a dissipative extension (denoted also by A) of A to the whole X. Set $A(0, -1) = (p, q)$ with $p, q \in R$. Then the dissipativity of A yields

$$\langle A(x, y) - A(0, -1), (x, y+1) \rangle \leqslant 0, \quad \forall x \in R, y \geqslant 1/2, \tag{4}$$

where $\langle .,. \rangle$ denotes the inner product of R^2. In view of (2), the inequality (4) becomes

$$(x^2/y)((1/y)-1) + px + q(y+1) \leqslant 0, \quad \forall x \in R, y \geqslant 1/2. \tag{5}$$

Substituting $y = 1/2$ into (5) we reach the absurdity

$$4x^2 + 2px + 3q \leqslant 0, \quad \forall x \in R \tag{6}$$

which completes the proof.

Remark 1. We have actually proved that grad f cannot be extended to R^2. It follows that if $F: R^2 \to R^2$ is a function of class C^1, such that its restriction F/D to a closed convex subset $D \subset R^2$ is convex, then F/D cannot be extended to R^2 as a convex function of class C^1. We have to recall that even if $X = R$, a continuous and convex function $F: D \subset R \to R$ (with D closed and convex) cannot be extended to R preserving both continuity and convexity. A standard counterexample in this direction is the following: $D = [-1, 1], F(x) = -(1-x^2)^{1/2}, x \in D$.

Remark 2. Obviously, A given by (2) is continuous and dissipative on the open subsets $U = \{(x, y); y > 0\} \supset D$. Consequently, the following problem arises: (P_1) Let D and A be as in (P) and let dim $X \geqslant 2$. Is it true that A cannot be extended to any open subset $U \supset D$ preserving both continuity and dissipativity?

ACKNOWLEDGEMENT. This paper was written at ICTP. The author would like to thank Professor Abdus Salam and Professor G. Vidossich, the International Atomic Energy Agency and UNESCO for very pleasant hospitality at the International Centre for Theoretical Physics, Trieste (Italy).

REFERENCES

1. P a v e l N. H. — *Analysis of some non-linear problems, in Banach Spaces and Applications*, University of Iași (România) (1982).
2. P a v e l N. H. — *Semilinear evolution equations with t-dependent domain perturbations*, Israel J. Math. (1983).
3. P a v e l N. H. — *Posibility and stability of some differential systems from biomathematics*, ICTP, Trieste, internal raport IC/83/101 (1983).
4. P a v e l N. H. — *Differential Equations, Flow-invariance and Applications*, Pitman, Research Notes in Math. 113 (1984).
5. P a v e l N. H. and V r a b i e, I.I. — *Differential equations with t-dependent domain operators*, Lectures Notes in Mathematics 737, 236—249 (1979).
6. R u b i n o v S. I. — *Introduction to Mathematical Biology*, Wiley-Interscience, New York, 1975
7. V o i c u M. — *Component-wise asymptotic stability of the linear constant dynamical systems*, Preprint Series in Mathematics 5, University of Iasi (România) (1982).

Received 6 IV 1985

Faculty of Mathematics
University of Iași
6600—Iași, România

Analele științifice ale Universității „Al. I. Cuza" din Iași
Tomul XXXII, s. I a, Matematică, 1986, f. 1

ON QUASICONFORMAL MAPPINGS IN RIEMANNIAN MANIFOLDS

BY

V. T. BORCEA and A. NEAGU

Introduction. In this paper, we consider quasiconformal homeomorphisms according to Gehring's metric definition and in Markushevich-Pesin's sense in Riemannian manifolds and we prove their equivalence in the case of differentiability with Jacobian different from zero.

1. Let us denote by V_n a Riemannian manifold, hence V_n is a topological space separate Hausdorff, paracompact, which has a countable basis, is of class C^∞ and has the partitions of unity of class C^∞. If V_n is connected, then it may be organized as a metric space. For this, let us consider the set $C(x, y)$ of all piecewise differentiable curves γ joining x and y of V_n. The distance from x to y is defined by:

$$d(x, y) = \inf_{\gamma \in C(x, y)} \int_a^b \left\| \frac{d\gamma}{dt} \right\| \mathrm{dt}, \tag{1}$$

where the norm $\|.\|$ is given by the Riemannian metric g on V_n and $\gamma: I \to V_n$, $[a, b] \subset I$, $\gamma(a) = x$, $\gamma(b) = y$ (g is a scalar product on the tangent bundle $T(V_n)$). The topology induced by the metric $d(x, y)$ on V_n coincides with the topology of the manifold.

Let g_{ij} be the associated matrix of g in a local chart (U, φ). There exists a unique linear connexion such that its coefficients in the local chart (U, φ) are given by:

$$\Gamma^h_{ij} = \frac{1}{2} \sum_{k=1}^{n} g^{hk} \left(\frac{\partial g_{jk}}{\partial x^i} + \frac{\partial g_{ki}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^k} \right), \tag{2}$$

called Levi-Civita connexion. A curve γ given in the local chart (U, φ) by:

$$x^i = x^i(t), \quad t \in I \tag{3}$$

will be a geodesic on V_n, if it satisfies the following system of differential equations:

$$\frac{d^2 x^i}{dt^2} + \sum_{j,k=1}^{n} \Gamma^i_{jk} \frac{dx^j}{dt} \frac{dx^k}{dt} = 0 \tag{4}$$

One proves that for every $x_0 \in U$, there exists a neighbourhood $U_1 \subset U$ of x_0 and $\varepsilon \in R_+$ such that for every $x(x^i) \in U_1$ and every $X(X^i) \in T_x(V_n)$

with $|X^i| < \varepsilon$, there exists a unique geodesic $\gamma_X : (-2,2) \to V_n$ which satisfies the initial conditions:

$$\gamma_X(0) = x_0, \qquad \left.\frac{d\gamma_X}{dt}\right|_0 = X.$$

In the local chart (U, φ) the geodesic γ_X which satisfies the initial conditions:

$$x^i(0, x_0, X) = x^i, \qquad \left.\frac{dx^i}{dt}\right|_{t=0} = X^i$$

is given by $x^i = x^i(t, x_0, X)$.

We observe that for every $\alpha \in R - \{0\}$, the curve $\gamma_{\frac{1}{\alpha}X} : (-2\alpha, \; 2\alpha) \to U_1$ defined by $x^i = x^i(\alpha t, x_0, \frac{1}{\alpha}X)$ is again a geodesic with the same initial conditions, hence:

$$\gamma_X(t) = \gamma_{\frac{1}{\alpha}X}(\alpha t). \tag{5}$$

The point $\gamma_X^{(1)} \in V_n$ is called the exponential of vector $X \in T_x(V_n)$ and will be denoted by $\exp_x X$. From the relation (5) (for $\alpha = 1/t$) we get:

$$\exp_x(tX) = \gamma_{tX}(1) = \gamma_X(t). \tag{6}$$

We obtain a differentiable map:

$$\exp_x : W \subset T(V_n) \to V_n \tag{7}$$

called the exponential map, where W is an open neighbourhood of null vector $0_x \in T_x(V_n)$ which depends on ε.

Then we can consider the local chart $(U, \exp_x^{-1})$ where $U = \exp_x(W)$, in which the coordinates of a point are called normal affine coordinates.

Because the images of lines which pass by the origin of $T_x(V_n)$ are geodesics passing by x, it follows that in normal coordinates the geodesics which pass through x are given by the equations:

$$x^i = tX^i. \tag{8}$$

A geodesic $\gamma : [a, b] \to V_n$ is called minimal, if its length is at most equal to the length of every piecewise smooth curve which joins its endpoints.

A Riemannian manifold V_n is called complete if any regular geodesic may be extended indefinitely. If V_n is a complete Riemannian manifold, then any two points of V_n may be joined by a minimal geodesic and any bounded subset has compact closure.

A Riemannian manifold V_n is complete as metric space iff it is a complet Riemannian manifold. A useful property in the sequel is that the exponential map has a behaviour of an isometry for the distances measured from 0_x:

$$\forall X \in T_x(V_n), \; d(0_x, X) = \| X \| = d(x, \exp_x X).$$

Let $f : V_n \to W_m$ be a map, with V_n and W_m manifolds of class C^l, $l \geq 1$. If we denote by (V, φ) and (W, ψ) two local charts in x_0 and $f(x_0)$ respectively, then f is differentiable of class C^k $(k \leq l)$, if the map:

$$F = \psi \circ f \circ \varphi^{-1} : \varphi(V) \to \psi(W)$$

is of class C^k. The Jacobian matrix of f in x_0 with respect to (V, φ) and (W, ψ) is $[\partial F^\alpha(x_0^i)/\partial x^i]$ where $y^\alpha = F^\alpha(x^i)$ is the expression of f in the considered local charts. The rank of f in x_0 (equal with the rank of the Jacobian matrix) does not depend on the choice of the local charts. If f is a diffeomorphism then: rank $f(x) = m = n$, hence $J_f(x) = \det [\partial F^\alpha(x^i)/\partial x^i] \neq 0$ for every $x \in V_n$;

2. In the sequel we will denote by V_n a complete Riemannian manifold and by $f : D \subset V_n \to D^* = f(D) \subset V_n$ a homeomorphism, where D is a domain in V_n.

Definition 1. *We call modulus of dilatation of the homeomorphism f at a point $x_0 \in D$ in a direction $S \in T_{x_0}(V_n)$ the quantity:*

$$\left| \frac{\partial f(x_0)}{\partial S} \right| = \lim_{\substack{x \to x_0 \\ x \in \gamma}} \frac{d(f(x), f(x_0))}{d(x, x_0)} \tag{9}$$

where γ is the geodesic which passes through x_0 and has tangent vector S.

Let W be an open neighbourhood of 0_{x_0} such that W and $U = (\exp_{f(x_0)}^{-1} \circ f \circ \exp_{x_0})(W)$ be contained in the domains of definition of the diffeomorphisms $\exp_{x_0}$ and $\exp_{f(x_0)}$, respectively.

Lemma 1. *The modulus of dilatation of the homeomorphism f in the point x_0, in the direction S, coincides with the modulus of dilatation of the map:*

$$F = \exp_{f(x_0)}^{-1} \circ f \circ \exp_{x_0} : W \to U \tag{10}$$

in the point $0_{x_0} \in T_{x_0}(V_n)$ and the direction S.

Proof. Let γ be a geodesic which passes through x_0 and has tangent vector $S \in W \subset T_{x_0}(V_n)$. If $x \in \gamma$ then there exists $X = tS$, such that

$$d(x, x_0) = d(X, 0_{x_0}) = \| X \| \text{ and } d(f(x), f(x_0)) = d(F(X), F(0_{x_0})) = \| F(X) \|$$

It follows that:

$$\left| \frac{\partial f(x_0)}{\partial S} \right| = \lim_{\substack{t \to 0 \\ X = S}} \frac{\| F(X) \|}{\| X \|} = \left| \frac{\partial F(0_{x_0})}{\partial S} \right|. \tag{11}$$

Definition 2. *An A-point (affinity point) of the map f, is a point x_0 characterized by the properties:*

a) *f is differentiable at x_0*

b) *$J_f(x_0) \neq 0$, where $J_f(x_0)$ is the Jacobian of f at x_0.*

Remark 1. If we use normal coordinates at x_0 and $f(x_0)$, we deduce that $J_f(x_0) = J_F(0_{x_0})$. If x_0 is an A-point, then the linear tangent map:

$$T_{x_0} f = DF(0_{x_0}) : T_{x_0}(V_n) \to T_{f(x_0)}(V_n) \tag{12}$$

is an affine transformation.

Remark 2. If x_0 is an A-point of f, then 0_{x_0} is an A-point of F.

Let us consider an ellipsoid $E_{x_0} \subset W$ with centre 0_{x_0}, semiaxes $a_1 \geq a_2 \geq, \ldots \geq a_n$ and characteristics :

$$(C) \qquad p_m = \frac{a_m}{a_n}(m=1, \ldots, n-1) \; ; \; \alpha_i^k$$

where α_i^k are the directing cosines of the axes . $p_1 = a_1/a_n$ is called the *principal characteristic parameter* of the ellipsoid E_{x_0}. We define an ellipsoid in V_n as being the submanifold $\mathfrak{E}_{x_0} = \exp_{x_0}(E_{x_0})$. We denote by $X_i \in W$, $(i = 1, \ldots, n)$ the vectors of an orthonormal base in $T_{x_0}(V_n)$ and by γ_i, $(i = 1, \ldots, n)$ the geodesics $\gamma_i(t) = \exp_{x_0}(tX_i)$ which pass through x_0 and are tangent to X_i. If $\pm Y_i$, $(i=1, \ldots, n)$ are vertices of the ellipsoid E_{x_0}, then the lengths of the semiaxes are $a_i = \| Y_i \|$ and the directing cosines of the semiaxes $\alpha_i^k = \cos(\widehat{X_i, Y_k})$.

By definition, the axes of the ellipsoid $\mathfrak{E}_{x_0}$ are the geodesic segments $\xi_i(t) = \exp_{x_0}(tY_i)$, $t \in [-1,1]$, and the vertices the points $x_i^{\pm} = \exp_{x_0}(\pm Y_i)$. It follows that the semiaxes of the ellipsoid $\mathfrak{E}_{x_0}$ will have the lengths :

$$d(x_0, x_i^{+}) = d(0_{x_0}, Y_i) = \| Y_i \| = a_i$$

and the directing cosines :

$$\cos(\widehat{\gamma_i, \zeta_k}) = \cos(\widehat{X_i, Y_k}) = \alpha_i^k$$

By these considerations it follows that E_{x_0} and $\mathfrak{E}_{x_0}$ have the same characteristics (C). An ellipsoid $\mathfrak{E}_{x_0}$ with characteristics (C) will be denoted by $\mathfrak{E}_{x_0}(C)$.

Definition 3. *A homeomorphism f is said to transform an infinitesimal ellipsoid $\mathfrak{E}_x(C)$ into an infinitesimall ellipsoid $\mathcal{S}_{f(x)}(C^*)$ if it transforms, for each x, the ellipsoid $\mathfrak{E}_x^h(C)$ with $a_n = h$, into a surface contained between two homothetical ellipsoids $\mathfrak{E}_{f(x)}^{h_1^*}(C^*)$, $\mathfrak{E}_{f(x)}^{h_2^*}(C^*)$ with minimal semiaxes h_1^* and h_2^* respectively, so that $\lim\limits_{h \to 0} h_1^*/h_2^* = 1$ as $\mathfrak{E}_x^h(C)$ shrinks homothetically to x.*

3. Let $f : D \subset V_n \to D^* = f(D) \subset V_n$ be a homeomorphism. We define :

$$(13) \qquad \begin{cases} L(x, r) = \sup\limits_{y \in S(x, r)} d(f(x), f(y)) \\ l(x, r) = \inf\limits_{y \in S(x, r)} d(f(x), f(y)) \end{cases}$$

where $S(x, r) = \{y : y \in V_n,\ d(x, y) = r\} \subset D$.

The linear dilatation of f at $x \in D$ is defined as :

$$(14) \qquad \delta(x) = \overline{\lim_{r \to 0}} \frac{L(x, r)}{l(x, r)}$$

Definition 5. *The homeomorphism f is said to be $K-qc$ (K-quasiconformal), $1 \leq K < \infty$, according to Gehrings's metric definition if $\delta(x) \leq K$, for every $x \in D$.*

A family of hipersurfaces (submanifolds of dimension $n-1$) $\{\sigma_\alpha\}$; $\alpha \in (0,1)$ of class C^p $(p \geq 0)$ is said to be regular of parameter $k = k(x_0)$ relatively to a point x_0, if $\sigma_\alpha = \varphi[S(0, \alpha)]$ — where φ is a homeomorphism of the unit ball $B(0, 1) \subset R^n$ onto a neighbourhood of the point $x_0 = \varphi(0)$ — is so that :

$$(15) \qquad \lim_{\alpha \to 0} \frac{\sup\limits_{x \in \sigma_\alpha} d(x, x_0)}{\inf\limits_{x \in \sigma_\alpha} d(x, x_0)} = k(x_0) < \infty .$$

A homeomorphism f is called regular at $x_0 \in D$, if for at least a regular family $\{\sigma_\alpha\}$ of parameter $k(x_0)$ relatively to x_0, the family $\{f(\sigma_\alpha)\}$ is a regular family of a parameter $k^*(x_0)$ relatively to $f(x_0)$. The homeomorphism f is said to be regular in a domain D if it is regular at each of its points.

The quantity :

$$(16) \qquad q(x_0) = \inf_{\{\sigma_\alpha\}} k(x_0) \,.\, k^*(x_0),$$

where the infimum is taken over all regular families relatively to x_0, is called the characteristic of the regular homeomorphism f at x_0.

Definition 6. *A homeomorphism f, regular in D, is called K-qc, $1 \leq K < \infty$, in Markushevich-Pesin's sense if $q(x) \leq K$ for every $x \in D$.*

Lemma 2. *At an A-point x_0 of a mapping f, the following double inequality holds*

$$(17) \qquad \frac{p_2 \ldots p_{n-1}}{p_1^{n-1}} \mid J_f(x_0) \mid \leq \left| \frac{\partial f(x_0)}{\partial S} \right| \leq p_1 \ldots p_n \mid J_f(x_0) \mid .$$

With equality on the left side for dilatations in the direction of the maximal axis of the ellipsoid according to the affine transformation $T_{x_0} f$ and with equality on the right side for dilatations in the direction of the minimal axis.

Proof. Let $\mathfrak{E}_{x_0}(C)$ be an ellipsoid in $\exp_{x_0}(W) \subset D$, with centre at x_0 and characteristics (C). Then, the ellipsoid $E_{x_0}(C) = \exp_{x_0}^{-1}(\mathfrak{E}_{x_0}(C))$ is an ellipsoid with centre at 0_{x_0} and characteristics (C). By Remark 2, Lemma 1 and [2] (Lemma 2, p. 131) we obtain (17).

Lemma 3. *At a point of differentiability $x_0 \in D$ of the homeomorphism we have :*

$$(18) \qquad \Lambda(x_0) = \sup_S \left| \frac{\partial f(x_0)}{\partial S} \right| .$$

$$(19) \qquad \lambda_f(x_0) = \inf_S \left| \frac{\partial f(x_0)}{\partial S} \right|$$

where :

$$(20) \qquad \Lambda_f(x_0) = \overline{\lim_{x \to x_0}} \frac{d(f(x), f(x_0))}{d(x, x_0)} \; ; \; \lambda_f(x_0) = \underline{\lim_{x \to x_0}} \frac{d(f(x), f(x_0))}{d(x, x_0)}$$

Proof. By Lemma 1 and [2] (Lemma 3, p. 132) it follows that :

$$\Lambda_f(x_0) = \overline{\lim_{x \to x_0}} \frac{d(f(x), f(x_0))}{d(x, x_0)} = \overline{\lim_{\substack{t \to 0 \\ X = tS}}} \frac{\| F(X) \|}{\| X \|} = \sup_S \left| \frac{\partial F(0_{x_0})}{\partial S} \right| = \sup_S \left| \frac{\partial f(x_0)}{\partial S} \right|$$

where x belongs to the geodesic $\gamma(t) = \exp_{x_0}(tS)$ and $X = \exp_{x_0}^{-1}(x)$. Analogously we prove (19).

Lemma 4. *At an A-point $x_0 \in D$ of a mapping $f: D \to D^*$ the following relations :*

$$\Lambda_f(x_0) = \sup_S \left| \frac{\partial f(x_0)}{\partial S} \right| = \sqrt[n]{p_1 \dots p_{n-1} \mid J_f(x_0) \mid}, \tag{21}$$

$$\lambda_f(x_0) = \inf_S \left| \frac{\partial f(x_0)}{\partial S} \right| = \sqrt[n]{\frac{p_2 \dots p_{n-1}}{p_1^{n-1}} \mid J_f(x_0) \mid}. \tag{22}$$

hold.

The proof is a consequence of the Lemmas 2 and 3.

Corollary. *At an A-point $x_0 \in D$ of a mapping f we have :*

$$\frac{\Lambda_f(x_0)}{\lambda_f(x_0)} = p_1(x_0), \tag{23}$$

where p_1 is the principal characteristic parameter of the affine transformation $T_{x_0} f$.

Remark 3. If $\{\delta_\alpha\}$ is a regular family of hipersurfaces of parameter k relative to a point x_0, then $\{\tau_\alpha\}$, $\tau_\alpha = \exp_{x_0}^{-1}(\sigma_\alpha)$ is a regular family of the same parameter k relatively to 0_{x_0} (because $\exp_{x_0}$ behaves as an isometry for the distances measured from 0_{x_0}).

$$F(\tau_\alpha) = (\exp_{f(x_0)}^{-1} o f o \exp_{x_0})(\tau_\alpha) = (\exp_{f(x_0)}^{-1} o f)(\sigma_\alpha) = \exp_{f(x_0)}^{-1}(f(\sigma_\alpha)) \ ;$$

Lemma 5. *Let $f: D \to D^*$ be a regular homeomorphism of class C^1. Then any regular hypersurface family $\{\sigma_\alpha\}$ of parameter k, $\alpha \in (0,1)$, relatively to an A-point x_0 is mapped into a regular hypersurface family of parameter k^* iff the affine transformation $T_{x_0} f$ maps the family $\{\tau_\alpha\}$, $\tau_\alpha = \exp_{x_0}^{-1}(\sigma_\alpha)$, into a regular hypersurface family of parameter k^*.*

Proof. If x_0 is an A-point of f, then 0_{x_0} is an A-point of F. Let $\{\sigma_\alpha\}$ be a regular hypersurface family of parameter k relatively to x_0, then $\{\tau_\alpha\}$ is a regular hypersurface family of parameter k relatively to 0_{x_0}. If $\{f(\sigma_\alpha)\}$ is a regular hypersurface family of parameter k^* relatively to $f(x_0)$, it follows that $\exp_{f(x_0)}^{-1}(f(\sigma_\alpha)) = F(\tau_\alpha)$ is a regular hypersurface family of parameter k relatively to $0_{f(x_0)}$. By [2] (Lemma 7, p. 147) it follows that the affine transformation $T_{x_0} f = DF(0_{x_0})$ maps the regular hypersurface family $\{\tau_\alpha\}$ of parameter k into a regular family of parameter k^*.

Conversely, if $\{\tau_\alpha\}$ is a regular hipersurface family of parameter k relatively to $0_{x_0} \in T_{x_0}(V_n)$ and the affine transformation $T_{x_0} f = DF(0_{x_0})$ maps $\{\tau_\alpha\}$ into a regular hypersurface family of parameter k^*, then by [2] (Lemma 7, p. 146) it follows that F maps the regular family $\{\tau_\alpha\}$ of parameter k relatively to 0_{x_0} into a regular family of parameter k^*, hence f has the same property.

Theorem 1. *A diffeomorphism $f: D \subset V_n \to D^* \subset V_n$ is K-qc $(1 \leqslant K < \infty)$ in Markushevich-Pesin's sense iff it is K-qc according to Gehring's metric definition.*

Proof. Let f be K-qc in Markushevich-Pesin's sense and $x_0 \in D$, it follows that x_0 is an A-point of f. By Remark 3 and Lemma 5 it follows that f and $T_{x_0} f$ have the same characteristic $q(x_0)$ in x_0. By [2] (Lemma 1, p. 130) we have $q(x_0) = p_1(x_0)$, where p_1 is the principal characteristic parameter of $T_{x_0} f$: hence $q(x) \leqslant K$ implies $p_1(x) \leqslant K$ in D. Let $S(x_0, r)$ be the sphere with centre x_0 and radius r, $S(x_0, r) \subset D$ and $x', x'' \in S(x_0, r)$ so that :

$$d(f(x'), f(x'')) > \sup_{x \in S(x_0, r)} d(f(x), f(x_0)) - \varepsilon r$$

$$d(f(x''), f(x_0)) < \inf_{x \in S(x_0, r)} d(f(x), f(x_0)) + \varepsilon r$$

$$\delta(x_0) = \overline{\lim_{r \to 0}} \frac{\sup_{x \in S(x_0, r)} d(f(x), f(x_0))}{\inf_{x \in S(x_0, r)} d(f(x), f(x_0))} < \overline{\lim_{r \to 0}} \frac{d(f(x'), f(x_0)) + \varepsilon r}{d(f(x''), f(x_0)) - \varepsilon r} \leqslant$$

$$\leqslant \left[\overline{\lim_{\substack{x \to x_0 \\ x \in S(x_0, r)}}} \frac{d(f(x), f(x_0))}{d(x, x_0)} + \varepsilon \right] \Big/ \underline{\lim_{\substack{x \to x^0 \\ x \in S(x_0, r)}}} \left[\frac{d(f(x), f(x_0))}{d(x, x_0)} - \varepsilon \right] = \frac{\Lambda_f(x_0) + \varepsilon}{\lambda_f(x_0) - \varepsilon}.$$

For $\varepsilon \to 0$ and by Corollary of Lemma 4 we have :

$$\delta(x_0) \leqslant \frac{\Lambda_f(x_0)}{\lambda_f(x_0)} = p_1(x_0) = q(x_0) \leqslant K,$$

and as x_0 was an arbitrary point of D, we are allowed to conclude the K-quasiconformality of f in D according to Gehring's metric definition.

Let us consider now that f is K-qc in the sense of Gehring's metric definition. Let $B(0_x, 1)$ be the unit ball and $B(O_x, a)$ the ball of radius a in the tangent space $T_x(V_n)$ with $x \in D$. Let the diffeomorphism $h: B(0_x, 1) \to B(0_x, a)$ be defined by $h(X) = a \cdot X$ for every $X \in B(O_x, 1)$. Let a be a fixed real sufficiently small number such that the map $\exp_x : B(O_x, a) \to V_n$ be well defined. Let us denote $\varphi = \exp_x o h : B(O_x, 1) \to V_n$. For every $\alpha \in (0,1)$, let us consider the hipersurface :

$$\sigma_\alpha = \varphi(S(O_x, \alpha)) = \exp_x(h(S(O_x, \alpha))) = \exp_x(S(O_x, a\alpha)) = \{y : d(x, y) = a\alpha\}.$$

It follows that :

$$\overline{\lim_{\alpha \to 0}} \frac{\sup_{y \in \sigma_\alpha} d(x, y)}{\inf_{y \in \sigma_\alpha} d(x, y)} = 1,$$

hence the family $\{\sigma_\alpha\}$ is regular relatively to x.

Set $\tau_\alpha = f(\sigma_\alpha) = (f o \exp_x o h)(S(O_x, \alpha))$, then :

$$k^*(x) = \overline{\lim_{\alpha \to 0}} \frac{\sup_{f(y) \in \tau_\alpha} d(f(x), f(y))}{\inf_{f(y) \in \tau_\alpha} d(f(x), f(y))} = \overline{\lim_{\alpha \to 0}} \frac{\sup_{d(x, y) = a} d(f(x), f(y))}{\inf_{d(x, y) = a} d(f(x), f(y))} = \delta(x) \leqslant K,$$

Hence $q(x) = \inf k(x) \cdot k^*(x) = \inf \delta(x) \leqslant K < \infty$, implying the K-quasiconformality in Markushevich-Pesin's sense.

Corollary 1. *A diffeomorphism K-qc according to Gehring's metric definition has the inverse K-qc according to this definition.*

The proof results from Theorem 1 and [1] (Remark 1)

Theorem 2. *Let $f: D \to D^*$ and $h: D^* \to D^{**}$ be diffeomorphisms K (respectively K')-qc according to Gehring's metric definition. Then, the diffeomorphism $u = h \circ f$ is $K \cdot K'$—qc according to the same definition.*

Proof.

$$\delta(x) = \overline{\lim_{r\to 0}} \frac{\sup\limits_{y\in S(x,r)} d(u(x), u(y))}{\inf\limits_{y\in S(x,r)} d(u(x), u(y))} \leqslant \frac{\overline{\lim\limits_{r\to 0}} \dfrac{\sup\limits_{y\in S(x,r)} d(u(x), u(y))}{r}}{\underline{\lim\limits_{r\to 0}} \dfrac{\inf\limits_{y\in S(x,r)} d(u(x), u(y))}{r}} = \frac{\overline{\lim\limits_{y\to x}} \dfrac{d(u(x), u(y))}{d(x, y)}}{\underline{\lim\limits_{y\to x}} \dfrac{d(u(x), u(y))}{d(x, y)}} =$$

$$= \frac{\overline{\lim\limits_{y\to x}} \dfrac{d(u(x), u(y))}{d(f(x), f(y))} \cdot \dfrac{d(f(x), f(y))}{d(x, y)}}{\underline{\lim\limits_{y\to x}} \dfrac{d(u(x), u(y))}{d(f(x), f(y))} \cdot \dfrac{d(f(x), f(y))}{d(x, y)}} \leqslant \frac{\overline{\lim\limits_{f(y)\to f(x)}} \dfrac{d(h(f(x)), h(f(y)))}{d(f(x), f(y))}}{\underline{\lim\limits_{f(y)\to f(x)}} \dfrac{d(h(f(x)), h(f(y)))}{d(f(x), f(y))}} \cdot$$

$$\cdot \frac{\overline{\lim\limits_{y\to x}} \dfrac{d(f(x), f(y))}{d(x, y)}}{\underline{\lim\limits_{y\to x}} \dfrac{d(f(x), f(y))}{d(x, y)}} = \frac{\Lambda_h(f(x))}{\lambda_h(f(x))} \cdot \frac{\Lambda_f(x)}{\lambda_f(x)} = p_1(x) \cdot p_1^*(f(x)) \leqslant K \cdot K'$$

REFERENCES

1. Borcea V. T. — *On quasiconformal mappings in Riemannian manifolds of dimension n*, to appear.
2. Caraman, P. — *n-dimensional quasiconformal mappings (QCf)*. Edit. Acad. București (Romania) and Abacus Tunbridge Wells, Kent (England), 1974
3. Caraman, P. — *Qasiconformal mappings in real normed spaces*, Rev. Roum. Math. Pures Appl., 25 (1979) 33—78.
4. Gehring, F. W. — *The Carathéodory convergence theorem for quasiconformal mappings in space*, Ann. Acad. Sci. Fenn. Ser. A 336 (1963), 1—21.
5. Gheorghiev, Gh., Oproiu, V. — *Geometrie diferențială*, Edit. Did. Ped. București (Romania), 1977.
6. Lans, S. — *Introduction to Differentiable Manifolds*, Intersience Publishers, New York, 1962

Received 9. III 1985

Department of Mathematics
Polytechnical Institute
6600 Iași, R.S. România

Analele științifice ale Universității „Al. I. Cuza" din Iași
Tomul XXXII, s. I a, Matematică, 1986, f. 1

ON A CERTAIN BOUNDARY VALUE PROBLEM FOR THIRD ORDER DIFFERENTIAL EQUATIONS

BY

B. G. PACHPATTE

1. Introduction. In this paper we shall be concerned with the study of nonlinear boundary value problem (BVR) for third order differential equations of the form

$$y''' = g(t, y, y', y'')\, F(t, y, y', y'', y''', \lambda), \qquad a \leqslant t \leqslant c, \tag{1}$$

$$y(a) = 0, \; y(b) = 0, \; y(c) = 0, \qquad a < b < c, \tag{2}$$

where λ is a real parameter, $g \in C[[a, c] \times R^3, R]$, $F \in C[[a, c] \times R^5, R]$, R denotes the set of real numbers. In the past several years, an enormous amount of attention has been given to the study of special forms of boundary value problems of the type (1)—(2), particularly for second order differential equations, under various types of boundary conditions (see for example [3, 9, 10, 14] and the references given therein). Many sufficient conditions are known for the existence and uniqueness of solutions of BVP (1)—(2) when $F = 1$ under various types of boundary conditions (see, [1, 2, 4, 5, 7]). The main purpose of this paper is to obtain sufficient conditions for the existence, uniqueness and continuous dependence on parameter λ of the solutions of BVP (1)—(2) on $[a, c]$. Our formulation of BVP (1)—(2) is motivated by the study of a very interesting class of boundary value problems by Nehari [10], Moroney [9], Hooker [3], Ullrich [14] and others and a class of boundary value problems studied by Kamenskii and Myshkis [6] for second order boundary value problems. The method employed in our analysis is an extension of the method used in [6] which in turn is based on the application of a suitable version of the well known Banach fixed point theorem.

2. Main results. Let $B[a, c]$ be the class of real valued functions $z(t) \in C^3[a, c]$ with norm

$$\| z \| = \max \left\{ \frac{81}{2(c-a)^3} \max_{a\leqslant t\leqslant c} |z(t)|, \; \frac{6}{(c-a)^2} \max_{a\leqslant t\leqslant c} |z'(t)|, \; \frac{3}{2(c-a)} \max_{a\leqslant t\leqslant c} |z''(t)|, \; \max_{a\leqslant t\leqslant c} |z'''(t)| \right\}. \tag{3}$$