2) *g is C^1 on every interval contained in the set D of the points where g' exists ;*
3) *the set of zeros of g' in every interval $I \subseteq D$ is J-measurable. Then g is Jordan measurable.*

Proof. Let $E=[a, b]\setminus D$. Then $m(E)=0$ and let $(I_k)_{k=\overline{1,n}}$ be open intervals covering E so that

$$\sum_{k=1}^{n} m(I_k)<\varepsilon/2.$$

If $F=\bigcup_{k=1}^{n} I_k$ then $L=[a, b]\setminus F$ is a finite union of disjoint closed intervals $([a_j, b_j])_{j=\overline{1,m}}$.

Let $M\subseteq R$ be a negligible compact. We shall prove that the measure of Fr $g^{-1}(M)$ is less than ε.

$$\text{Fr } g^{-1}(M)=(\text{Fr } g^{-1}(M)\cap F)\cup(\text{Fr } g^{-1}(M)\cap L)=F_1\cup L_1. \tag{3.9}$$

First, $me\,(F_1)\leqslant \varepsilon/2$, from (3.8). Denoting by $g_j=g\,|_{[a_j, b_j]}$ we infer from the hypoteses and from theorem 3 that

$$L_1=\bigcup_{j=1}^{m} \cdot A_j,\ A_j=\text{Fr} g^{-1}(M)\cap[a_j, b_j].$$

Because g_j is J-function we infer that A_j is J-negligible. Hence L_1 has the outer J-measure less than $\varepsilon/2$ and now the corollary follows from (3.9) and (3.10).

We have heitherto given only examples of J-functions. Does there exist non J-functions ?

A simple criterium of non J-measurability is *„If Z_g is not J-measurable then g is not J-measurable"*

It is known the following result of W h i t n e y :
„For every compact $K\subseteq \mathbf{R}^n$ there is a C^∞ function $g:\mathbf{R}^n\to\mathbf{R}$ so that $Z_g=K$".
Hence there exists non J-functions.

4. Epilogue. The formula of changeing variables in Riemann's integral generated the works [2], [5] and the present one. The best result in this direction was established by C. D e s c h a m p s in 1980. It states that equality (S) holds even when g is merely C^1.

Acknowledgements. I wish to thank prof. S. Frunză for the assistance lended to me. I am also indebted to prof. A. Haimovici for his useful remarks concerning the paper.

REFERENCES

1. C. D e s c h a m p s — *Un enoncé simple du théorème de changement de variables dans une integrale de Riemann*. Rev. Math. Spec. (1980—81), 7, p. 301—302.
2. S. M a r c u s — *Sur la superposition de deux fonctions integrables au sens de Riemann*. Rev. Roum. Math. Pures et Appl. 1959, Tome IV, No. 3, p. 381—389.
3. M. N i c o l e s c u — *Manual de analiză matematică*, Edit. Did. Ped., Bucureşti, 1980, vol. 1.
4. M. S p i v a k — *Calculus on Manifolds*, W.A. Benjamin, 1965.
5. I. D. Z a s l a v s k i — *On R-integrability of superposition of R-integrable functions* (in Russian), Vestnik Leningrad Univ., 1953, 11, 49—55.

Received 29 XI 1984

Faculty of Mathematics
University of Iaşi
6600 Iaşi, R. S. România

Analele ştiinţifice ale Universităţii „Al. I. Cuza" din Iaşi
Tomul XXXII, s. I a, Matematică, 1986, f. 1

A DETERMINING FUNCTION FOR A PAIR OF SELF-ADJOINT OPERATORS WITH A COMMUTATOR WHICH IS NOT OF TRACE CLASS

BY

SHIV PRASAD and R. N. MUKHERJEE

1. Introduction. P i n c u s has characterized determining functions for a pair of bounded self-adjoint operators whose commutator is a trace class operator [2], [3]. In this exposition we establish the existence of a $G(\nu, \mu)$ function which is a complete unitary invariant for the given pair of operators. This extends the theory of determining functions to a larger class of operators. Our extension is suitable for application purposes in the following sense : We extend the theory of deficiency subspaces of symmetric singular integral operators given by P i n c u s [5]. Our motivation in such an extension can be described as follows. We assume that $[U, V]$ is a pair of unbounded self-adjoint operators with U self-adjoint and V symmetric. Let the commutator of $\{U, V\}$ be given by $VU-UV=[1/(\pi i)]\,D$, where D may not be an operator of one dimensional range (or even for that matter, not even a trace class operator) as in the case treated by P i n c u s [5]. Then we characterize the deficiency subspaces of V $((m, n)$ with $m, n<\infty)$ with some restrictions on the operator D to be explained later in the sequel. Although in such a situation V is not a singular integral operator, but because of our assumption on D it will follow that the Cayley transform of V is a singular integral operator.

We invoke some necessary details from earlier works of P i n c u s [2], [3] and state our theorems in the present work with or without proofs, proofs being given only in the cases when modifications are substantial.

2. Assumptions. For a pair of bounded self adjoint operators U and V defined on a separable Hilbert space H we define the commutator of U and V as $[V, U]=VU-UV$. Let $[V, U]=[1/(\pi i)]\,C$, where C is a trace class operator on H. We further have the following assumptions. Let U be having a purely absolutely continuous spectral measure. Also let the smallest invariant subspace for both U and V which contains the range of C be whole of H.

Let h be a fixed l_2 space whose dimension is equal to the maximum of the dimension of the range of C and the spectral multiplicity of U.

Let the spectral representation of C be given by $C=\Sigma(\text{sgn } c_n)\lambda_n^2\varphi_n$, where $\{\varphi_n\}$ denotes the orthonormalized set of eigenvectors of C, completed if necessary if C has a null space, and $c_n=(\text{sgn } c_n)\lambda_n^2$ is the n-th eigenvalue of C.

(We use throughout the notation $c_n/|c_n|=\operatorname{sgn} c_n$ if $c_n\neq 0$ and $\operatorname{sgn} c_n=0$ if $c_n=0$).

Let $\{\theta_j\}$ be a fixed complete orthonormal set in h. Define $k^*: h\to H$ by setting $k^*\theta_j=\lambda_j\theta_j$, and $k: H\to h$ by setting $k\theta_j=\lambda_j\theta_j$. We further define an (indefinite metric) operator $\operatorname{sgn} c$ on h by setting $\operatorname{sgn} c\ \theta_n=\operatorname{sgn} c_n\theta_n$. Finally let us introduce the operator $J: h\to h$ defined by setting $J\theta_n=\varepsilon_n\theta_n$, where $\varepsilon_n=1$ if $\operatorname{sgn} c_n=1$, $\varepsilon_n=0$ if $\operatorname{sgn} c_n=-1$ and $\varepsilon_n=0$ if $\operatorname{sgn} c_n=0$.

Definition. *The determining function of the pair $\{U, V\}$ is defined as a map from h into h by setting*

$$E(l, z)=1+\frac{1}{\pi i}JK(V-l)^{-1}(U-z)^{-1}K^*J$$

for all l and z not in $\sigma(V)$, $\sigma(U)$, where 1 is the identity on h. The following results are in Pincus [3].

Theorem 2.1. *Suppose $\{U', V'\}$ is another pair satisfying our hypothesis with $V'U'-U'V'=(1/(\pi i))C'$. Then $T'=U'+iV'$ is unitarily equivalent to $T=U+iV$ if and only if $E'(l, z)$ on h is unitarily equivalent $t_0E(l, z)$ on h.*

The next theorem gives a representation of the determining function $E(l, z)$.

Theorem 2.2. $\det E(l, z)=\exp\left(\frac{1}{2\pi i}\iint G(\nu, \mu)\frac{d\nu}{\nu-l}\frac{d\mu}{\mu-z}\right)$ *for* $\operatorname{Im}(l)\neq 0$, $\operatorname{Im}(z)\neq 0$, *where $G(\nu, \mu)\in L^1(d\nu\times d\mu)$ and $d\nu\times d\mu$ is the planar Lebesgue measure supported on $\sigma(V)\times\sigma(U)$.*

The function $G(\nu, \mu)$ is called a complete unitary invariant for the pair $\{U, V\}$. The reason for its complete unitary invariant property follows from theorem 2.1.

3. In this section we consider a pair $\{U, V\}$ such that $[V, U]=(1/\pi i)D$, where D is an operator which is not necessarily a trace class operator but we will have some further restriction on D which will be revealed soon. Let $D=K^*K$ where K is a bounded operator on H. Also let λ be a real number such that $\lambda\in\rho(V)$, the resolvent set of V. We observe that the pair $\{U, (\lambda-V)^{-1}\}$ is a self-adjoint pair with a commutator given in the following lemma which can be obtained by an algebraic manipulation of easy nature.

Lemma 3.1. $[(\lambda-V)^{-1}, U]=-(\lambda-V)^{-1}\frac{K^*K}{\pi i}(\lambda-V)^{-1}$.

Next we assume that $-(\lambda-V)^{-1}K^*K(\lambda-V)^{-1}$ is a trace class operator which we denote by C_λ. Also we denote $(\lambda-V)^{-1}$ by T_λ. Then we define a determining function $E_\lambda(l, z)$ for the pair $\{U, T_\lambda\}$ in the following way,

$$E_\lambda(l, z)=1+(1/\pi i)Jk(T_\lambda-l)^{-1}(U-z)^{-1}k^*J,$$

where k, k^*, J are all operators defined analogously as in section 2 for the operator C of our case. In above we also assume that the pair $\{U, T_\lambda\}$ satisfies the hypothesis stated as in paragraph one in section 2. The $E_\lambda(l, z)$ function thus defined will correspond to the pair $\{U, V\}$ even if D is not of trace class. This turns out to be an extension of the concept of determining function $E(l, z)$ of section 2.

Theorem 3.1. *Let $\{U, V\}$ and $\{U', V'\}$ be two pairs such that* $[V, U]=(1/\pi i)C$ and $[V', U']=(1/\pi i)C'$.

Let $\lambda\in R$ such that $\lambda\in\rho(V)\cap\rho(V')$. Also let the pair $\{U, T_\lambda\}$ and $\{U', T'_\lambda\}$ be obtained as mentioned above with $[T_\lambda, U]=(1/\pi i)C_\lambda$ and $[T'_\lambda, U']=(1/\pi i)C'_\lambda$ which satisfy the hypothesis cited before. Then the operator $U+iV$ is unitarily equivalent to $U'+iV'$ if and only if $E_\lambda(l, z)$ on h is unitarily equivalent to $E'_\lambda(l, z)$ on h.

Proof. We observe that $U+i(V-\lambda)^{-1}$ is unitarily equivalent to $U'+i(V'-\lambda)^{-1}$ if and only if $U+iV$ is unitarily equivalent to $U'+iV'$ where we set $T_\lambda=(V-\lambda)^{-1}$ and $T'_\lambda=(V'-\lambda)^{-1}$. Then applying theorem 2.1 to the operators $U+iT_\lambda$ and $U'+iT'_\lambda$ we get our result.

Theorem 3.2. $\det E(l, z)=\exp\left(\frac{1}{2\pi i}\iint G_\lambda(\nu, \mu)\frac{d\nu}{\nu-l}\frac{d\mu}{\mu-z}\right)$ *for* $\operatorname{Im}(l)\neq 0$, $\operatorname{Im}(z)\neq 0$, *where $G_\lambda(\nu, \mu)\in L^1(d\nu\times d\mu)$ where $d\nu\times d\mu$ is the planar Lebesgue measure supported on $\sigma(T_\lambda)\times\sigma(U)$.* The function $G_\lambda(\nu, \mu)$ will be called a complete unitary invariant for the pair $\{U, V\}$.

The following theorem gives a formula to recover the trace of the operator C_λ from the $G_\lambda(\nu, \mu)$ function. Proof is similar to theorem 10 of [3].

Theorem 3.3. Trace $C_\lambda=\iint G_\lambda(\nu, \mu)\,d\nu\,d\mu$.

§4. Let $\{U, V\}$ be a pair of bounded operators in which U is a self adjoint operator and V is a symmetric operator with finite deficiency indices of the type (m, n). Let the commutator relation $VU-UV=(1/\pi i)D$ be formally true with D not necessarily of trace class (D being a bounded self-adjoint and positive operator).

Lemma 4.1.

$$WU=[U+(l-\bar{l})/\pi i]\ (V-l)^{-1}D(V-\bar{l})^{-1}]W. \tag{4.1}$$

Proof. Let $R(l)=(V-l)^{-1}$ be the generalized resolvent of the symmetric operator V for $\operatorname{Im}(l)\neq 0$. If $W=(V-\bar{l})\ (V-l)^{-1}$ with $\operatorname{Im}(l)>0$, then W is known as Cayley transform (which is an isometry) of the symmetric operator V. From $VU-UV=(1/\pi i)D$, we get $(V-l)U-V(V-\bar{l})=(1/\pi i)D$. Then $(V-l)^{-1}$ from the left and $(V-\bar{l})^{-1}$ from the right to the last relation we get

$$U(V-\bar{l})^{-1}-(V-l)^{-1}U=\frac{1}{\pi i}(V-l)^{-1}D(V-\bar{l})^{-1}. \tag{4.2}$$

Using $W=(V-\bar{l})((V-l)^{-1}$ and the resolvent identify from relation (4.2) one can arrive at (4.1).

Let h be the Hilbert space of doubly infinite sequence of complex numbers.

$$\tilde{\alpha}=\{\ldots, \alpha_{-2}, \alpha_{-1}, \alpha_0, \alpha_1, \alpha_2,\ldots\},$$

with $\|\tilde{\alpha}\|^2=\sum_{j=-\infty}^{\infty}|\alpha_j|^2$.

Let $\tilde{K}^*: h\to\tilde{H}$ and $\tilde{K}: \tilde{H}\to h$ be chosen so that $\tilde{B}=(1/\pi)\tilde{K}^*\tilde{K}$. We define the determining function of the triplet $[\tilde{W}, \tilde{U}, \tilde{U}+\tilde{B}]$, as $\tilde{\varphi}(l, z)$ by setting

$$\tilde{\varphi}(l, z)=1+\frac{1}{\pi}\tilde{K}\tilde{W}(\tilde{W}-z)^{-1}(\tilde{U}-l)^{-1}\tilde{K}^*.$$

for $|z| \neq 1$ and Im $(l) \neq 0$ (See [4]). The operator $\tilde{W}$ in the triplet is the minimal unitary dilation of the isometry W which intertwines U and $U+B$, i.e. $WU=(U+B)W$. $\tilde{U}$ and $\tilde{B}$ are the operators which are defined in the extended Hilbert space $\tilde{H}$ such that they are self-adjoint and their restrictions to the space $\tilde{H}$ being the operators U and B. From Lemma 4.1 we get a triplet $[\tilde{W}, \tilde{U}, \tilde{U}+\tilde{B}]$ where $\tilde{W}$ is the minimal unitary dilation of W on $\tilde{H}$, $\tilde{U}$ is the self-adjoint operator on $\tilde{H}$ whose projection on H is the operator U and $\tilde{B}$ is the self-adjoint and positive operator on $\tilde{H}$ whose projection on H is a self-adjoint operator B on H. We make an assumption that $\tilde{B}$ is of the one dimensional range, i.e. $\tilde{B}=\tilde{K}\otimes\tilde{K}$. The procedure for dilation of these operators is the same as followed in [4].

In our special case, since B is of one dimensional range, it follows from [4] that $\tilde{\Phi}(l, z)$ considered as an operator on h, has only one eigenvalue which is not identically one. Let us denote this eigenvalue by $\Phi(l, z)$. The above considerations would lead to the following results.

Lemma 4.2.
$$\Phi(l, z) = \exp\left(\frac{1}{2\pi i}\int_0^{2\pi}\int_R g(\nu, \tau)\,\frac{d\nu}{\nu-l}\,\frac{d\tau}{\tau-z}\right),$$
where $0 \leqslant g(\nu, \tau) \leqslant 1$, *and* $g(\nu, \tau)$ *is the complete unitary invariant for the triplet* $[\tilde{W}, \tilde{U}, \tilde{U}+\tilde{B}]$.

Lemma 4.3.
$$\det\,(1+\pi^{-1}k(U-l)^{-1}k^*)=\det\,(1+B(U-l)^{-1})=\exp\left(\int_R \delta(\nu)\,\frac{d\nu}{\nu-l}\right).$$

Lemma 4.4.
$$\delta(\nu)=\frac{1}{2\pi}\int_0^{2\pi} g(\nu, e^{i\theta})\,d\theta,$$
$$\text{Trace } B=\frac{1}{2\pi}\int_R\int_0^{2\pi} g(\nu, e^{i\theta})\,d\theta\,d\nu.$$

Remark 1. In lemma 4.3 the function $\delta(\nu)$ satisfies $0 \leqslant \delta(\nu) \leqslant 1$ and is called the phase shift corresponding to the perturbation problem $\tilde{U} \to \tilde{U}+\tilde{B}$. For a self-adjoint operator with spectral resolution $T=\int dE_\lambda$, the set of elements x in H for which $\|E_\lambda x\|^2$ is an absolutely continuous function of λ is a closed subspace of H, which we will denote by $H_{ac}(T)$. The orthocomplement of $H_{ac}(T)$ will be denoted by $H_s(T)$. The restriction of T to $H_s(T)$ is called the singular part of T. Let $P_{ac}(T)$ and $P_s(T)$ denote the orthogonal projections onto $H_{ac}(T)$ and $H_s(T)$ respectively. With this we state the following lemma [4].

Lemma 4.5. *In the special case where* $H_{ac}(U)=H$, *the deficiency of the isometric operator* W *is precisely the dimensionality of* $H_s(U+B)$.

We next characterize the deficiency subspaces of the isometric operator V, underlying assumption being that the singularly component spectral measure of $\tilde{U}+\tilde{B}$ has only finite number of atoms.

Lemma 4.6. dim $H_s(U+B)=$dim $L_2\,(d\mu_+^s)=m$.

Proof. $R_0(l)=(U-l)^{-1}$ and $R(l)=(U+B-l)^{-1}$ are the resolvents of U and $U+B$ respectively.

Let $D(l)=\det\,(1+B(U-l)^{-1})$. Then

$$(U+B-l)^{-1}=R_0(l)+\frac{1}{\pi}\,\frac{1}{D(l)}\,R_0(l)K(., R_0(l)K) \tag{4.3}$$

from which it easily follows that the atoms in the singularly continuous component of the spectral measure of $U+B$ are given by the zeros of the determinant $D(l)$. From the considerations made in [4] by invoking M. G. K r e i n's results, it can be shown that there exists a positive Borel measure $d\mu_+$ such that

$$\frac{1}{D(l)}=1+\int\frac{d\mu_+(t)}{t-l}\,.$$

Let $d\mu_+^s$ (the singularly continuous component of $d\mu_+$) have atoms (finite in number) at $t_1^+, t_2^+, \ldots, t_m^+$. Then the poles of $D(l)$ are exactly at those points. The proof is thus complete.

Lemma 4.7. *The finite dimensional subspace spanned by the vectors* $\{k(t)/(t-t_r^+)\}_{r=1}^m$ *is the deficiency subspace of the isometric operator* W.

Some results of the dual character can be given by considering the Cayley transform $W=(V-l)\,(V-\bar{l})^{-1}$ (with Im $(l)<0$) and proceeding in an analogous way. The statements are as follows.

Lemma 4.8. $W'U=[U+[(\bar{l}-l)^2/\pi i](V-\bar{l})^{-1}D(V-l)^{-1}]W'$ *where we put*

$$B'=[(\bar{l}-l)^2/\pi i](V-\bar{l})^{-1}D(V-l)^{-1},$$

and we require that $\tilde{B}'$ *be of one dimensional range.*

In fact $\tilde{B}'$ is already a self-adjoint and positive operator and we express $\tilde{B}'$ as $\tilde{B}'=\Phi\otimes\Phi$. Then if $D'(l)$ is defined in an analogous way $d\mu$ is the measure involved in the representation of $1/D'(l)$, then we have the following lemma.

Lemma 4.9. dim $H_s(U+B')=$dim $L_2(d\mu^s)=n$,
where underlying assumption is that $d\mu'$ *have finite number of atoms at the points* $t_1^-, t_2^-, t_3^-, \ldots, t_n^-$.

Lemma 4.10. *The finite dimensional subspace spanned by the vectors* $\{\Phi(t)/(t-t_r^-)\}_{r=1}^n$ *is the deficiency subspace of the isometric operator* W'.

Finally we get the following theorem about the deficiency indices of the symmetric operator V.

Theorem 4.1. *The deficiency indices of the symmetric operator* V *are given by* (m, n), *where* $m=$dim $L_2(d\mu_+^s)$ *and* $n=$dim $L_2(d\mu_-^s)$.

Proof. Proof follows from lemma 4.6 and lemma 4.9.

Remark 2. We observe that the results on deficiency indices of the operator V is an extension of the previous result of Pincus [5] in the sense that here we do not require V to be a singular integral operator instead our assumption apparently forces W to be a singular integral operator because $WU-UW=BW$, where BW is of trace class. Similar is the case for W'.

Remark 3. The necessary and sufficient condition for a real number μ to be an atom of the measure $d\mu_+^e$ is that for some $\varepsilon>0$ the integral

$$\int_{\mu-\varepsilon}^{\mu}\frac{1-\delta(\nu)}{\mu-\nu}\,d\nu+\int_{\mu}^{\mu+\varepsilon}\frac{\delta(\nu)}{\nu-\mu}\,d\nu$$

be finite, where $\delta(\nu)$ is the phase shift corresponding to the perturbation problem $\tilde{U}\to\tilde{U}+\tilde{B}$ (see Carey [1]) appearing in lemma 4.3.

REFERENCES

1. Carey R. W. — *A unitary invariant for pairs of self-adjoint operators*, Preprint, Univ. of Kentucky, Lexington, 1974.
2. Pincus J. D. — *Commutators and system of singular integral equations*, I. Acta-Math. (1968), pp. 219—259.
3. Pincus J. D. — *The determining function method in the treatment of commutator systems* Colloq. Math. Soc. Janos. Bolyai. 5 Hilbert space operators, Tihany, Hungary 1970.
4. Pincus J. D. and Carey R. W. — *The structure of intertwining isometries*, Indiana Univ. Math. J., 22 (1973), 679—703.
5. Pincus J. D. — *Symmetric singular integral operators with arbitrary deficiency. Topics in Functional Analysis* (Essays dedicated to M. G Krein on the occasion of his 70 th Birthday), (Advances in Mathematics, Supplementary Studies), Academic Press, New York, 1978.

Received 21 I 1985

Applied Mathematics Section
School of Applied Sciences
Institute of Technology
Banaras Hindu University
Varanasi, 221005 India

Analele ştiinţifice ale Universităţii „Al. I. Cuza" din Iaşi
Tomul XXXII, s. I a, Matematică, 1986, f. 1

INVARIANT BOXES AND STABILITY OF SOME SYSTEMS FROM BIOMATHEMATICS AND CHEMICAL REACTIONS

BY

N. H. PAVEL

1. **Introduction.** In the first part of this paper (Sec. 2) we recall a general result (Theorem 2.1) on the flow-invariance of a time-dependent rectangular box with respect to a differential system. In Sec. 3 we apply Theorem 3.1 to the study of some differential systems from biomathematics and chemical reactions. In Sec. 4 we characterize the matrices A for which the corresponding linear system $x'=Ax$ is component-wise positive asymptotically stable (CWPAS). For the proof of Theorem 2.1, as well as for other problems in flow-invariance theory, the reader may consult the papers in the references, in particular Refs. [4], [5] and [7]. In the Appendix we give a partial answer to an open problem in the theory of the flow-invariance of a set with respect to an abstract differential equation in a Banach space.

2. **Statement of the general result.** Let us consider the differential system

$$x_i'=f_i(t, x_1, \ldots, x_m),\quad t\geqslant 0, \tag{2.1}$$

with the initial condition

$$x_i(t_0)=x_i^0,\quad x_i^0\in R,\quad i=1, \ldots, m, \tag{2.2}$$

where $f_i:[0, +\infty)\times R^m\to R$ are continuous and guarantee the uniqueness of solution to the Cauchy problems (2.1) and (2.2).

Recall that a time-dependent subset $D(t)\subset R^m$ is said to be flow-invariant (or simply-invariant) with respect to (2.1) if for each $t_0\geqslant 0$ and $x^0=(x_1^0, \ldots, x_m^0)\in D(t_0)$, the corresponding solution $x=x(t)$ to (2.1) with $x(t_0)=x^0$ also satisfies

$$x(t)\in D(t),\quad t\geqslant t_0, \tag{2.3}$$

(i.e. $x(t)$ remains in $D(t)$ as long as it exists to the right of t_0). The characterization of t-dependent flow-invariant sets in very general cases can be found in Refs. [2], [4] and [5]. Here we recall only the particular case of rectangular boxes below

$$D(t)=\prod_1^m[a_i(t), b_i(t)]=\{x=(x_1, \ldots, x_m)\in R^m;\ a_i(t)\leqslant x_i\leqslant b_i(t),\ 0\leqslant t,\ i=1, \ldots, m\}, \tag{2.4}$$

where $a_i, b_i:[0, +\infty)\to R$ are continuously differentiable functions, with $a_i(t)<b_i(t)$, for all $i=1, \ldots, m$ and $t\geqslant 0$. The characterization of the flow-