Corollary. *Let M be a connected Riemannian n-manifold whose sectional curvature is positive everywhere. If $(\nabla_X L)\, Y = (\nabla_Y L)\, X$ and $(\bar{\nabla}_X L^{-1})\, Y = (\bar{\nabla}_Y L^{-1})\, X$ for all $X, Y \in TM$, then M is an Einstein manifold.*

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NEW LIFTING PROPERTIES OF WEAK FIBRATIONS

BY

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In this note we study weak versions of lifting properties introduced by A. Strom in [3], and further investigated in [2], to obtain new characterizations of weak fibrations in the sense of A. Dold (see [1] for definitions, where they are called „h-Faserungen").

Throughout this paper we work in the category of topological spaces and continuous maps.

Definition 1. *A map $p : E \to B$ has the weak right lifting property (WRLP) with respect to a map $i : A \to X$, if for every commutative square*

$$(*)\qquad \begin{array}{ccc} A & \xrightarrow{f} & E \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle p} \\ X & \xrightarrow[g]{} & B \end{array}$$

in Top *there exists a map $h : X \to E$ such that $p \circ h = g$ and $h \circ i \underset{p}{\simeq} f$* (i.e. *there exists a homotopy H from $h \circ i$ to f such that $pH(a, t) = pH(a, o)$ for all $(a, t) \in A \times I$, in short a fiberhomotopy from $h \circ i$ to f*).

Let **G** denote the class of weak fibrations, **C** the class of closed cofibrations, **D** the class of weak cofibrations and **W** the class of homotopy equivalences (see [1] for definitions).

Theorem 1. *For a map $p : E \to B$, the following are equivalent:*

(i) $p \in \mathbf{G}$
(ii) *p has the WRLP w.r.t. all $i \in \mathbf{D} \cap \mathbf{W}$,*
(iii) *p has the WRLP w.r.t. all $i \in \mathbf{C} \cap \mathbf{W}$,*
(iv) *p has WRLP w.r.t. all $i \in \mathbf{W}$.*

Proof. (i)$\Rightarrow$(ii) : Consider a commutative square (*) with $p \in \mathbf{G}$ and $i \in \mathbf{D} \cap \mathbf{W}$. By [1] (2.14) and (2.29) $i(A)$ is a strong deformation retract of X, i.e. there exists a retraction $r : X \to A$ and a homotopy $\varphi : X \times I \to X$ from $i \circ r$ to 1_X, rel $i\,(A)$. Then $g \circ \varphi : X \times I \to B$ satisfies $g\varphi(x, 0) = gir\,(x) = p\,fr\,(x)$.

Since $p \in G$, by [1] (6.1) there exists a homotopy $F : X \times I \to E$ such that $p \circ F = g \circ \varphi$ and $F_0 \underset{p}{\simeq} f \circ r$.

Define $h : X \to E$ by $h(x) = F(x, 1)$, then $ph(x) = pF(x, 1) = g\varphi(x, 1) = g(x)$ and one verifies easily that the map

$$H : A\times I \to E : (a,t)\to \begin{cases} F(i(a),\ 1-2t) & \text{for } t\in[0,\ 1/2] \\ L(i(a),\ 2t-1) & \text{for } t\in[1/2,\ 1] \end{cases}$$

(where $L : F_0 \underset{p}{\simeq} f o r$) is a fiberhomotopy from $h o i$ to f.

Hence p has the WRLP w.r.t. i.

(ii)$\Rightarrow$(iii) : Trivial since $\mathbf{C}\subset\mathbf{D}$.

(iii)$\Rightarrow$(iv) : Consider a commutative square (*) with $i\in\mathbf{W}$. By [1] (1.26), i can be factored as $i=q o k_1$ with $k_1\in\mathbf{C}$ and $q\in\mathbf{W}$. More precisely, $k_1 : A\to Z_i : a\to[a,1]$, where Z_i is the mapping cylinder of i (see [1] (1.11)) and $q : Z_i\to X$ is defined by $q[x]=x$ and $q[a,t]=i(a)$.

(**)
$$\begin{tikzcd} & A \arrow[r, "f"] \arrow[dl, "k_1"'] \arrow[d, "i"] & E \arrow[d, "p"] \\ Z_i \arrow[r, "q"'] & X \arrow[r, "g"'] & B \end{tikzcd}$$

Let $j : X\to Z_i : X\to[X]$. Then $q o j=1_X$ and $K : A\times I\to Z_i : (a,t)\to[a,t]$ defines a fiberhomotopy $j o i \underset{q}{\simeq} k_1$.

Since i and $q\in\mathbf{W}$, also $k_1\in\mathbf{W}$. Hence $k_1\in\mathbf{C}\cap\mathbf{W}$. Consequently p has the WRLP w.r.t. k_1. Let $h' : Z_i\to E$ be a map such that $p o h'=g o q$ and $h' o k_1 \underset{p}{\simeq} f$. Define $h : X\to E$ by $h=h' o j$. Then $p o h=p o h' o j=$ $=g o q o j=g$ and one verifies easily that the map

$$H : A\times I\to E : (a,t)\to \begin{cases} h'K(a,\ 2t) & \text{for } t\in[0,\ 1/2] \\ L(a,\ 2t-1) & \text{for } t\in[1/2,\ 1] \end{cases}$$

(where $L : h' o k_1 \underset{p}{\simeq} f$) is a fiberhomotopy from $h o i$ to f.

(iv)$\Rightarrow$(i) : The WRLP of p w.r.t. the homotopy equivalence $i : X\to X\times I : x\to(x,\ 0)$ coincides with the weak covering homotopy property (WCHP) of p w.r.t. X (see [1] (6.1)). Hence $p\in\mathbf{G}$.

Theorem 2. *For a map $p : E\to B$, the following are equivalent :*

(i) $p\in\mathbf{G}\cap\mathbf{W}$,

(ii) *p has the WRLP w.r.t. all $i\in\mathbf{C}$,*

(iii) *p has the WRLP w.r.t. all maps i,*

(iv) *p has the WRLP w.r.t. all $i\in\mathbf{D}$.*

Proof. (i)$\Rightarrow$(ii) : Consider a commutative square (*) with $p\in\mathbf{G}\cap\mathbf{W}$ and $i\in\mathbf{C}$. By [1] (6.26) p is shrinkable, i.e. there exists a section $s : B\to E$ of p and a fiberhomotopy $L : s o p \underset{p}{\simeq} 1_E$. By [1] (3.20) and [4] lemma 2 the inclusion map $\tilde{i} : X\times 0\cup A\times I\to X\times I$ (associated with i) belongs to $\mathbf{C}\cap\mathbf{W}$. By theorem 1 p has the WRLP w.r.t. $\tilde{i}$. Hence, in the commutative square

$$\begin{tikzcd} X\times 0\cup A\times I \arrow[r, "F"] \arrow[d, "\tilde{i}"] & E \arrow[d, "p"] \\ X\times I \arrow[r, "G"'] & B \end{tikzcd}$$

with F defined by $F(x,\ 0)=sg(x)$, $F(a,t)=L(f(a),\ t)$ and G by $G(x,t)=$ $=g(x)$, there exists a map $H : X\times I\to E$ such that $p o H=G$ and $H o \tilde{i} \underset{p}{\simeq} F$.

Define $h : X\to E : x\to H(x,\ 1)$, then $ph(x)=pH(x,\ 1)=G(x,\ 1)=g(x)$ and $hi(a)=H(i(a),\ 1)=H\tilde{i}(a,\ 1)$ so that clearly $h o i$ is fiberhomotopic to the map $A\to E : a\to F(a,\ 1)=L(f(a),\ 1)=f(a)$. Hence p has the WRLP w.r.t. i.

(ii)$\Rightarrow$(iii) : Consider again the construction (**) made in the proof of theorem 1. Then p has the WRLP w.r.t. $k_1\in\mathbf{C}$ and the rest of the proof is the same as in part (iii)$\Rightarrow$(iv) of theorem 1.

(iii)$\Rightarrow$(iv) : Trivial.

(iv)$\Rightarrow$(i) : As in theorem 1, (iv)$\Rightarrow$(i), the WRLP w.r.t. the map $i : X\to X\times I : x\to(x,\ 0)$ (which obviously belongs to $\mathbf{C}\subset\mathbf{D}$) ensures that $p\in\mathbf{G}$.

If we consider
$$\begin{tikzcd} \varnothing \arrow[r] \arrow[d] & E \arrow[d, "p"] \\ B \arrow[r, "1_B"'] & B \end{tikzcd}$$
(Note that $\varnothing\to B$ also belongs to $\mathbf{C}$)

the WRLP of p ensures the existence of a section $s : B\to E$ of p.

Finally, considering
$$\begin{tikzcd} E\times\{0,\ 1\} \arrow[r, "F"] \arrow[d, "j"] & E \arrow[d, "p"] \\ E\times I \arrow[r, "G"'] & B \end{tikzcd}$$

with $F(e,\ 0)=sp(e)$, $F(e,\ 1)=e$, $G(e,t)=p(e)$ and j the obvious inclusion map (which belongs to $\mathbf{C}$ since $\{0,\ 1\}\subset I$ belongs to $\mathbf{C}$), we obtain a map $H : E\times I\to E$ such that $p o H=G$ and $H o j \underset{p}{\simeq} F$. Then it easily follows that $s o p=F_0\simeq(H o j)_0=H_0\simeq H_1=(H o j)_1\simeq F_1=1_E$. Hence $p\in\mathbf{W}$.

Definition 2. *If $i : A\to X$ is an inclusion map, we say that a map $p : E\to B$ has the weak lifting homotopy extension property (WLHEP) with respect to i, if for every commutative square*

(***)
$$\begin{tikzcd} X\times O\cup A\times I \arrow[r, "F"] \arrow[d, "\tilde{i}"] & E \arrow[d, "p"] \\ X\times I \arrow[r, "G"'] & B \end{tikzcd}$$

in Top, *with G semi-constant (i.e. $G(x,t)=G(x,0)$ for all $x\in X$ and all $t\in[0,\ 1/2]$), there exists a map $H : X\times I\to E$ such that $p o H=G$ and $H o \tilde{i} \underset{p}{\simeq} F$, rel $X\times O$.*

Theorem 3. *$p\in\mathbf{G}\Leftrightarrow p$ has the WLHEP w.r.t. any closed inclusion map $i : A\to X$.*

Proof. „$\Rightarrow$" : One verifies easily that $\tilde{i} : X\times O\cup A\times I\to X\times I$ is a homotopy equivalence, the map $r : X\times I\to X\times O\cup A\times I$: $(x,t)\to(x,\ 0)$

being a homotopy inverse of $\tilde{i}$. By theorem 1 p has the WRLP w.r.t. $\tilde{i}$. Hence, in diagram (***), there exists a map $H' : X \times I \to E$ such that $p \circ H' = G$ and a fiberhomotopy $K : H' \circ \tilde{i} \underset{p}{\simeq} F$.

Define $H : X \times I \to E$ by

$$H(x, s) = \begin{cases} K(x, 0, 1-3s) & \text{for } s \in [0, 1/3] \\ H'(x, 3s-1) & \text{for } s \in [1/3, 1/2] \\ H'(x, s) & \text{for } s \in [1/2, 1]. \end{cases}$$

An easy verification, using the semi-constancy of G, shows that $p \circ H = G$. Moreover $H(x, 0) = F(x, 0)$ and

$$H(a, s) = \begin{cases} K(a, 0, 1-3s) & \text{for } s \in [0, 1/3],\ a \in A \\ K(a, 3s-1, 0) & \text{for } s \in [1/3, 1/2],\ a \in A \\ K(a, s, 0) & \text{for } s \in [1/2, 1],\ a \in A. \end{cases}$$

Define $L : (X \times O \cup A \times I) \times I \to E$ by
$L(x, 0, t) = K(x, 0, 1) = F(x, 0)$ for $x \in X$, $t \in I$ and

$$L(a, s, t) = \begin{cases} K(a, 0, 1-3s+6st) & \text{for } a \in A,\ s \in [0, 1/3],\ t \in [0, 1/2] \\ K(a, 3s-1, 2t) & \text{for } a \in A,\ s \in [1/3, 1/2],\ t \in [0, 1/2] \\ K(a, s, 2t) & \text{for } a \in A,\ s \in [1/2, 1],\ t \in [0, 1/2] \\ K(a, 0, 1) & \text{for } a \in A,\ s \in [0, 1/3(2-2t)],\ t \in [1/2, 1] \\ K\left(a, \dfrac{3s+2t-2}{4t-1}, 1\right) & \text{for } a \in A,\ s \in [1/3(2-2t), 1/2],\ t \in [1/2, 1] \\ K(a, s, 1) & \text{for } a \in A,\ s \in [1/2, 1] \text{ and } t \in [1/2, 1] \end{cases}$$

(Continuity verifications are left to the reader).

One verifies easily (using again the semi-constancy of G) that $L : H \circ \tilde{i} \underset{p}{\simeq} F$, rel $X \times O$. Hence p has the WLHEP w.r.t. i.

"$\Rightarrow$": The WLHEP of p w.r.t. $i : \varnothing \to X$, combined with theorem [1] (6.12), ensures that $p \in \mathrm{G}$.

Remark. If we drop the semi-constancy of G in the definition of the WLHEP, we obtain a characterization of the class of (Hurewicz) fibrations. This can be proved using the RLP of A. Strøm [3].

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COURBES DANS DES ESPACES LAGRANGE GÉNÉRALISÉS

PAR

ELENA MACOVEI

H. Rund ([4] chap. V) a développée la théorie des courbes dans un espace de Finsler $F^n = (M, L)$ et il a observé que les formules de Frenet sont de la même forme que dans le cas Riemannien, mais il n'y a pas de théorème fondamental pour elles.

Cette Note contient la détermination d'un repère invariant et du système complet d'invariants, les équations fondamentales et un théorème fondamental pour des courbes dans l'espace de Lagrange généralisé.

Le problème étudié appartient à la géométrie des espaces de Lagrange [3], des espaces de Finsler [4] et à la théorie des configurations Myller dans ces espaces.

1. Soit $M^n = (M, g_{ij}(x, y))$ un espace de Lagrange généralisé où M est une variété differentiable réelle n-dimensionnelle et $g_{ij}(x, y)$ est un champ de tenseurs Lagrange du type (0,2) symétrique, non dégénéré et défini positif. En M^n il y a une seule connexion Lagrange métrique $F\Gamma = (N, F, C)$ [3], théorème 3.2) qui a les propriétés suivantes :

(1)
1. la connexion nonlinéaire N est fixée apriori dans TM,
2. les h- et v-dérrivées covariantes de g_{ij} par rapport à $F\Gamma$ sont nulles,
3. les torsions T^i_{jk} et S^i_{jk} de $F\Gamma$ sont identiquement nulles.

Si $(\bar{C})$ est une courbe donnée sur la variété M par $x^i = x^i(t)$, $t \in (a, b)$ alors, on peut considérer une courbe (C) de TM^n ayant les équations

$$x^i = x^i(t), \quad y^i = y^i(t), \quad t \in (a, b), \tag{2}$$

qui a la propriété d'être projetée sur la courbe $(\bar{C})$ de M. Dans ces équations $y(y^i(t))$ est l'élément support de l'espace M^n.

Le carré de l'élément d'arc de la courbe (C) est défini par :

$$ds^2(t) = g_{ij}(x(t), y(t)) \frac{dx^i}{dt} \frac{dx^j}{dt} dt^2$$

et la longueur de la courbe $(\bar{C})$ est donnée par :

$$s = \int_{t_0}^{t_1} \sqrt{g_{ij}(x(t), y(t)) \frac{dx^i}{dt} \frac{dx^j}{dt}}\, dt, \quad (t_0, t_1) \subseteq (a, b). \tag{3}$$