \in S^{**}$ be an element which is not a supremum of elements from S. If we denote : $\beta = \bigvee \{\tilde{s} \mid s \in S, \tilde{s} \leqslant \alpha\}$ then $\beta \leqslant \alpha$ and $\varphi(\alpha) = \varphi(\beta)$. If S is specifically solid in S^{**}, then φ is a (finite) morphism of H-cones. Moreover, if for each $s \in S$, $s \neq 0$ there exists $\mu \in S^*$ such that $\mu(s) \neq 0$, then we have a commutative diagramm :

$$\begin{tikzcd} & S \arrow[dl] \arrow[dr] & \\ S^{**} \arrow[rr, "\varphi"] & & \hat{S} \end{tikzcd}$$

If φ is surjective in general, remains an open question.

b) S being dense, hence solid in $H(S)$ [3, prop 2.2.] it follows that $H(S) = H(\hat{S})$, hence S and $\hat{S}$ are isomorphic H-cones.

c) The cone S° defined in [4] can be also compared with $H(S^*)$ as follows. There exists two maps : $\varphi : S^\circ \to H(S^*)$, $\psi : H(S^*) \to S^\circ$ defined by : $\varphi(\mu) = \{\nu \in S^* \mid \nu \leqslant \mu\}$; $\psi(A)(s) = \sup_{\nu \in A} \nu(s)$. φ is correctly defined and bimonotone ; ψ is correctly defined and monotone ; moreover $\psi \circ \varphi = 1_{S^\circ}$. However, $\varphi \circ \psi = 1_{H(S^\circ)}$ if and only if S° satisfies ($H4$) from [2]. It is an open question if this holds always. By [4, prop. 1.8], when S is an H-cone of functions, increasingly dense in S^{**} (in particular, when S is a standard H-cone), then $(S^*)^\circ$ satisfies ($H4$), hence $(S^*)^\circ$ is isomorphic in this case to $H(S)$.

It is hence preferably, in general, to use in the definition of a morphism of H-cone, the hyperharmonic cone $H(S)$ from [3], instead of S°, since the former always contains S (as a solid and dense part), satisfies $(H\,4)$ and has a universality property. This definition coincides with the one given in [4], for example when S is an H-cone of functions, dense in S^{**}.

§3. We consider next the following categories and functors. ε is the category having as objects those hyperharmonic cones, for which the set of cancellable elements is dense. If $V, W \in \mathrm{Ob}\,\varepsilon$, the set $\varepsilon(V, W)$ of morphisms is formed of all hyperharmonic morphisms [2], from V to W, which preserve the cancellable elements.

$\mathcal{H}$ is the category having as objects the H-cones such that $S=\hat{S}$ and as morphisms, the finite morphisms of H-cones [4]. Accordingly, $\mathcal{H}(S, T)$ will stand for the set of all morphisms from S to T in the category $\mathcal{H}$. We define next two functors:

$H : \mathcal{H} \to \varepsilon$ is defined on objects in §1. By [2, th. 6.20] any $\varphi \in \mathcal{H}(S, T)$ extends uniquely to a hyperharmonic morphism, denoted here by $H(\varphi)$) from $H(S)$ to $H(T)$. It is immediate that H is a correctly defined functor.

$G : \varepsilon \to \mathcal{H}$ is defined on objects as : $G(V)$ is the H-cone of all cancellable elements from V [2, th. 3.13]; and, for $\varphi \in \varepsilon(V, W)$, $G(\varphi)=\varphi\,|_{G(V)}$. Again, we have defined a functor. Clearly $G \circ H = 1_{\mathcal{H}}$.

Theorem 4. *H is the adjoint of G.*

Proof. We show more, namely that there exists a natural bijection :

$$\varepsilon(H(S), V) \to \mathcal{H}(S, G(V)), \quad \forall S \in \mathrm{Ob}\,\mathcal{H},\ \forall V \in \mathrm{Ob}\,\varepsilon$$

which is bimonotone and preserves the addition and multiplication by positive scalars. This facts result from the following universality property :

Lemma 5. *Let $S \in \mathrm{Ob}\,\mathcal{H}$, $V \in \mathrm{Ob}\,\varepsilon$ and $\varphi : S \to V$ be a map with properties :*

$\forall s, t \in S : \varphi(s+t) = \varphi(s)+\varphi(t)$; $s \leqslant t \Rightarrow \varphi(s) \leqslant \varphi(t)$; $\forall F \subseteq S$ *increasing and bounded* $\Rightarrow \varphi(\bigvee F) = \bigvee_{s \in F} \varphi(s)$ $\forall s \in S$, $\varphi(s)$ *is cancellable in* V.

Then, there exists a unique morphism $\tilde{\varphi} \in \varepsilon(H(S), V)$ such that the following diagramm commutes :

$$\begin{tikzcd} S \arrow[rr, "i"] \arrow[dr, "\varphi"'] & & H(S) \arrow[dl, "\tilde{\varphi}"] \\ & V & \end{tikzcd}$$

Proof. The uniqueness is implied by the density of S in $H(S)$. Let $A \in H(S)$ and define : $\tilde{\varphi}(A) = \bigvee_{s \in A} \varphi(s)$. The definition is correct and the diagramm is commutative. The fact that $\tilde{\varphi}$ is additive follows from the fact that A is increasing. Clearly $\tilde{\varphi}$ is monotone and by construction, $\forall s \in \hat{S}$, $\tilde{\varphi}(s)=\varphi(s)$ is cancellable, since $S=\hat{S}$. Let $(A_i)_{i \in I}$ be an increasing family from $H(S)$. We have to prove only the inequality : $\bigvee_{i\in I} \tilde{\varphi}(A_i) \geqslant \tilde{\varphi}(\bigvee_{i \in I} A_i)$.

Using lemma 2, $s \in \bigvee_{i\in I} A_i = (\bigcup_{i \in I} A_i)$, means that there exists an increasing family $(t_j)_{j \in J}$ such that : $t_j \in \bigcup_{i \in I} A_i$, $\forall j \in J$ and $s \leqslant \bigvee_{j \in J} t_j$. Then :

$$\bigvee_{i \in I} \tilde{\varphi}(A_i) = \bigvee_{i\in I} \bigvee_{t \in A_i} \varphi(t) = \bigvee_{t \in \cup A_i} \varphi(t) \geqslant \varphi(s) \ \forall s \in \bigvee_{i \in I} A_i,$$

$$\text{and so : } \bigvee_{i \in I} \varphi(A_i) \geqslant \bigvee_{s \in \vee A_i} \varphi(s) = \tilde{\varphi}(\bigvee_{i \in I} A_i).$$

Now, we have a bijection, associating with $\varphi \in \varepsilon(H(S), V)$ the morphism $\varphi\,|_S \in \mathcal{H}(S, G(V))$. Indeed, the choice of morphisms assures that the definition is correct, while the lemma proves that $\varphi \to \tilde{\varphi}$ gives the inverse map. Since $\tilde{\varphi} = \varphi \circ i$, it follows the other asserted properties of this bijection.

Remarks. From the general theory [5], it follows that :

a) H commutes with the colimits and G commutes with the limits (if they exists).

b) There exists natural arrows :

$$u : 1_{\mathcal{H}} \to G \circ H \, ; \quad v : H \circ G \to 1_\varepsilon \, ;$$

u is in fact identity. If $W \in \mathrm{Ob}\,\varepsilon$, then $v_W : H(G(W)) \to W$ is the canonical projection, i.e. if $A \subseteq W$ is a non-void increasing subset of cancellable elements then : $v_W(A) = \bigvee_{s \in A} s$, (in W).

One verifies easily (using lemma 4) that the next compositions are the identities :

$$G(W) \xrightarrow{uG} GHG(W) \xrightarrow{Gv} G(W), \quad \forall W \in \mathrm{Ob}\,\varepsilon$$

and :

$$H(S) \xrightarrow{Hu} HGH(S) \xrightarrow{vH} H(S), \quad \forall S \in \mathrm{Ob}\,\mathcal{H}$$

c) $\mathcal{H}$ is isomorphic with the full subcategory of ε, having as objects the hyperharmonic cones, which satisfy also $(H4)$ from [2]. Since there exists hyperharmonic cones V, such that the cancellable elements are dense in V, but V does not satisfy $(H4)$, it follows that ε and $\mathcal{H}$ are not equivalent categories. Also, this fact proves that there is no hope to obtain an adjunction situation, when finite morphisms are replaced by arbitrary ones, despite the fact that the functor H can be extended to general morphisms.

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