Proof. Proof follows from lemma 4.6 and lemma 4.9.

Remark 2. We observe that the results on deficiency indices of the operator V is an extension of the previous result of Pincus [5] in the sense that here we do not require V to be a singular integral operator instead our assumption apparently forces W to be a singular integral operator because $WU-UW=BW$, where BW is of trace class. Similar is the case for W'.

Remark 3. The necessary and sufficient condition for a real number μ to be an atom of the measure $d\mu_+^a$ is that for some $\varepsilon>0$ the integral

$$\int_{\mu-\varepsilon}^{\mu} \frac{1-\delta(\nu)}{\mu-\nu}\, d\nu + \int_{\mu}^{\mu+\varepsilon} \frac{\delta(\nu)}{\nu-\mu}\, d\nu$$

be finite, where $\delta(\nu)$ is the phase shift corresponding to the perturbation problem $\tilde{U}\to\tilde{U}+\tilde{B}$ (see Carey [1]) appearing in lemma 4.3.

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INVARIANT BOXES AND STABILITY OF SOME SYSTEMS FROM BIOMATHEMATICS AND CHEMICAL REACTIONS

BY

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1. **Introduction.** In the first part of this paper (Sec. 2) we recall a general result (Theorem 2.1) on the flow-invariance of a time-dependent rectangular box with respect to a differential system. In Sec. 3 we apply Theorem 3.1 to the study of some differential systems from biomathematics and chemical reactions. In Sec. 4 we characterize the matrices A for which the corresponding linear system $x'=Ax$ is component-wise positive asymptotically stable (CWPAS). For the proof of Theorem 2.1, as well as for other problems in flow-invariance theory, the reader may consult the papers in the references, in particular Refs. [4], [5] and [7]. In the Appendix we give a partial answer to an open problem in the theory of the flow-invariance of a set with respect to an abstract differential equation in a Banach space.

2. **Statement of the general result.** Let us consider the differential system

$$x_i'=f_i(t, x_1, \ldots, x_m),\ t\geq 0, \tag{2.1}$$

with the initial condition

$$x_i(t_0)=x_i^0,\ x_i^0\in R,\ i=1, \ldots, m, \tag{2.2}$$

where $f_i : [0, +\infty)\times R^m\to R$ are continuous and guarantee the uniqueness of solution to the Cauchy problems (2.1) and (2.2).

Recall that a time-dependent subset $D(t)\subset R^m$ is said to be flow-invariant (or simply-invariant) with respect to (2.1) if for each $t_0\geq 0$ and $x^0=(x_1^0, \ldots, x_m^0)\in D(t_0)$, the corresponding solution $x=x(t)$ to (2.1) with $x(t_0)=x^0$ also satisfies

$$x(t)\in D(t),\ t\geq t_0, \tag{2.3}$$

(i.e. $x(t)$ remains in $D(t)$ as long as it exists to the right of t_0). The characterization of t-dependent flow-invariant sets in very general cases can be found in Refs. [2], [4] and [5]. Here we recall only the particular case of rectangular boxes below

$$D(t)=\prod_1^m [a_i(t), b_i(t)]=\{x=(x_1, \ldots, x_m)\in R^m ;$$
$$a_i(t)\leq x_i\leq b_i(t),\ 0\leq t,\ i=1, \ldots, m\}, \tag{2.4}$$

where $a_i, b_i : [0, +\infty)\to R$ are continuously differentiable functions, with $a_i(t)<b_i(t)$, for all $i=1, \ldots, m$ and $t\geq 0$. The characterization of the flow-

invariance of the above $D(t)$ is given by the following theorem (for its proof see P a v e l [3], [4] or V o i c u [7]).

Theorem 2.1. *Let $f_i : [0, +\infty) \times R^m \to R$ and $a_i, b_i : [0, +\infty) \to R$ satisfy the above hypotheses. Then for each $t_0 \geq 0$ and $x_i = (x_1^0, \ldots, x_m^0)$ with $a_i(t_0) \leq x_i^0 \leq b_i(t_0)$, $i = 1, 2, \ldots, m$ the corresponding solution $x = (x_i(t), \ldots, x_m(t))$ to the problems (2.1) and (2.2) satisfies*

$$(2.5) \qquad a_i(t) \leq x_i(t) \leq b_i(t), \quad \forall t \geq t_0, \ i = 1, \ldots, m,$$

if and only if

$$(2.6) \qquad \begin{aligned} &f_i(t, x_1, \ldots, x_{i-1}, a_i(t), x_{i+1}, \ldots, x_m) \geq a_i'(t), \\ &f_i(t, x_1, \ldots, x_{i-1}, b_i(t), x_{i+1}, \ldots, x_m) \leq b_i'(t), \end{aligned}$$

$\forall t \geq t_0$; $i = 1, \ldots, m$; $x_j \in [a_j(t), b_j(t)]$; $j = 1, \ldots, i-1, i+1, \ldots m$ (with the convention $x_0 \equiv x_1$, $x_{m+1} \equiv x_m$).

3. Invariant rectangular boxes. We first consider the enzymatic reactions

$$E + S \rightleftarrows C \to E + P,$$

where E, S, C and P denote enzyme, substrate complex and product, respectively. It is known that the evolution of the substrate and complex concentrations (denoted by $x_1(t)$ and $x_2(t)$, respectively) is described by the following system (see R u b i n o v [6])

$$(3.1) \qquad x_1' = -x_1 + (x_1 + a)\, x_2, \quad r x_2' = x_1 - (x_1 + b)\, x_2,$$

where $0 < a \leq b$ and r is a small positive constant.

Another system we want to study here is the system of F i e l d and N o y e s (see Ref. [4] for details)

$$(3.2) \quad x_1' = s(x_2 - x_1 x_2 - x_1 - q x_1^2), \quad x_2' = s^{-1}(f x_3 - x_2 - x_1 x_2), \quad x_3' = w(x_1 - x_3),$$

where s, w, f and q are positive constants. It is essential to note that q is less than one (precisely $q = 8.375 \cdot 10^{-6}$). The differential system (3.2) is the kinetic system for the five irreversible steps model of Field and Noyes

$$(3.3) \qquad A + Y \to X;\ X + Y \to P;\ B + X \to 2X + Z;\ 2X \to Q;\ \ Z \to fY,$$

where f is a stoichiometric factor, P, Q products and

$$X = HBrO_2,\ Y = Br^-,\ Z = Ce\ (IV),\ A = B = BrO_3^-.$$

Actually, (3.3) is the quantitative model for the chemical mechanism in the Belousov reaction.

The third non-linear system we want to discuss here is

$$(3.4) \qquad \begin{aligned} x_1' &= x_1 - \frac{1}{10} x_1^2 - x_1 x_2 + \frac{1}{10} x_2^2, \\ x_2' &= x_1 x_2 - \frac{1}{10} x_2^2 - x_2 x_3 + \frac{1}{10} x_3^2, \\ x_3' &= x_2 x_3 - \frac{1}{10} x_3^2 - x_1 x_3 + \frac{5}{100} x_1. \end{aligned}$$

This is the kinetic system for the chemical scheme (of H a n u s s e see Ref. [4]) of two processes A and B with three intermediate species X, Y, Z. In connection with (3.1) it was proved in Refs. [3] and [4] that all rectangles of the form

$$(3.5) \qquad D_2 = [0, b_1] \times [0, b_2], \quad b_1, b_2 > 0,$$

are given by

$$(3.6) \qquad a b_2 (1 - b_2)^{-1} \leq b_1 \leq b b_2 (1 - b_2)^{-1},$$

for all $b_2 \in (0, 1)$. This can be proved by applying Theorem 2.1 with $m = 2$, $a_1(t) = a_2(t) = 0$, $b_1(t) = b_1 = \text{const}$, $b_2(t) = b_2 = \text{const}$. for all $t \geq 0$.

It was shown by M u r r a y (see Ref. [4]) that the rectangular boxes

$$(3.7) \qquad D_3 = \{(x_1, x_2, x_3) ;\ 1 \leq x_1 \leq q^{-1},\ y_1 \leq x_2 \leq y_2 ;\ 1 \leq x_3 \leq q^{-1}\}$$

have the property that every solution to (3.2) in the positive octant R_3^+ eventually enters the box D_3 and no trajectory can leave D, provided that

$$(3.8) \qquad y_1 = qf/(1 + q), \quad y_2 = f/2q.$$

In what follows we shall give more precise informations on the systems above. To do this, let us also consider the rectangular box

$$(3.9) \qquad \tilde{D}_3 = \{(x_1, x_2, x_3) ;\ 1/3 \leq x_1 \leq 10,\ 1/8 \leq x_2 \leq 100,\ 1/20 \leq x_3 \leq 1\ 000\}.$$

The main result of this section is given by

Theorem 3.1. 1) *The positive octant R_3^+ is a flow-invariant set with respect to each of the systems* (3.1), (3.2) *and* (3.4). 2) *The necessary and sufficient condition in order for the system* (3.1) *to admit an invariant set of the form*

$$(3.10) \qquad \tilde{D}_2 = [a_1, b_1] \times [a_2, b_2], \quad 0 < a_i b_i, \ i = 1, 2$$

is $a = b$. All rectangles of the form (3.10) *which are invariant with respect to* (3.1) *are given by*

$$(3.11) \qquad a_2 = a_1 (a_1 + a)^{-1}, \quad b_2 = b_1 (b_1 + a)^{-1}$$

with arbitrary $a_1, b_1 > 0$. 3). *All rectangular boxes D_3 of the form* (3.7) *which are invariant with respect to* (3.2) *correspond to*

$$(3.12) \qquad y_1 \leq qf/(1 + q), \quad y_2 = \leq f/2q.$$

4) *The set $\tilde{D}_3$ (sec* (3.9) *is invariant with respect to the system* (3.4)).

Proof. We shall apply Theorem 2.1. For the part 1), set $x_1 = 0$ in the right-hand side of the first equation of (3.4) (e.g.). We get $1/10\, x_2^2$ which is greater (or equally) than zero for all x_2 (in particular for $x_2 \geq 0$) and so on. 2) In this case, by Theorem 2.1, the invariances of $\tilde{D}_2$ is equivalent to (3.6) and

$$(3.13) \qquad \begin{aligned} &0 \leq -a_1 + (a_1 + a)\, x_2, \qquad \forall x_2 \in [a_2, b_2], \\ &0 \leq x_1 - (x_1 + b)\, a_2, \qquad \forall x_1 \in [a_1, b_1]. \end{aligned}$$

Obviously, (3.13) are equivalent to $a_1(a_1 + a)^{-1} \leq a_2 \leq a_1 (a_1 + b)^{-1}$, which implies $a \geq b$. But (3.6) gives $a \leq b$ so $a = b$ and the result follows.

3) One applies Theorem 2.1 with $a_1=a_3=1$, $a_2=y_1$, $b_1=b_3=1/q$, $b_2=y_2$ and $n=3$. Now the conditions (2.6) become

$$fx_3\geqslant(1+x_1)\,y_1,\ fx_3\leqslant(1+x_1)\,y_2,$$

for all $1\leqslant x_1\leqslant 1/q$ and $1\leqslant x_3\leqslant 1/q$ which means just (3.12). 4) In this case we apply Theorem 2.1 with $n=3$, $a_1=1/3$, $b_1=10$, $a_2=1/8$, $b_2=100$, $a_3=1/20$ and $b_3=1\ 000$. The tangential conditions (2.6) become

$$\text{(C1)}\qquad \frac{1}{3}-\frac{1}{10}\,\frac{1}{9}-\frac{1}{3}\,x_2+\frac{1}{10}\,x_2^2\geqslant 0$$

$$\text{(C2)}\qquad \frac{1}{8}\,x_1-\frac{1}{10}\,\frac{1}{64}-\frac{1}{8}\,x_3+\frac{1}{10}\,x_3^2\geqslant 0,$$

$$\text{(C3)}\qquad \frac{1}{20}\,x_2-\frac{1}{10}\,\frac{1}{400}-\frac{1}{20}\,x_1+\frac{5}{100}\,x_1\geqslant 0,$$

$$\text{(C4)}\qquad -10\,x_2+\frac{1}{10}\,x_2^2\leqslant 0,$$

$$\text{(C5)}\qquad 100\,x_1-1\ 000-100\,x_3+\frac{1}{10}\,x_3^2\leqslant 0,$$

$$\text{(C6)}\qquad 1\ 000\,x_2-10^5-10^3\,x_1+\frac{5}{100}\,x_1\leqslant 0,$$

for all $x_1\in\left[\frac{1}{3},\ 10\right]$, $x_2\in\left[\frac{1}{8},\ 100\right]$, $x_3\in\left[\frac{1}{20},\ 1\ 000\right]$. The inequality (C1) holds true since

$$\min_{x_2\in R}\left\{\frac{1}{10}\,x_2^2-\frac{1}{3}\,x_2+\frac{29}{90}\right\}=\frac{4}{40}.$$

To verify (C2) it sufficies to see that for $x_1=1/3$ we have

$$\frac{1}{8}\cdot\frac{1}{3}\geqslant\frac{1}{8}\max\left\{\frac{1}{640}+\frac{x_3}{8}-\frac{1}{10}\,x_3^2\right\}=\frac{1}{8}\,\frac{13}{45}.$$

The remaining conditions above are obviously verified. The proof is complete.

Corollary 3.1. *All solutions to the systems* (3.1), (3.2) *and* (3.4) *starting from* $\tilde{D}_2$ (*or* D_2), D_3 *and* $\tilde{D}_3$ *respectively are defined on the whole semiaxis* $R_+=[0,\ +\infty)$.

Proof. Let x be a solution (e.g.) to (3.4) with $x(0)=x^0\in\tilde{D}_3$. Then $x(t)\in\tilde{D}_3$ for all $t\in[0,\ t_{\max})$ and therefore x is bounded on $[0,\ t_{\max})$. By a classical result, this implies $t_{\max}=+\infty$ which concludes the proof.

Note that the assertion of Corollary 3.1 is not trivial. Indeed, even for very simple ordinary differential equations (e.g. $x'=x^2$) the solution (i.e. $x(t)=1/(1-t)$, $x(0)=1$) may blow up in finite time. In our example here, $t_{\max}=1$.

4. Component-wise positive asymptotic stability (CWPAS). A special class of Hurwitz matrices. In this section we will study the flow-invariance of $D(t)$ in the case $a_i(t)=0$, i.e.

$$D_0(t)=\prod_{i=1}^{m}\,[0,\ b_i(t)],\ 0<b_i(t),\ i=1,\ \dots,\ m\ ;\ t\geqslant 0 \tag{4.1}$$

with respect to the linear homogeneous system with constant coefficients

$$x'(t)=Ax(t),\ t\geqslant 0, \tag{4.2}$$

where $A=(a_{ij})$, $a_{ij}\in R$, $i, j=1, \dots, m$ and $x=(x_1, \dots, x_m)$. Clearly, the flow-invariance of $D_0(t)$ means that for each initial condition $x(t_0)=x^0$ with $0\leqslant x(t_0)\leqslant b(t_0)$, the corresponding solution to (4.2) satisfies

$$0\leqslant x(t)\leqslant b(t),\ t\geqslant t_0, \tag{4.3a}$$

where $b(t)=b_1(t), \dots, b_m(t)$. As usual, this means

$$0\leqslant x_i(t)\leqslant b_i(t),\ i=1,\ \dots,\ m. \tag{4.3b}$$

In view of Theorem 2.1 we can prove

Theorem 4.1. *The necessary and sufficient conditions for the flow-invariance of* $D_0(t)$ (*i.e. for* (4.3a)) *to be satisfied, are given by*

1°) $\quad a_{ij}\geqslant 0,\ i\neq j,\ i,\ j=1,\ \dots,\ m$

2°) $\quad b'(t)\geqslant Ab(t),\ \ \forall t\geqslant 0.$

In the sequel we keep the condition $b_i(t)>0$ and in addition we will introduce the asymptotic stability condition

$$\lim_{t\to\infty} b_i(t)=0,\ i=1,\ \dots,\ m. \tag{4.4}$$

Definition 4.1. *The system* $x'=Ax$ *is said to be* (*CWPAS*) *if there exists a m by* 1 *vector* $b(t)>0$ *such that* (4.3a) *and* (4.4) *are satisfied.*

The theorem below characterizes the matrices A for which $x'=Ax$ is (CWPAS).

Theorem 4.2. *The system* $x'=Ax$ *is* (*CWPAS*) *if and only if* 1) $a_{ij}\geqslant 0$, $\forall i\neq j$, $i, j=1, \dots, m$. 2) $a_{ii}<0$, $i=1, \dots, m$. 3) A *has the Hurwitz's property* (*the real part of each eigenvalue is negative*).

Necessity. The condition 1) is equivalent to the non-negativity of solutions starting from R_+^m (One applies (2.6) with $a_i(t)=0$). Furthermore, from (4.4) it follows that there are $t_i>0$ such that $b_i'(t)<0$. By Theorem 4.1 we now have

$$0>a_{ii}b_i(t_i)+\sum_{j=1}^{m}a_{ij}b_j(t_i),\ j\neq i,\ i=1,\ \dots,\ m \tag{4.5}$$

which yields 2). Clearly, 3) is a consequence of (4.3b) and (4.4) in conjunction with a classical result.

Sufficiency. If (1)—(3) are fulfilled, then the system (4.2) is (CWPAS) with $b(t)=x_0e^{At}$, for each m by 1 vector $x_0>0$. This is because (1)—(3) implies $b(t)>0$ as well as (4.4) (so we take $b(t)=x(t)=x_0e^{At}$). The proof is complete.

Remark 4.1. Actually, (CWPAS) of (4.2) is equivalent to the existence of some $m\times 1$ vectors v of components $v_i>0$ and of some numbers $c>0$ such that (4.2) is (CWPAS) with $b(t)=v\,e^{-ct}$. We will simply say that (4.2) is $(\text{CWPAS})_{c,v}$. Indeed, Theorem 4.2 can be restated as :

Theorem 4.3. *The system* (4.2) *is* (*CWPAS*) *iff*: 1) $a_{ij}\geq 0$, $\forall i\neq j$, $i,j=1,\ldots,m$. 2) $a_{ii}<0$, $i=1,\ldots,m$. 3) $Av<0$,

$$0<c\leq \min_{i=\overline{1,m}}\left\{-a_{ii}-\frac{1}{v_i}\sum_{j=1}^{m}a_{ij}v_j\ ;\ j\neq i\right\}.$$

Remark 4.2. It follows that under hypotheses 1) and 2), the matrix A has the Hurwitz's property if and only if there are some m by 1 vectors $v>0$, such that $Av<0$. A simple algorithm (which can be readily implemented by computer [4]) to compute such vectors v is given by

Theorem 4.4. *Let* $A=(a_{ij})$ *be a* $m\times m$ *matrix with the property*

$$\text{(P)}\qquad a_{ii}<0\ ;\ a_{ij}\geq 0,\ i\neq j,\ i,j=1,\ldots,m.$$

Then a necessary and sufficient condition for the existence of some $m\times 1$ *vectors* $v>0$ *such that* $Av<0$ (*or equivalently for* A *to have Hurwitz's property*) *is that*

$$\text{(4.6)}\qquad d_{p,p}^{(k)}>0,\ k=1,\ldots,m-1\ ;\ p=1,\ldots,m-k$$

where $d_{p,q}^{(k)}$ *are inductively defined by*

$$\text{(4.7)}\qquad d_{i,j}^{(0)}=a_{i,j},\ d_{p,q}^{(k)}=\begin{vmatrix} d_{p,q}^{(k-1)} & d_{p,m-k+1}^{(k-1)} \\ d_{m-k+1,q}^{(k-1)} & d_{m-k+1,\,m-k+1}^{(k-1)}\end{vmatrix}$$

$i,j=1,\ldots,m$; $k=1,\ldots,m-1$; $p,q=1,\ldots,m-k$.

Proof. We have to find some positive numbers $v_1,\ldots,v_m$ which satisfy the system of inequalities

$$\text{(4.8)}\qquad a_{i1}v_1+\ldots+a_{im}v_m<0,\ i=1,\ldots,m$$

(under hypothesis (P)). Because $a_{mm}<0$ and $a_{pm}\geq 0$, $\forall p=1,\ldots,m-1$ (4.8) is equivalent to (if $a_{pm}>0$)

$$\text{(4.9)}\qquad \frac{a_{m1}}{a_{mm}}v_1+\ldots+\frac{a_{m\,m-1}}{a_{mm}}v_{m-1}<v_m<\frac{-a_{p1}}{a_{pm}}v_1+\ldots+\frac{-a_{p\,m-1}}{a_{pm}}v_{m-1}.$$

We conclude that (even if $a_{pm}=0$) $v_1,\ldots,v_{m-1}$ have actually to satisfy (necessarily and sufficient)

$$\text{(4.10)}\qquad -d_{p,1}^{(1)}v_1-\ldots-d_{p,m-1}^{(1)}v_{m-1}<0,\ p=1,\ldots,m-1,$$

where

$$\text{(4.11)}\qquad d_{p,q}^{(1)}=\begin{vmatrix} a_{pq} & a_{pm}\\ a_{mq} & a_{mm}\end{vmatrix},\quad p,q=1,\ldots,m-1$$

hence $d_{p,q}^{(1)}\leq 0$, for $p\neq q$. Accordingly, the conditions $d_{p,q}^{(1)}>0$ ($p=1,\ldots,m$) are necessary. The main remark now is that the matrix in (4.10) also satisfies a condition of type (P) (with $-d_{p,q}^{(1)}$ in place of a_{pq}). Therefore we can

eliminate v_{m-1} from (4.10) as v_m was eliminated from (4.8), and so on. Thus, after the elimination of v_m, v_{m-1}, ..., v_3 the remaining variables v_1, v_2 satisfy

$$\text{(4.12)}\qquad \frac{-d_{2,1}^{(m-2)}}{d_{2,2}^{(m-2)}}v_1<v_2<\frac{d_{1,1}^{(m-2)}}{-d_{1,2}^{(m-2)}}v_1,$$

with $d_{1,2}^{(m-2)}$, $d_{2,1}^{(m-2)}\geq 0$; $d_{1,1}^{(m-2)}$, $d_{2,2}^{(m-2)}>0$. In these conditions, (4.12) is equivalent to

$$\text{(4.13)}\qquad \begin{vmatrix} d_{1,1}^{(m-2)} & d_{1,2}^{(m-2)}\\ d_{2,1}^{(m-2)} & d_{2,2}^{(m-2)}\end{vmatrix}v_1=-d_{1,1}^{(m-1)}v_1<0,$$

hence $d_{1,1}^{(m-1)}>0$ is a necessary and sufficient condition for the existence of some $v_1>0$, satisfying (4.13). Obviously, in (4.13) v_1 may be assigned positive values at pleasure. Then we take a $v_2>0$ satisfying (4.12). In such a manner, going back from (4.13) to (4.10) and (4.9), we prove the existence (sufficiency part) of $v_1,\ldots,v_{m-1}$, $v_m>0$ (with $v_1>0$, arbitrary) satisfying (4.8). This completes the proof.

Remark 4.3. A Fortran programme to compute such $v_1,\ldots,v_m$ was given by G h e r a s i m (see the references in [4]).

APPENDIX : On the extendability of continuous and dissipative functions. The theory of differential equations on closed subsets (on which this paper is based), leads to the following problem (see author's paper [4]). (P) Let D be a non-empty closed subset of the Banach space X and let $A:D\to X$ be a continuous and dissipative function, which is tangent to D (see [4], (3.9), Ch. 2). Can A be extended to X preserving both continuity and dissipativity ? The answer to this question is given by

Theorem 1. *Let* $A:D\to X$ *be as in* (*P*). *If* $X=R$ (*real axis*) *then* A *can be extended to* X *preserving both continuity and dissipativity. This fact may not be true if the dimension of* X *is greater than one.*

Proof. Let $X=R$. In this case the fact that $A: D\subset R\to R$ is dissipative means that A is non-increasing. Denote by D_1 the complementary $R\setminus D$ of D. Take an arbitrary open interval $]a,b[\subset D_1$ with $-\infty<a<b<+\infty$. Hence $a,b\in D$ and $Aa\geq Ab$. Set

$$\text{(1)}\qquad Ax=Aa+(b-a)^{-1}(Ab-Aa)(x-a),\ x\in[a,b].$$

In such a manner, A can be extended to R as a continuous and non-increasing function, q.e.d. If the dimension of X is greater than one (i.e. $\dim X\geq 2$), then we have the following counterexample to the extendability of A to X, namely :

$$\text{(2)}\qquad X=R^2,\ D=\{(x,y)\ ;\ x\in R,\ y\geq 1/2\},\ A(x,y)=(2x/y,\ x^2/y^2)$$

Clearly, $-A$ is the gradient of the convex function

$$\text{(3)}\qquad f(x,y)=x^2/y,\ x\in R,\ y>0$$

and therefore A is dissipative. Moreover, A is tangent to D. Indeed, we have $(x,1/2)+hA(x,1/2)\in D$ for all $x\in R$ and $h>0$, hence (3.9)' in [4] Chapter 2 is verified. We now prove that A cannot be extended to X as a dissipative function. To this goal, assume, by contradiction that there exists a dissipative extension (denoted also by A) of A to the whole X. Set $A(0,-1)=(p,q)$ with $p,\ q\in R$. Then the dissipativity of A yields

$$\langle A(x,y)-A(0,-1),\ (x,y+1)\rangle\leq 0,\quad \forall x\in R,\ y\geq 1/2, \tag{4}$$

where $\langle .,.\rangle$ denotes the inner product of R^2. In view of (2), the inequality (4) becomes

$$(x^2/y)\,((1/y)-1)+px+q(y+1)\leq 0,\quad \forall x\in R,\ y\geq 1/2. \tag{5}$$

Substituting $y=1/2$ into (5) we reach the absurdity

$$4x^2+2px+3q\leq 0,\quad \forall x\in R \tag{6}$$

which completes the proof.

Remark 1. We have actually proved that grad f cannot be extended to R^2. It follows that if $F: R^2\to R^2$ is a function of class C^1, such that its restriction F/D to a closed convex subset $D\subset R^2$ is convex, then F/D cannot be extended to R^2 as a convex function of class C^1. We have to recall that even if $X=R$, a continuous and convex function $F: D\subset R\to R$ (with D closed and convex) cannot be extended to R preserving both continuity and convexity. A standard counterexample in this direction is the following: $D=[-1,\ 1]$, $F(x)=-(1-x^2)^{1/2}$, $x\in D$.

Remark 2. Obviously, A given by (2) is continuous and dissipative on the open subsets $U=\{(x,y);\ y>0\}\supset D$. Consequently, the following problem arises: (P_1) Let D and A be as in (P) and let dim $X\geq 2$. Is it true that A cannot be extended to any open subset $U\supset D$ preserving both continuity and dissipativity?

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ON QUASICONFORMAL MAPPINGS IN RIEMANNIAN MANIFOLDS

BY

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Introduction. In this paper, we consider quasiconformal homeomorphisms according to Gehring's metric definition and in Markushevich-Pesin's sense in Riemannian manifolds and we prove their equivalence in the case of differentiability with Jacobian different from zero.

1. Let us denote by V_n a Riemannian manifold, hence V_n is a topological space separate Hausdorff, paracompact, which has a countable basis, is of class C^∞ and has the partitions of unity of class C^∞. If V_n is connected, then it may be organized as a metric space. For this, let us consider the set $C(x,y)$ of all piecewise differentiable curves γ joining x and y of V_n. The distance from x to y is defined by:

$$d(x,y)=\inf_{\gamma\in C(x,y)}\int_a^b\left\|\frac{d\gamma}{dt}\right\|\,\mathrm{dt}, \tag{1}$$

where the norm $\|.\|$ is given by the Riemannian metric g on V_n and $\gamma: I\to V_n$, $[a,b]\subset I$, $\gamma(a)=x$, $\gamma(b)=y$ (g is a scalar product on the tangent bundle $T(V_n)$). The topology induced by the metric $d(x,y)$ on V_n coincides with the topology of the manifold.

Let g_{ij} be the associated matrix of g in a local chart (U,φ). There exists a unique linear connexion such that its coefficients in the local chart (U,φ) are given by:

$$\Gamma^h_{ij}=\frac{1}{2}\sum_{k=1}^n g^{hk}\left(\frac{\partial g_{jk}}{\partial x^i}+\frac{\partial g_{ki}}{\partial x^j}-\frac{\partial g_{ij}}{\partial x^k}\right), \tag{2}$$

called Levi-Civita connexion. A curve γ given in the local chart (U,φ) by:

$$x^i=x^i(t),\quad t\in I \tag{3}$$

will be a geodesic on V_n, if it satisfies the following system of differential equations:

$$\frac{d^2x^i}{dt^2}+\sum_{j,k=1}^n\Gamma^i_{jk}\frac{dx^j}{dt}\frac{dx^k}{dt}=0 \tag{4}$$

One proves that for every $x_0\in U$, there exists a neighbourhood $U_1\subset U$ of x_0 and $\varepsilon\in R_+$ such that for every $x(x^i)\in U_1$ and every $X(X^i)\in T_x(V_n)$