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ALMOST SEMI-INVARIANT SUBMANIFOLDS IN COSYMPLECTIC MANIFOLDS OF CONSTANT F-SECTIONAL CURVATURE

BY

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§ 1. Introduction. Taking into account the fundamental results on the geometry of almost semi-invariant submanifolds in a Sasakian manifold obtained in [1] by A. Bejancu and N. Papaghiuc and also the study of A. Bejancu and D. Smaranda in [3], we have considered in [5] the case of almost semi-invariant submanifolds in cosymplectic manifolds.

The present paper continues the study of almost semi-invariant submanifolds in cosymplectic manifolds which are supposed now of constant F-sectional curvature. The study of cosympcctic manifolds of constant F-sectional curvature is not far from the case of Sasakian space forms.Consequently many of the following results will be proved by methods similar to those used in [8] by N. Papaghiuc.

§ 2. Preliminaries. Let $\tilde{M}$ be a $(2n+1)$ - dimensional almost contact metric manifold with (F, ξ, η, g) as the almost contact metric structure, where F is a tensor field of type $(1,1)$, ξ is a vector field, η is a 1-form and g is a Riemannian metric on $\tilde{M}$ satisfying :

$$F^2X=-X+\eta\otimes\xi\ ;\quad F\xi=0\ ;\quad \eta\circ F=0\ ;\quad \eta(\xi)=1 \text{ and} \tag{2.1}$$

$$g(FX,\ FY)=g(X,\ Y)-\eta(X)\ \eta(Y),\quad \forall\ X,\ Y\in\Gamma(T\tilde{M}). \tag{2.2}$$

Thus we get :

$$g(X,\ FY)+g(FX,\ Y)=0,\qquad \forall\ X,\ Y\in\Gamma(T\tilde{M}). \tag{2.3}$$

Let Φ be the fundamental two-form associated to the structure F, defined by $\Phi(X, Y)=g(X, FY)$, $\forall\ X, Y\in\Gamma(T\tilde{M})$. Suppose that all the additional conditions are satisfied on the almost contact manifold $\tilde{M}$, such that $\tilde{M}$ becomes a cosymplectic manifold (see [3]).

If $\tilde{M}$ is a cosymplectic manifold, we have :

$$\tilde{\nabla}_X F=0 \text{ and} \tag{2.4}$$

$$\tilde{\nabla}_X\xi=0,\quad \forall X\in\Gamma(T\tilde{M}),\text{ where } \tilde{\nabla} \text{ is the Levi-Civita connection on } \tilde{M}. \tag{2.5}$$

Now, let M be a Riemannian manifold isometrically immersed in a cosymplectic manifold $\tilde{M}$ and suppose $\xi \in \Gamma(TM)$. Then we denote by the same symbol g the Riemannian metric on M. Denote also, by ∇ the Levi-Civita connection on M with respect to g and by $\nabla^{\perp}$ the linear connection induced by $\tilde{\nabla}$ on the normal bundle $TM^{\perp}$. For each bundle K, let $\Gamma(K)$ be the module of its differentiable sections. Then the equations of Gauss and Weingarten are:

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y) \text{ and} \tag{2.6}$$

$$\tilde{\nabla}_X N = -A_N X + \nabla^{\perp}_X N, \quad \forall X, Y \in \Gamma(TM), \ N \in \Gamma(TM^{\perp}), \tag{2.7}$$

where h is the second fundamental form on M and A_N is the fundamental tensor of Weingarten with respect to the normal section N, satisfying:

$$g(h(X, Y), N) = g(A_N X, Y) = g(A_N Y, X), \quad \forall X, Y \in \Gamma(TM), \quad N \in \Gamma(TM^{\perp}). \tag{2.8}$$

Denote by $\{\xi\}$ the 1-dimensional distribution spanned by ξ on M and by $\{\xi\}^{\perp}$ the complementary orthogonal distribution to $\{\xi\}$ in TM.

For any $X \in \Gamma(TM)$, we have $g(FX, \xi) = 0$, then we put

$$FX = bX + cX, \tag{2.9}$$

where $bX \in \Gamma(\{\xi\}^{\perp})$ and $cX \in \Gamma(TM^{\perp})$.

Definition 2. 1. *The submanifold M is called an almost semi-invariant submanifold of $\tilde{M}$ if its tangent bundle TM has the decomposition $TM = D \oplus D^{\perp} \oplus \tilde{D} \oplus \{\xi\}$, where:*

i) *D is an invariant distribution on M, that is $F(D) = D$,*

ii) *$D^{\perp}$ is an anti-invariant distribution on M, that is $F(D^{\perp}) \subset TM^{\perp}$,*

iii) *$\tilde{D}$ is the complementary orthogonal distribution to $D \oplus D^{\perp} \oplus \{\xi\}$ in TM.*

It is easily seen that the decomposition of TM is an orthogonal one.

From the definition of $\tilde{D}$, we get that b and c are two automorphisms of $\tilde{D}$.

We notice the following particular cases of almost semi-invariant submanifolds:

1. Semi-invariant submanifold, when $\tilde{D} = \{0\}$ (and it is a proper one if $D \neq \{0\}$, $D^{\perp} \neq \{0\}$);
2. Invariant submanifold, when $\tilde{D} = D^{\perp} = \{0\}$ (and it is a proper one if $D \neq \{0\}$),
3. Anti-invariant submanifold if $D = \tilde{D} = \{0\}$ (and it is a proper one if $D^{\perp} \neq \{0\}$),
4. Almost invariant submanifold if $D^{\perp} = \{0\}$ (and it is a proper one if $D \neq \{0\}$, $\tilde{D} \neq \{0\}$),
5. Almost anti-invariant submanifold when $D = \{0\}$ (and it is a proper one if $D^{\perp} \neq \{0\}$ and $\tilde{D} \neq \{0\}$),
6. Pseudo-invariant submanifold when $D = D^{\perp} = \{0\}$ (and it is a proper one if $\tilde{D} \neq \{0\}$),

7. Proper almost semi-invariant submanifold when $D \neq \{0\}$, $D^{\perp} \neq \{0\}$, $\tilde{D} \neq \{0\}$.

Denote by ν the orthogonal complementary vector bundle to $FD^{\perp} \oplus c\tilde{D}$ in $TM^{\perp}$. Then we have the following orthogonal decomposition:

$$TM^{\perp} = FD^{\perp} \oplus c\tilde{D} \oplus \nu. \tag{2.10}$$

Let M be an almost semi-invariant submanifold of a cosymplectic manifold $\tilde{M}$. Recall the following results obtained in [5]:

Lemma 2. 1. [5]. *If K is one of the distributions D, $D^{\perp}$, $\tilde{D}$, or $\{\xi\}$, then we have:*

$$\nabla_X \xi = 0, \ h(X, \xi) = 0 \text{ and } \nabla_{\xi} X \in \Gamma(K), \quad \forall X \in \Gamma(K). \tag{2.11}$$

Proposition 2. 1, [5]. *Let K be one of the distributions D, $D^{\perp}$, $\tilde{D}$, or every direct sum of them. Then we have:*

i) *K is integrable if and only if $K \oplus \{\xi\}$ is integrable,*

ii) *K is integrable and its leaves are totally geodesic immersed in M if and only if $K \oplus \{\xi\}$ has the same property.*

Theorem 2. 1, [5]. 1) *D is integrable if and only if:*

$$h(X, FY) = h(Y, FX), \quad \forall X, Y \in \Gamma(D). \tag{2.12}$$

2) *$D^{\perp}$ is always integrable.*

3) *$\tilde{D}$ is integrable if and only if:*

$$\nabla_Y bX - \nabla_X bY + A_{cY} X - A_{cX} Y \in \Gamma(\tilde{D}) \text{ and} \tag{2.13}$$

$$h(bX, Y) - h(X, bY) + \nabla^{\perp}_Y cX - \nabla^{\perp}_X cY \in \Gamma(c\tilde{D}), \quad \forall X, Y \in \Gamma(\tilde{D}). \tag{2.14}$$

4) *$D \oplus D^{\perp}$ is integrable if and only if:*

$$A_{FY} X + \nabla_Y FX \in \Gamma(D), \quad \forall X \in \Gamma(D), \ Y \in \Gamma(D^{\perp}). \tag{2.15}$$

5) *$D \oplus \tilde{D}$ is integrable if and only if:*

$$h(bX, Y) - h(X, bY) + \nabla^{\perp}_Y cX - \nabla^{\perp}_X cY \in \Gamma(c\tilde{D}), \quad \forall X, Y \in \Gamma(D \oplus \tilde{D}). \tag{2.16}$$

Theorem 2. 2, [5]. 1) *D is integrable and its leaves are totally geodesic immersed in M if and only if:*

$$h(X, Y) \in \Gamma(\nu), \quad \forall X, Y \in \Gamma(D) \text{ and} \tag{2.17}$$

$$\nabla_X Z \in \Gamma(D^{\perp} \oplus \tilde{D}), \quad \forall X \in \Gamma(D), \quad Z \in \Gamma(\tilde{D}). \tag{2.18}$$

2) *$D \oplus D^{\perp}$ is integrable and its leaves are totally geodesic immersed in M if and only if*

$$h(X, bZ) + \nabla^{\perp}_X cZ \in \Gamma(c\tilde{D} \oplus \nu) \text{ and} \tag{2.19}$$

$$\nabla_X bZ - A_{cZ} X \in \Gamma(D^{\perp} \oplus \tilde{D}), \quad \forall x \in \Gamma(D \oplus D^{\perp}), \quad Z \in \Gamma(\tilde{D}). \tag{2.20}$$

3) *$D \oplus \tilde{D}$ is integrable and its leaves are totally geodesic immersed in M if and only if*

$$h(X, bY) + \nabla^{\perp}_X cY \in \Gamma(c\tilde{D} \oplus \nu), \quad \forall X, Y \in \Gamma(D \oplus \tilde{D}). \tag{2.21}$$

Recall that the sectional curvature $\tilde{K}(X, Y)$ of the 2-plane of $\tilde{M}$ formed by the orthonormal basis $\{X, Y\}$, is given by $(\tilde{K}(X, Y)=g(\tilde{R}(X, Y)\, Y, X)$, see [6]). If X is a unit vector orthogonal to ξ, we say that X and FX span a F-section. If the sectional curvatures $\tilde{K}(X)$ of all F-sections are independent of X, we say $\tilde{M}$ is of constant F-sectional curvature. D. E. Blair has shown (unpublished) that in a cosymplectic manifold of constant F-sectional curvature k, we have (see [6]):

$$\begin{aligned}\tilde{R}(X, Y)\, Z= & \frac{k}{4}\{g(Y, Z)X-g(X, Z)Y-\eta(X)g(Y, Z)\xi-\eta(Y)\eta(Z)X+\\ & +\eta(X)\, \eta(Z)\, Y+\eta(Y)\, g(X, Z)\, \xi+\Phi(Z, Y)\, FX+\Phi(X, Z)\, FY+\\ & +2\, \Phi(X, Y)\, FZ\}, \quad \forall X, Y, Z\in\Gamma(TM).\end{aligned} \tag{2.22}$$

Then the equation of Codazzi is given by:

$$(D_Xh)(Y,Z)-(D_Yh)(X,Z)=\frac{k}{4}\{g(Z, bY)cX-g(Z, bX)\, cY-2g(bX, Y)cZ\}, \tag{2.23}$$

$\forall X, Y, Z\in\Gamma(TM)$, where $(D_Xh)(Y, Z)$ is defined by:

$$(D_Xh)(Y, Z)=\nabla^{\perp}_X(h(Y, Z))-h(\nabla_XY, Z)-h(Y, \nabla_XZ), \quad \forall X, Y, Z\in\Gamma(TM). \tag{2.24}$$

§ **3. Some basic results.** Denote by H, H_1 or H_2 one of the distributions D, $D^{\perp}$, $\tilde{D}$, $\{\xi\}$ or every direct sum of them.

Remark 3.1. If $X\in\Gamma(H)$ then $bX\in\Gamma(H)$.

Definition 3.1. *An almost semi-invariant submanifold of a cosymplectic manifold $\tilde{M}$ is said to be (H_1, H_2)-geodesic if $h(X, Z)=0$, $\forall X\in\Gamma(H_1)$, $Z\in\Gamma(H_2)$.*

Remark 3.2. By using (2.11) we obtain that M is (H_1, H_2)-geodesic if and only if M is $(H_1\oplus\{\xi\}, H_2\oplus\{\xi\})$-geodesic.

Lemma 3.1. *Let M be a (H_1, H_2)-geodesic almost semi-invariant submanifold of a cosymplectic manifold $\tilde{M}$. Then we have:*

i) *Every $X\in\Gamma(H_1)$ and $Z\in\Gamma(H_2)$ satisfy:*

$$b\nabla_XZ=\nabla_XbZ-A_{cZ}X \text{ and} \tag{3.1}$$

$$c\nabla_XZ=\nabla^{\perp}_XcZ. \tag{3.2}$$

ii) *A_UX is a section of the direct complement of $H_2\oplus\{\xi\}$ in TM, $\forall X\in\Gamma(H_1)$, $U\in\Gamma(TM^{\perp})$.*

Proof. i) From (2.4), (2.6), (2.7), (2.9) we get (3.1) and (3.2), since M is (H_1, H_2)-geodesic in $\tilde{M}$.

ii) We complete the proof by using (2.8) and taking into account the hypothesis.

Lemma 3.2. *Let M be a (H_1, H_2)-geodesic almost semi-invariant submanifold in a cosymplectic manifold $\tilde{M}$ of constant F-sectional curvature k and suppose H_1 and H_2 are orthogonal distributions. If H_1 is integrable, then we have:*

$$(k/2)\, g(bX, Y)\, cZ=h(Y, \nabla_XZ)-h(X, \nabla_YZ), \quad \forall X, Y\in\Gamma(H_1),\ Z\in\Gamma(H_2). \tag{3.3}$$

Proof. Let $X, Y\in\Gamma(H_1)$ and $Z\in\Gamma(H_2)$ be arbitrary vector fields. From (2.24) we get:

$$(D_Xh)(Y, Z)=-h(\nabla_XY, Z)-h(Y, \nabla_XZ), \tag{3.4}$$

since M is (H_1, H_2)-geodesic in $\tilde{M}$. By using (3.4) and the integrability of H_1, we obtain:

$$(D_Xh)(Y, Z)-(D_Yh)(X, Z)=h(X, \nabla_YZ)-h(Y, \nabla_XZ), \tag{3.5}$$

since M is (H_1, H_2)-geodesic in $\tilde{M}$. Taking into account the remark 3.1. in (2.23) and by using the orthogonality of H_1 and H_2, we get:

$$(D_Yh)(X, Z)-(D_Xh)(Y, Z)=(k/2)\, g(bX, Y)\, cZ. \tag{3.6}$$

From (3.5) and (3.6) we find (3.3).

Remark 3.3. By using (2.8) in (3.3) we obtain:

$$\frac{k}{2}g(bX, Y)\, \|cZ\|^2=g(A_{cZ}Y, \nabla_XZ)-g(A_{cZ}X, \nabla_YZ), \quad \forall X, Y\in\Gamma(H_1),\ Z\in\Gamma(H_2). \tag{3.7}$$

Remark 3.4. If we take in the lemma 3.2 $H_1=D^{\perp}$, then we obtain $h(X, \nabla_YZ)=h(Y, \nabla_XZ)$, $\forall X, Y\in\Gamma(D^{\perp})$, $Z\in\Gamma(H_2)$.

Definition 3.2. *Let M be an almost semi-invariant submanifold of a cosymplectic manifold $\tilde{M}$. Then*

i) *H_2 is said to be H_1-parallel if we have:*

$$\nabla_XY\in\Gamma(H_2), \quad \forall X\in\Gamma(H_1),\ Y\in\Gamma(H_2). \tag{3.8}$$

ii) *$FD^{\perp}$ is said to be H-parallel if we have:*

$$\nabla^{\perp}_XN\in\Gamma(FD^{\perp}), \quad \forall X\in\Gamma(H),\ N\in\Gamma(FD^{\perp}). \tag{3.9}$$

Remark 3.5. Using (2.10), (2.7), (3.9), we obtain that (3.9) is equivalent with:

$$\nabla^{\perp}_XN\in\Gamma(c\tilde{D}\oplus\nu), \quad \forall X\in\Gamma(H),\ N\in\Gamma(c\tilde{D}\oplus\nu). \tag{3.10}$$

Lemma 3.3. *Let M be a (H_1, H_2)-geodesic almost semi-invariant submanifold in a cosymplectic manifold $\tilde{M}$ of nonzero constant F-sectional curvature $(k\neq 0)$ and suppose H_1 and H_2 are orthogonal distributions. If H_1 is integrable and H_2 is H_1-parallel, then either $bH_1=\{0\}$ or $cH_2=\{0\}$.*

Proof. Using (3.8) in (3.3), we get :

$$\frac{k}{2}g(bX,\ Y)\,cZ=0, \qquad \forall\, X,\ Y\in\Gamma(H_1),\ Z\in\Gamma(H_2). \tag{3.11}$$

From (3.11), by using the remark 3.1, we get :

$$\frac{k}{2}\|\,bX\,\|^2\ \|\,cZ\,\|^2=0, \qquad \forall X\in\Gamma(H_1),\ Z\in\Gamma(H_2). \tag{3.12}$$

Thus the proof is complete.

Remark 3.6. By using (2.11) we obtain that the condition "H_2 is H_1 - paralell" can be replaced by :

$$\nabla_X Z\in\Gamma\,(H_2\oplus\{\xi\}), \qquad \forall X\in\Gamma\,(H_1),\ Z\in\Gamma(H_2). \tag{3.13}$$

§ 4. Almost semi-invariant submanifolds in cosymplectic manifolds of constant F-sectional curvature. From now on, suppose $\tilde{M}$ be a cosymplectic manifold of constant F-sectional curvature k. All the following results will be obtained replacing H_1 and H_2 from §3, by some particular distributions. Consider the following condition :

(*C*1) *M* is a $(D,\ D^{\perp}\oplus\tilde{D})$ - geodesic submanifold of $\tilde{M}$ and suppose D is an integrable distribution.

Lemma 4. 1. *If the condition* (*C*1) *is satisfied, then we have* :

$$\begin{gathered}\|\nabla_X bZ\,\|^2=\|\,A_{cZ}X\,\|^2+\|\,b\nabla_X Z\,\|^2+\frac{k}{2}\|\,X\,\|^2\ \|\,cZ\,\|^2,\\ \forall X\in\Gamma(D),\ Z\in\Gamma(D^{\perp}\oplus\tilde{D}).\end{gathered} \tag{4.1}$$

Proof. In the lemma 3.1, we take $H_1=D$ and $H_2=D^{\perp}\oplus\tilde{D}$. Using the result obtained in ii) of the lemma 3.1, we get :

$$A_{cZ}X\in\Gamma(D), \qquad \forall X\in\Gamma(D),\ Z\in\Gamma(D^{\perp}\oplus\tilde{D}). \tag{4.2}$$

From (2.2), (2.6), (2.7), (4.2), (2.3), (2.9), we have :

$$\begin{gathered}g(A_{cZ}Y,\ \nabla_X Z)=g(A_{cZ}Y,\ \tilde{\nabla}_X Z)=-g(A_{cZ}Y,\ F\tilde{\nabla}_X FZ)=\\ =g(FA_{cZ}Y,\ \nabla_X bZ)-g(FA_{cZ}Y,\ A_{cZ}X), \qquad \forall X,\ Y\in\Gamma(D),\ Z\in\Gamma(D^{\perp}\oplus\tilde{D}).\end{gathered} \tag{4.3}$$

Using (2.8), (2.12) and (4.2), we find :

$$A_{cZ}FX=-FA_{cZ}X, \qquad \forall X\in\Gamma\ (D),\ Z\in\Gamma(D^{\perp}\oplus\tilde{D}). \tag{4.4}$$

Substituting (4.4) in (4.3) we get :

$$\begin{gathered}g(A_{cZ}Y,\ \nabla_X Z)=g(FA_{cZ}Y,\ \nabla_X bZ)+g(A_{cZ}FY,\ A_{cZ}X),\\ \forall X,\ Y\in\Gamma(D),\ Z\in\Gamma(D^{\perp}\oplus\tilde{D}).\end{gathered} \tag{4.5}$$

Changing X with Y in (4.5), we have also :

$$\begin{gathered}g(A_{cZ}X,\ \nabla_Y Z)=g(FA_{cZ}X,\ \nabla_Y bZ)+g(A_{cZ}FX,\ A_{cZ}Y),\\ \forall X,\ Y\in\Gamma(D),\ z\in\Gamma(D^{\perp}\oplus\tilde{D}).\end{gathered} \tag{4.6}$$

Substituting (4.5) and (4.6) in (3.7), then taking $Y=FX$ and using (2.1), (2.2), (4.4), (4.2), we get :

$$\begin{gathered}\frac{k}{2}\|\,X\,\|^2\ \|\,cZ\,\|^2=g(A_{cZ}X,\ \nabla_X bZ)-g(FA_{cZ}X,\ \nabla_{FX} bZ)-2\,\|\,A_{cZ}X\,\|^2,\\ \forall X\in\Gamma(D),\ Z\in\Gamma(D^{\perp}\oplus\tilde{D}).\end{gathered} \tag{4.7}$$

Since $FA_{cZ}X$ and bZ are orthogonal, using (2.3) and (4.2), we have :

$$\begin{gathered}g(FA_{cZ}X,\ \nabla_{FX} bZ)=-g(\tilde{\nabla}_{FX}(FA_{cZ}X),\ bZ)=-g(F\tilde{\nabla}_{FX}(A_{cZ}X),\ bZ)=\\ =g(\tilde{\nabla}_{FX}(A_{cZ}X),\ FbZ)=g([FX,\ A_{cZ}X]+\tilde{\nabla}_{A_{cZ}X}FX,\ FbZ)=\\ =g(\tilde{\nabla}_{A_{cZ}X}FX,\ FbZ)=g(F\tilde{\nabla}_{A_{cZ}X}X,\ FbZ)=g(\tilde{\nabla}_{A_{cZ}X}X,\ bZ)=\\ =g(\tilde{\nabla}_{A_{cZ}X}X,\ bZ)=g([A_{cZ}X,\ X]+\tilde{\nabla}_X(A_{cZ}X),\ bZ)=g(\tilde{\nabla}_X(A_{cZ}\ X),\ bZ)=\\ =g(\nabla_X(A_{cZ}X),\ bZ)=-g(A_{cZ}X,\ \nabla_X bZ), \qquad \forall X\in\Gamma(D),\ Z\in\Gamma(D^{\perp}\otimes\tilde{D}).\end{gathered} \tag{4.8}$$

Substituting (4.8) in (4.7), we get :

$$\begin{gathered}\frac{k}{2}\|\,X\,\|^2\ \|\,cZ\,\|^2=2g(A_{cZ}X,\ \nabla_X bZ)-2\ \|\,A_{cZ}X\,\|^2,\\ \forall X\in\Gamma(D),\ Z\in\Gamma(D^{\perp}\oplus\tilde{D}).\end{gathered} \tag{4.9}$$

From (3.1) we have :

$$\begin{gathered}2g(A_{cZ}X,\ \nabla_X bZ)=\|\nabla_X bZ\,\|^2+\|\,A_{cZ}X\,\|^2-\|\,b\nabla_X Z\,\|^2,\\ \forall x\in\Gamma(D),\ Z\in(D^{\perp}\oplus\tilde{D}).\end{gathered} \tag{4.10}$$

Substituting (4.10) in (4.9) we get (4.1).

Theorem 4.1. *Suppose M satisfies the condition* (*C*1). *If the distribution $\tilde{D}$ is D-parallel and $k\neq 0$, then we have: either M is an almost anti-invariant submanifold of $\tilde{M}$, or M is a semi-invariant submanifold of $\tilde{M}$. If M is a proper semi-invariant submanifold, then we have $k<0$.*

Proof. In the lemma 3.1 and the definition 3.2, we take *If* $H_1=D$ and $H_2=D^{\perp}\oplus\tilde{D}$. From (4.2), (3.1) and (3.8), we obtain :

$$A_{cY}X=0 \ \text{ and } \ \nabla_X bY=b\nabla_X Y, \qquad \forall X\in\Gamma(D),\ Y\in\Gamma(\tilde{D}) \text{ and} \tag{4.11}$$

$$A_{cZ}X=-b\nabla_X Z, \qquad \forall X\in\Gamma(D),\ Z\in\Gamma(D^{\perp}). \tag{4.12}$$

Then (4.1) becomes :

$$\frac{k}{2}\|\,X\,\|^2\ \|\,cY\,\|^2=0, \qquad \forall X\in\Gamma(D),\ Y\in\Gamma(\tilde{D}) \text{ and respectively} \tag{4.13}$$

$$(4.14)\qquad 2\,\|A_{cZ}X\|^2+\frac{k}{2}\|X\|^2\,\|cZ\|^2=0,\qquad \forall X\in\Gamma(D),\ Z\in\Gamma(D^\perp).$$

Then the assertion of the theorem follows from (4.13) and (4.14).

Theorem 4.2. *Suppose M satisfies the condition (C1). If $D\neq\{0\}$ and $k>0$, then M is an almost invariant submanifold of $\tilde{M}$.*

Proof. Suppose $D^\perp\neq\{0\}$ and let $Z\in\Gamma(D^\perp)$ be a nonzero vector field. Then $bZ=0$ and $cZ=FZ$. By applying the lemma 4.1, we get:

$$(4.15)\qquad \|A_{FZ}X\|^2+\|b\nabla_X Z\|^2+\frac{k}{2}\|X\|^2\,\|Z\|^2=0,\qquad \forall X\in\Gamma(D).$$

From (4.15) it follows $k<0$. This contradiction completes the proof.

Theorem 4.3. *Suppose M satisfies the condition (C1). If the leaves of the distribution D are totally geodesic immersed in M and $k\neq0$, then M is either an invariant submanifold of $\tilde{M}$, or an almost anti-invariant submanifold of $\tilde{M}$.*

Proof. Let $X\in\Gamma(D)$ and $Z\in\Gamma(D^\perp\oplus\tilde{D})$ be arbitrary vector fields. Since M is $(D,\ D^\perp\oplus\tilde{D})$-geodesic, from (2.8) and (2.17) (see the theorem 2.2), we obtain:

$$(4.16)\qquad A_{cZ}X=0.$$

In the lemma 3.1, we consider $H_1=D$ and $H_2=D^\perp\oplus\tilde{D}$. Then from (3.1) we get:

$$(4.17)\qquad \nabla_X bZ=b\nabla_X Z.$$

By using (4.16) and (4.17) in (4.1) we obtain:

$$(4.18)\qquad (k/2)\,\|X\|^2\,\|cZ\|^2=0.$$

From (4.18) we get either $D=\{0\}$ or $D^\perp\oplus\tilde{D}=\{0\}$ and the proof is complete.

Consider the following condition:

(C2) M is a $(D\oplus\tilde{D},\ D^\perp)$-geodesic almost semi-invariant submanifold of $\tilde{M}$, the distribution $D\oplus\tilde{D}$ is integrable and the subbundle $FD^\perp$ is $D\oplus\tilde{D}$-parallel.

Lemma 4.2. *If M satisfies the condition (C2), then we have:*

$$(4.19)\qquad A_N bX+bA_N X=0,\qquad \forall X\in\Gamma(TM),\ N\in\Gamma(FD^\perp).$$

Proof. Taking $H=D\oplus\tilde{D}$ in (3.10) we get:

$$(4.20)\qquad \nabla^\perp_X cY\in\Gamma(c\tilde{D}\oplus\nu),\qquad \forall X,\ Y\in\Gamma(D\oplus\tilde{D}).$$

Using (4.20) in (2.16) and taking account of (2.11) and of the definition of $D^\perp$, we obtain:

$$(4.21)\qquad g(h(X,bY),N)=g(h(Y,bX),N),\qquad \forall X,\ Y\in\Gamma(TM),\ N\in\Gamma(FD^\perp).$$

Then, from (2.8), (2.9) and (4.21) we get (4.19).

Theorem 4.4. *If M is a proper almost semi-invariant submanifold satisfying the condition (C2), then we have $k\leqslant0$.*

Proof. Taking $Z=-FN\in\Gamma(D^\perp)$ and $Y=bX\in\Gamma(D\oplus\tilde{D})$ in (3.3), we get:

$$(4.22)\qquad \begin{gathered}(k/2)\,\|bX\|^2\,N=h(X,\nabla_{bX}FN)-h(bX,\nabla_X FN),\\ \forall X\in\Gamma(D\oplus\tilde{D}),\ N\in\Gamma(FD^\perp).\end{gathered}$$

Using (2.3), (2.6), (2.7) and (3.9) we obtain

$$(4.23)\qquad \nabla_X FN=-bA_N X+F\nabla^\perp_X N,\qquad \forall X\in\Gamma(D\oplus\tilde{D}),\ N\in\Gamma(FD^\perp).$$

Substitute (4.23) and (4.19) in (4.22). Since M is $(D\oplus\tilde{D},\ D^\perp)$-geodesic in $\tilde{M}$, we find:

$$(4.24)\qquad \begin{gathered}(-k/2)\,\|bX\|^2\,N=h(bX,\ A_N bX)+h(X,\ bA_N bX),\\ \forall X\in\Gamma(D\oplus\tilde{D}),\ N\in\Gamma(FD^\perp).\end{gathered}$$

From (4.24) we get:

$$(4.25)\qquad \begin{gathered}(-k/2)\,\|bX\|^2\,\|N\|^2=2\,\|A_N bX\|^2,\\ \forall X\in\Gamma(D\oplus\tilde{D}),\ N\in\Gamma(FD^\perp).\end{gathered}$$

Then we obtain $k\leqslant0$.

Corollary 4.1. *If the condition (C2) is satisfied and $k>0$, then M is either an anti-invariant submanifold of $\tilde{M}$, or an almost invariant submanifold of $\tilde{M}$.*

Corollary 4.2. *If M satisfies the condition (C2) and the leaves of the distribution $D\oplus\tilde{D}$ are totally geodesic immersed in M, then we have $k=0$.*

Theorem 4.5. *Let M be a $(\tilde{D},\ D\oplus D^\perp)$-geodesic almost semi-invariant submanifold of $\tilde{M}$ and suppose $k\neq0$. If the distribution $\tilde{D}$ is integrable and the subbundle $FD^\perp$ is $\tilde{D}$-parallel, then M is either a semi-invariant submanifold of $\tilde{M}$ or a pseudo-invariant submanifold of $\tilde{M}$.*

Proof. In the lemma 3.1, the lemma 3.3, and the remark 3.6, we take $H_1=\tilde{D}$ and $H_2=D\oplus D^\perp$. Since $FD^\perp$ is $\tilde{D}$-parallel, by using (3.2), we obtain:

$$(4.26)\qquad \nabla_X Z\in\Gamma(D\oplus D^\perp\oplus\{\xi\}),\qquad \forall X\in\Gamma(\tilde{D}),\ Z\in\Gamma(D\otimes D^\perp).$$

By using the remark 3.6, we obtain either $\tilde{D}=\{0\}$ or $D\oplus D^\perp=\{0\}$ and thus the proof is complete.

Theorem 4.6. *Let M be a $(D\oplus D^\perp,\ \tilde{D})$-geodesic almost semi-invariant submanifold of $\tilde{M}$ and suppose $k\neq0$. If $D\oplus D^\perp$ is integrable and its leaves are totally geodesic immersed in M, then M is either a semi-invariant submanifold of $\tilde{M}$ or a pseudo-invariant submanifold of $\tilde{M}$.*

Proof. In the lemma 3.1., the lemma 3.3 and the remark 3.6, we take $H_1=\tilde{D}$. Since M is $(D\oplus D^\perp,\ \tilde{D})$-geodesic in $\tilde{M}$, by using (2.19) in (3.1) and (2.20) in (3.2), we obtain:

$$\nabla_X Z \in \Gamma(\tilde{D} \oplus \{\xi\}), \qquad \forall X \in \Gamma(D \oplus D^{\perp}),\ Z \in \Gamma(\tilde{D}). \tag{4.27}$$

By using the remark 3.6 in the lemma 3.3, we get either $D \oplus D^{\perp} = \{0\}$, or $\tilde{D} = \{0\}$. Thus the proof is complete.

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SOME THEOREMS ON SEMI-INVARIANT SUBMANIFOLDS OF A SASAKIAN MANIFOLD

BY

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Let $\tilde{M}$ be a $(2n+1)$-dimensional Sasakian manifold with the Sasakian structure (φ, ξ, η, g), where φ is a tensor field of type $(1, 1)$ on $\tilde{M}$, ξ is a vector field, η is a 1-form and g is the Riemannian metric on $\tilde{M}$, these tensor fields being related by

$$\varphi^2 X = -X + \eta(X)\,\xi\ ;\ \varphi\xi = 0\ ;\ \eta(\varphi X) = 0\ ;\ \eta(\xi) = 1, \tag{1}$$

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\,\eta(Y) \qquad \text{and} \tag{2}$$

$$(\tilde{\nabla}_X \varphi)\, Y = g(X, Y)\,\xi - \eta(Y)\, X \tag{3}$$

for any vector fields X, Y tangent to $\tilde{M}$, where $\tilde{\nabla}$ denotes the Riemannian connection on $\tilde{M}$. From (3) we also have

$$\tilde{\nabla}_X \xi = -\varphi X \ \text{for any vector field } X \text{ tangent to } \tilde{M}. \tag{4}$$

Let M be an m-dimensional Riemannian manifold isometrically immersed in $\tilde{M}$ and suppose that the structure vector field ξ of $\tilde{M}$ is tangent to M, We denote by TM and $TM^{\perp}$ the tangent bundle to M and respectively the normal bundle to M. Then, the submanifold M of $\tilde{M}$ is called a *semi-invariant submanifold* if it is endowed with a pair of distributions $(D, D^{\perp})$ satisfying the conditions:

(i) $TM = D \oplus D^{\perp} \oplus \{\xi\}$, where $\{\xi\}$ denotes the distribution spanned by ξ,
(ii) the distribution D is invariant by φ, that is, $\varphi(D_x) = D_x$, for each $x \in M$.
(iii) the distribution $D^{\perp}$ is anti-invariant by φ, that is, $\varphi(D_x^{\perp}) \subset T_x M^{\perp}$, for each $x \in M$.

Remark. From the definition, it follows that the distributions D, $D^{\perp}$ and $\{\xi\}$ are mutually orthogonal to each other.

The semi-invariant submanifolds of a Sasakian manifold have been introduced and studied by A. Bejancu and the present author in [2], [3], [6]. They are corresponding to CR-submanifolds of Kaehlerian manifolds introduced by A. Bejancu in [1] and studied then by many authors. In this note, we obtain some results concerned with the maximal invariant subspaces D_x of $T_x M$ and respectively with the maximal anti-invariant subspaces $D_x^{\perp}$ of $T_x M$, where M is a submanifold of a Sasakian manifold $\tilde{M}$