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# SOME PROPERTIES OF JORDAN MEASURABLE FUNCTIONS

BY

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**0. Preliminaries.** The classical formula of changing the variables in Riemann integral, namely

„If $g:[a,b]\to[c,d]$ is a $C^1$ and strictly monotone function and if $f:[c,d]\to\mathbb{R}$ is Riemann integrable, then

$$\int_{g(a)}^{g(b)} f(x)dx=\int_a^b f(g(t))g'(t)dt\text{''}, \tag{S}$$

involves the monotony assumption about $g$.

Some more weaker conditions are found in S. Marcus [2], I. D. Zaslavski [5]. On this occasion arises the problem to know when $f\circ g$ is Riemann integrable (or simply, $R$-function). Our paper provides necessary and sufficient conditions in order to answer this question.

Let $g:[a,b]\to\mathbb{R}$ be a $R$-function, $m_g=\inf_{x\in[a,b]} g(x)$, $M_g=\sup_{x\in[a,b]} g(x)$. Following Zaslavski [5] we say that $g$ is of type $R_*$ if for every Riemann integrable function $f:[m_g,M_g]\to\mathbb{R}$, $f\circ g$ is $R$-function.

Let $V_g$ denote the set of points of $[a,b]$ which are not interior to the intervals where $g$ is constant. Let $\mu(t)$ denote the Lebesgue measure of the set

$$V_g\cap\{x\in[a,b]\;;\;g(x)\leqslant t\},\ t\in\mathbb{R}.$$

Then, the following theorems hold:

**Theorem A** (Zaslavski). *$g$ is of type $R_*$ if and only if for every negligible closed set $F$ contained in $[m_g,M_g]$ the set $V_g\cap g^{-1}(F)$ is negligible.*

**Theorem B.** *$g$ is of type $R_*$ if and only if $\mu(t)$ is absolutely continuous on $[m_g,M_g]$ and $\mu(m_g)=0$.*

**Theorem C** (Zaslavski). *Any continuous analytic function is of type $R_*$.*

However the theorems $A$ and $B$ are difficult to handle. In the present paper we study the continuous functions of type $R_*$ and we prove that these are precisely the continuous Jordan measurable functions. We find theorem C as a corollary of a criterium of Jordan measurability.

**1. A Characterization of J-functions.** A function $g:[a,b]\to\mathbb{R}$ is called Jordan measurable ($J$-function) iff $g^{-1}(M)$ is Jordan measurable for every measurable closed set $M$.

**Theorem 1.** *A continuous function $g:[a,b]\to\mathbb{R}$ is $J$-function if and only if for every negligible compact $M\subseteq\mathbb{R}$, $g^{-1}(M)$ is Jordan measurable.*

*Proof.* The „only if" part is straightforward. To prove the „if" part one can easily see that if $M$ is a closed set then

$$\mathrm{Fr}\, g^{-1}(M) = \mathrm{Fr}\, g^{-1}(\mathrm{Fr}\, M).$$

Now let $M$ be a Jordan measurable compact. Then $\mathrm{Fr}\, M$ is a negligible compact [3]. Hence $g^{-1}(\mathrm{Fr}\, M)$ is $J$-measurable so that $\mathrm{Fr}\, g^{-1}(M) = \mathrm{Fr}\, g^{-1}(\mathrm{Fr} M)$ is of zero Jordan measure. q.e.d.

From this point we shall deal only with continuous J-functions so that one may take Theorem 1 as definition of the J-functions.

**2. The Fundamental Property of Continuous J-functions.** A theorem much alike theorem A is the following:

**Theorem 2.** *Let $g : [c, d] \to [a, b]$ be a continuous function: then $g$ is $J$-function if and only if $g$ is of type $R_*$.*

*Proof.* Let us prove the „only if" part. We have to prove that if $f : [a, b] \to \mathbb{R}$ is a $R$-function then $f \circ g$ is a $R$-function provided that $g$ is $J$-function.

We shall need the set

$$D = \{x \in [a, b] : f \text{ is discontinuous at } x\}.$$

For each $h : [a, b] \to \mathbb{R}$ we denote by $\omega(h, x)$ the oscillation of $h$ at $x$. Also $\mathrm{Disc}\; h = \{x \in [a, b]\; ;\; \omega(h, x) > 0\}$. We shall use the operator

$$\mathrm{disc}_f \; 2^{[a, b]} \to 2^{[c, d]}$$

$$\mathrm{disc}_f(M) = g^{-1}(M) \cap \mathrm{Disc}\;\; f \circ g.$$

A simple but useful property of this operator is that it preserves the unions, viz.

(2.1) $$\forall (M_i)_{i \in I} \subseteq D, \;\; \mathrm{disc}_f\Big(\bigcup_{i \in I} M_i\Big) = \bigcup_{i \in I} \mathrm{disc}_f(M_i).$$

Clearly

(2.1') $$\mathrm{disc}_f(M) \subseteq g^{-1}(M), \;\; \forall M \subset [a, b].$$

The importance of the operator $\mathrm{disc}_f$ derives from the following equality $\mathrm{Disc}\; f \circ g = \mathrm{disc}_f(D)$ which is straightforward in view of the definition of $\mathrm{disc}_f$.

**Lemma 2. 1.** *For every $M \subseteq D$ we have*

$$\mathrm{disc}_f(M) \subseteq g^{-1}(M) - \mathrm{Int}\;\; g^{-1}(M).$$

*Proof.* According to (2.1') we have to prove only that the sets $\mathrm{disc}_f(M)$ and $\mathrm{Int}\; g^{-1}(M)$ are disjoint. Let us suppose the contrary, that is, there is $(u, v) \subseteq g^{-1}(M)$ so that $(u, v) \cap \mathrm{disc}_f(M) \neq \emptyset$. $K = g[(u, v)]$ is a connected set, $K \subseteq D$. So $K$ is either an interval or a point. But $K$ cannot be an interval for in this case $D$ would not be Lebesgue negligible and hence $f$ would not be Riemann integrable. Therefore $K = \{y\}$. Let $x \in (u, v) \cap \mathrm{disc}_f(M)$. Then there is $x_n \to x$ so that $f \circ g(x_n) \not\to f \circ g(x)$. Because $x \in (u, v)$ there is a $n_0 \in N$ so that $\forall n \geq n_0\; x_n \in (u, v) \Rightarrow g(x_n) = y = g(x)$. We infer that $f \circ g(x_n) = f \circ g(x) \to \to f \circ g(x)$ which is a contradiction.



*Remark.* In the proof of this lemma we have not used the $J$-function assumption about $g$. Hence it is true for every continuous $g$. We denote $E_n = \{x \in [c, d] : \omega(f, x) \geq 1/n\}$, $n \in N^*$. $E_n$ are negligible compact sets and $D = \bigcup_{n \geq 1} E_n$ [3]. From (2.1) we infer that

$$\mathrm{disc}_f\; (D) = \bigcup_{n \geq 1} \mathrm{disc}_f(E_n).$$

$$\mathrm{Disc}\;\; f \circ g \subseteq \bigcup_{n \geq 1} [g^{-1}(E_n) - \mathrm{Int}\; g^{-1}(E_n)] \subseteq \bigcup_{n \geq 1} \mathrm{Fr}\; g^{-1}(E_n).$$

But $g$ is $J$-function and $E_n$ are Jordan measurable so that $\mathrm{Fr}\, g^{-1}(E_n)$ is Lebesgue negligible and hence $\mathrm{Disc}\; f \circ g$ is Lebesgue negligible. According to Lebesgue's $R$-integrability criterium we infer that $f \circ g$ is $R$-integrable. Q.e.d.

Now we prove the „if" part. We have to prove that for every negligible compact $K \subseteq R$, $g^{-1}(K)$ is Jordan measurable, given that $g$ is of type $R_*$. Let $K$ be a negligible compact and $\chi_K : \mathbb{R} \to \mathbb{R}$ be its characteristic function.

If $C_K = [a, b] - K$ then $C_K$ is dense in $[a, b]$ because $K$ is negligible. The set of discontinuities of $\chi_K$ is $K$ so that $\chi_K$ is $R$-integrable [4]. For the sake of simplicity, we denote $f = \chi_K$. According to lemma 2.1 we infer

(2.4) $$\mathrm{Disc}\;\; f \circ g = \mathrm{disc}_f(K),$$

(2.5) $$\mathrm{disc}_f(K) \subseteq g^{-1}(K) - \mathrm{Int}\;\; g^{-1}(K).$$

We shall prove that in (2.5) „$\subseteq$" may be replaced by „$=$".

**Lemma 2.2.**

(2.6) $$\mathrm{disc}_f(K) = g^{-1}(K) - \mathrm{Int}\; g^{-1}(K) \;\; (= \mathrm{Fr}\; g^{-1}(K)).$$

*Proof.* According to (2.5) we have to prove only that

$$g^{-1}(K) - \mathrm{Int}\;\; g^{-1}(K) \subseteq \mathrm{disc}_f(K).$$

Let $x \in g^{-1}(K) - \mathrm{Int}\; g^{-1}(K)$. Then for every neighbourhood $V$ of $x$

$$V \cap g^{-1}(K) \neq \emptyset \;\; \text{and} \;\; V - g^{-1}(K) \neq \emptyset.$$

Hence $g$ is not constant on any neighbourhood of $x$. We denote $U_n = (x - 1/n, x + 1/n)$, $n \in N^*$. $g(U_n)$ is a connected set and because of those previous said, we infer that $g(U_n)$ is a non-degenerate interval containing $x$. We know that $C_K$ is dense in $[a, b]$ and hence $C_K \cap g(U_n) \neq \emptyset$, $\forall n$. Let $y_n \in C_K \cap g(U_n)$. There is $x_n \in U_n$ so that $g(x_n) = y_n$. Because of the uniform continuity of $g$ we infer that $\mathrm{diam}\; g(U_n) \to 0$ when $n \to \infty$. Therefore $y_n \to y = g(x)$ when $n \to \infty$. $f \circ g$ is discontinuous at $x$. Indeed there is $x_n \to x$ so that

$$f \circ g(x_n) = f(y_n) = \chi_K(y_n) = 0 \not\to = \chi_K = f \circ g(x).$$

The proof is complete. q.e.d.

Now because $g$ is of type $R_*$ this implies that $f \circ g = \chi_K \circ g$ is $R$-integrable (we remember that $\chi_K$ was $R$-integrable). Hence $\mathrm{Disc}\; f \circ g$ is Lebesgue negligible. From (2.4) and (2.6) it follows that $\mathrm{Disc}\; f \circ g = \mathrm{Fr}\; g^{-1}(K)$. Hence the compact $\mathrm{Fr}\; g^{-1}(K)$ is negligible and hence $g^{-1}(K)$ is $J$-measurable.

We have given a necessary and sufficient condition for a function to be $J$-measurable. But this result is still inapplicable because we have not a concrete image about these functions. Moreover we have not given yet examples of $J$-functions. The next section solves all these problems.

**3. A Criterium of Jordan Measurability**. The main result of this section is the following:

**Theorem 3.** *Let $g$ be of class $C^1$ on $[a, b]$. If the set $Z_{g'}$ of the zeros of its derivative is Jordan measurable then $g$ is Jordan measurable.*

*Proof.* We have to prove that for every negligible compact $M \in R$ $\mathrm{Fr}\, g^{-1}(M)$ is negligible. Let $\varepsilon > 0$. Because $Z_{g'}$ is Jordan measurable we may choose some intervals (eventually degenerated) $I_1,\dots,I_n \subset Z_{g'}$ so that

$$m(Z_{g'} - \bigcup_{k=1}^{n} I_k) < \frac{\varepsilon}{4}. \tag{3.1}$$

($m(S)$ denotes the Jordan measure of the set $S$). Therefore we can cover the set $Z_{g'} - \bigcup_{1}^{n} I_k$ with a finite family of open intervals $(J_s)_{s=\overline{1,m}}$ so that

$$\sum_{s=1}^{m} m(J_s) < \varepsilon/3. \tag{3.2}$$

We have chosen the intervals $J_s$ so that they should cover the degenerated intervals among the $I_k$-s. We denote $W = \bigcup_{s=1}^{m} J_s$, $D = \bigcup_{k=1}^{n} I_k$, $L = [a, b] \setminus (W \cup D)$. Clearly

$$[a, b] = L \cup W \cup D.$$

$L$ is a disjoint union of closed intervals. Hence $L = \bigcup_{j=1}^{p} [a_j, b_j]$. If $M$ is a negligible compact then

If $M$ is a negligible compact teen

$$\mathrm{Fr}\, g^{-1}(M) = (\mathrm{Fr}\, g^{-1}(M) \cap L) \cup (\mathrm{Fr}\, g^{-1}(M) \cap W) \cup (\mathrm{Fr}\, g^{-1}(M) \cap D) = \tag{3.3}$$
$$= L_1 \cup W_1 \cup D_1.$$

Our aim is to prove that each of these sets $L_1$, $W_1$, $D_1$ has the outer Jordan measure less then $\varepsilon/3$. This will imply that $\mathrm{Fr}\, g^{-1}(M)$ is negligible and the theorem will be proved. First we should observe that

$$\mathrm{Fr}\, g^{-1}(M) \cap D = \mathrm{Fr}\, D. \tag{3.4}$$

Let us suppose the contrary, i.e. that $\mathrm{Fr}\, g^{-1}(M) \cap (\bigcup_{k=1}^{n} I_k) \neq \mathrm{Fr}(\bigcup_{k=1}^{n} I_k)$. Then there is an open interval $I \subset D$ so that $I \cap \mathrm{Fr}\, g^{-1}(M) \neq \varnothing$. Let $x \in I \cap \mathrm{Fr}\, g^{-1}(M)$. The neighbourhood $I$ of $x$ contains both points of $g^{-1}(M)$ and of $\complement g^{-1}(M)$. Hence $g$ is not constant on $I$. Thus (3.4) is proved.

$D$ is Jordan measurable and hence $\mathrm{Fr}\, D$ is negligible. Therefore taking (3.4) into account, we infer that the outer Jordan measure of $D_1$ is

$$m_e(D_1) = m_e(\mathrm{Fr}\, g^{-1}(M) \cap D). \tag{3.4}$$

$W_1 \subset W$ and (3.2) assures that $m(W) < \varepsilon/3$. Whence

$$m_e(W_1) < \varepsilon/3.$$



The only thing left to be proved is that $m_e(L_1) < \varepsilon/3$. $L$ is a compact set and $g'$ has no zero in $L$. Therefore

$$q = \min \{ |g'(x)| ;\ x \in L \} \neq 0.$$

Because $M$ is negligible it has a covering with a finite number of intervals (open) $(H_i)_{i=\overline{1,r}}$ so that

$$\sum_{i=1}^{r} m(H_i) < \varepsilon q(3p). \tag{3.6}$$

(we remember that $L = \bigcup_{j=1}^{p} [a_j, b_j]$). We denote

$$A = g^{-1}(M) \cap L,\ A_j = A \cap [a_j, b_j].$$

If we prove that the outer measure of $A_j$ is less than $\varepsilon(3p)$ then the outer measure of $A$ would be less than $\varepsilon/3$ and the theorem would be then proved.

Let us consider $g_j = g\,|_{[a_j, b_j]}$. One can easily see that $A_j = g_j^{-1}(M)$. Because $g_j'$ has no zero on $[a_j, b_j]$, we infer that $g_j$ is a diffeomorphism. If $T_i = (u_i, v_i) = H_i \cap \mathrm{Im}\ g_j$ then

$$m(g_j^{-1}(T_i)) \leqslant \frac{1}{\min\limits_{x \in L} |g'(x)|}\, m(T_i) \leqslant \frac{1}{q}\, m(H_i). \tag{3.7}$$

But

$$m_e(A_j) \leqslant m\left(\bigcup_{i=1}^{r} g_j^{-1}(T_i)\right) \leqslant \frac{1}{q} \sum_{i=1}^{r} m(H_i) \leqslant \frac{\varepsilon}{3p}. \qquad \text{q.e.d.}$$

In the same direction S. M a r c u s proved in [2] the following.

**Theorem D.** *Let $g$ be continuous and strictly monotone. Then $g$ is of type $R_*$ (hence $J$-function) if and only if the set*

$$E = \{x ;\ g'(x) \text{ exists and } g'(x) = 0\},$$

*is Lebesgue negligible.*

The theorems A, D, 3 make plausible the following

**Conjecture.** *A $C^1$ function $g$ is Jordan measurable if and only if $Z_{g'}$ is Jordan measurable.*

We state here some corollaries of Theorem 3. First we shall make some usefull remarks.

1) The composed of two $J$-functions is a $J$-function.

2) A piecewise $J$-function is a global $J$-function.

**Corollary 1.** *Every analytic function is $J$-function. Every elementary function is $J$-function.*

**Corollary 2.** *Spline functions are $J$-functions.*

**Corollary 3.** *The primitives of a $J$-function are $J$-functions.*

*Proof.* Let $g$ be a continuous $J$-function, $G' = g$. Then $Z_{G'} = Z_g = g^{-1}(0)$. Or $g$ is $J$-function and consequentely $Z_g$ is $J$-measurable. q.e.d.

The next result is stronger then Theorem 3.

**Corollary 4.** *Let $g : [a, b] \to \mathbb{R}$ be a continuous function with the following properties:*

*1) $g$ is everywhere differentiable with the possible exception of a Jordan negligible set;*

2) *g is $C^1$ on every interval contained in the set $D$ of the points where $g'$ exists ;*
3) *the set of zeros of $g'$ in every interval $I \subseteq D$ is J-measurable. Then g is Jordan measurable.*

*Proof.* Let $E=[a, b]\setminus D$. Then $m(E)=0$ and let $(I_k)_{k=\overline{1,n}}$ be open intervals covering $E$ so that

$$\sum_{k=1}^{n} m(I_k)<\varepsilon/2.$$

If $F=\bigcup_{k=1}^{n} I_k$ then $L=[a, b]\setminus F$ is a finite union of disjoint closed intervals $([a_j, b_j])_{j=\overline{1,m}}$.

Let $M \subseteq R$ be a negligible compact. We shall prove that the measure of Fr $g^{-1}(M)$ is less than $\varepsilon$.

$$\text{(3.9)} \qquad \operatorname{Fr} g^{-1}(M)=(\operatorname{Fr} g^{-1}(M)\cap F)\cup(\operatorname{Fr} g^{-1}(M)\cap L)=F_1\cup L_1.$$

First, $me\,(F_1)\leqslant \varepsilon/2$, from (3.8). Denoting by $g_j=g\mid_{[a_j, b_j]}$ we infer from the hypoteses and from theorem 3 that

$$L_1=\bigcup_{j=1}^{m} \cdot A_j, \quad A_j=\operatorname{Fr} g^{-1}(M)\cap[a_j, b_j].$$

Because $g_j$ is $J$-function we infer that $A_j$ is $J$-negligible. Hence $L_1$ has the outer $J$-measure less than $\varepsilon/2$ and now the corollary follows from (3.9) and (3.10).

We have heitherto given only examples of $J$-functions. Does there exist non $J$-functions ?

A simple criterium of non $J$-measurability is *„If $Z_g$ is not J-measurable then g is not J-measurable"*

It is known the following result of W h i t n e y :
*„For every compact $K \subseteq \mathbf{R}^n$ there is a $C^\infty$ function $g: \mathbf{R}^n \to \mathbf{R}$ so that $Z_g=K$".*
Hence there exists non $J$-functions.

**4. Epilogue.** The formula of changeing variables in Riemann's integral generated the works [2], [5] and the present one. The best result in this direction was established by C. D e s c h a m p s in 1980. It states that equality (S) holds even when $g$ is merely $C^1$.

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# A DETERMINING FUNCTION FOR A PAIR OF SELF-ADJOINT OPERATORS WITH A COMMUTATOR WHICH IS NOT OF TRACE CLASS

BY

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**1. Introduction.** P i n c u s has characterized determining functions for a pair of bounded self-adjoint operators whose commutator is a trace class operator [2], [3]. In this exposition we establish the existence of a $G(\nu, \mu)$ function which is a complete unitary invariant for the given pair of operators. This extends the theory of determining functions to a larger class of operators. Our extension is suitable for application purposes in the following sense : We extend the theory of deficiency subspaces of symmetric singular integral operators given by P i n c u s [5]. Our motivation in such an extension can be described as follows. We assume that $[U, V]$ is a pair of unbounded self-adjoint operators with $U$ self-adjoint and $V$ symmetric. Let the commutator of $\{U, V\}$ be given by $VU-UV=[1/(\pi i)]\,D$, where $D$ may not be an operator of one dimensional range (or even for that matter, not even a trace class operator) as in the case treated by P i n c u s [5]. Then we characterize the deficiency subspaces of $V$ (($m, n$) with $m, n<\infty$) with some restrictions on the operator $D$ to be explained later in the sequel. Although in such a situation $V$ is not a singular integral operator, but because of our assumption on $D$ it will follow that the Cayley transform of $V$ is a singular integral operator.

We invoke some necessary details from earlier works of P i n c u s [2], [3] and state our theorems in the present work with or without proofs, proofs being given only in the cases when modifications are substantial.

**2. Assumptions.** For a pair of bounded self adjoint operators $U$ and $V$ defined on a separable Hilbert space $H$ we define the commutator of $U$ and $V$ as $[V, U]=VU-UV$. Let $[V, U]=[1/(\pi i)]\,C$, where $C$ is a trace class operator on $H$. We further have the following assumptions. Let $U$ be having a purely absolutely continuous spectral measure. Also let the smallest invariant subspace for both $U$ and $V$ which contains the range of $C$ be whole of $H$.

Let $h$ be a fixed $l_2$ space whose dimension is equal to the maximum of the dimension of the range of $C$ and the spectral multiplicity of $C$.

Let the spectral representation of $C$ be given by $C=\Sigma(\operatorname{sgn} c_n)\lambda_n^2\varphi_n$, where $\{\varphi_n\}$ denotes the orthonormalized set of eigenvectors of $C$, completed if necessary if $C$ has a null space, and $c_n=(\operatorname{sgn} c_n)\lambda_n^2$ is the $n$-th eigenvalue of $C$.