3. Si $g_{ij}(x, y)=(1/2)\,[\partial^2 L/\partial y^i \partial y^j)^2$, où $L^n=(M, L)$ est un espace Lagrange alors nous avons la théorie des courbes de L^n. En particulier, lorsque $L(x, y)$ est un lagrangien homogène, nous obtenons la théorie des courbes dans les espaces Finsler F^n([4]).

4. Le théorème fondamental démonstré pour le cas de l'espace Lagrange généralisé $M^n=(M, g_{ij}(x, y))$ n'a pas été démonstré jusqu'ici ni dans le cas classique (M, L) ([3]), ni pour les espaces classiques Finsler ([4]).

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THE INTRINSIC GAUSS AND CODAZZI EQUATIONS FOR THE CARTAN CONNECTION AND THEIR APPLICATIONS

BY

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Recently, M. Matsumoto developed a systematic theory of Finsler hypersurfaces in [10]. His theory is based on a Finsler connection on the ambient space and the induced connection. And R. Miron found a new method which is based on an intrinsic Finsler connection in a hypersurface and its lifted connection in the ambient space [11]. In the present paper we will set up the direct relation between the Cartan connection in the ambient space and the intrinsic Cartan connection in a hypersurface, especially Grauss and Codazzi equations.

The other purpose of the present paper is a generalization of a concept „totally umbilic" in the Riemannian geometry. And we will examine what properties can be inherited to a „totally umbilic" hypersurface. We need intrinsic Grauss and Codazzi equations because many geometrical properties of a Finsler space are expressed by the Cartan connection. The author wishes to express his sincere gratitude to Professor M. Matsumoto for his valuable suggestions and encouragement.

§ 1. **The intrinsic Gauss and Codazzi equations.** Let $F^n=(M^n, L(x, y))$ be an n-dimensional Finsler space, where M^n is an underlying manifold and L is a fundamental function. We put

$$(1.1)\qquad l_i=\dot\partial_i L,\ g_{ij}=\dot\partial_i\dot\partial_j L^2/2,\ h_{ij}=g_{ij}-l_i l_j,\ C_{ijk}=\dot\partial_k g_{ij}/2,$$

where $\dot\partial_i=\partial/\partial y^i$ and indices $i, j, \ldots$ run over the range $1, 2, \ldots, n$. Let a hypersurface M^{n-1} of M^n be given by $x^i=x^i(u^\alpha)$. We put

$$(1.2)\qquad B^i_\alpha=\partial x^i/\partial u^\alpha,\qquad B^i_{\alpha\beta}=\partial_\beta B^i_\alpha,\qquad B^{ij\ldots}_{\alpha\beta\ldots}=B^i_\alpha B^j_\beta\ldots,$$

where Greek indices $\alpha, \beta, \ldots$ run over the range $1, 2, \ldots, n-1$. In the hypersurface theory, the supporting element y^i is assumed to be tangent to M^{n-1}, so that we may write

$$(1.3)\qquad y^i(u, v)=B^i_\alpha(u)v^\alpha.$$

Then the function $L(u, v)=L(x(u), y(u, v))$ gives rise to a fundamental function of M^{n-1} and we get a Finsler hypersurface $F^{n-1}=(M^{n-1}, L(u, v))$, where $L(u, v)$ may be written as merely $L(u, v)$ in the sequel. We have easily

$$(1.4)\qquad l_\alpha=l_i B^i_\alpha,\quad g_{\alpha\beta}=g_{ij}B^{ij}_{\alpha\beta},\quad h_{\alpha\beta}=h_{ij}B^{ij}_{\alpha\beta},\quad C_{\alpha\beta\gamma}=C_{ijk}B^{ijk}_{\alpha\beta\gamma}.$$

We define a unit normal vector $B^i(u, v)$ by

$$(1.5)\qquad g_{ij}B^i_\alpha B^j=0, \qquad g_{ij}B^iB^j=1,$$

and denote the inverse matrix of (B^i_α, B^i) by (B^α_i, B_i). If we put

$$(1.6)\qquad M_{\alpha\beta}=C_{ijk}B^{ij}_{\alpha\beta}B^k, \qquad M_\alpha=C_{ijk}B^i_\alpha B^jB^k, \qquad M=C_{ijk}B^iB^jB^k,$$

then we have ;

$$(1.7)\qquad \dot\partial_\beta B^i_\alpha=0, \qquad \dot\partial_\beta B^i=-2M^\alpha_\beta B^i_\alpha-M_\beta B^i.$$

The Cartan connection $C\Gamma=(\dot\Gamma^i_{jk}, \dot\Gamma^i_j, C^i_{jk})$ on F^n is uniquely determined by the metric function $L(x, y)$ and the (intrinsic) Cartan connection $C\Gamma==(\dot\Gamma^\alpha_{\beta\gamma}, \dot\Gamma^\alpha_\beta, C^\alpha_{\beta\gamma})$ on F^{n-1} is also determined by $L(u, v)$. We define N^α_β, H_β by

$$(1.8)\qquad N^\alpha_\beta=B^\alpha_i(B^i_{0\beta}+\dot\Gamma^i_jB^j_\beta), \qquad H_\beta=B_i(B^i_{0\beta}+\dot\Gamma^i_jB^j_\beta),$$

where the index 0 means contraction by the supporting element y^i or v^α. Then from $\dot\Gamma^i_j=\gamma^i_{0j}-C^i_{jk}\gamma^k_{00}$ (cf. (17. 7) of [9]), we have

$$(1.9)\qquad \dot\Gamma^\alpha_\beta=N^\alpha_\beta+H_0M^\alpha_\beta.$$

Let δ_i and δ_β be respective δ-defferentiations with respect to $C\Gamma$ and $C\Gamma$. Then we have

$$(1.10)\qquad \delta_\beta=B^i_\beta\delta_i+H^i_\beta\dot\partial_i, \qquad H^i_\beta=H_\beta B^i-H_0M^\varepsilon_\beta B^i_\varepsilon.$$

Here we define $F^\alpha_{\beta\gamma}$ and $\overline{H}_{\beta\gamma}$ by

$$(1.11)\qquad F^\alpha_{\beta\gamma}=B^\alpha_i(B^i_{\beta\gamma}+\dot\Gamma^i_{jk}B^{jk}_{\beta\gamma}), \qquad \overline{H}_{\beta\gamma}=B_i(B^i_{\beta\gamma}+\dot\Gamma^i_{jk}B^{jk}_{\beta\gamma}).$$

Then we have

$$(1.12)\qquad \dot\Gamma^\alpha_{\beta\gamma}=F^\alpha_{\beta\gamma}-H_0(C^\alpha_{\beta\varepsilon}M^\varepsilon_\gamma+C^\alpha_{\gamma\varepsilon}M^\varepsilon_\beta-C^\varepsilon_{\beta\gamma}M^\alpha_\varepsilon)+(M^\alpha_\beta H_\gamma+M^\alpha_\gamma H_\beta-M_{\beta\gamma}H^\alpha),$$

from $\dot\Gamma^i_{jk}=\gamma^i_{jk}-C^i_{jm}\dot\Gamma^m_k-C^i_{km}\dot\Gamma^m_j+C_{jkm}\dot\Gamma^m_lg^{li}$ (cf. (17.3') of [9]).

Relative h - and v-covariant derivatives of a mixed tensor, for example Y^i_α, are defined as follows :

$$(1.13)\qquad Y^i_{\alpha|\beta}=\delta_\beta Y^i_\alpha+Y^j_\alpha(\dot\Gamma^i_{jk}B^k_\beta+C^i_{jk}H^k_\beta)-Y^i_\gamma\dot\Gamma^\gamma_{\alpha\beta},$$

$$(1.14)\qquad Y^i_\alpha|_\beta=\dot\partial_\beta Y^i_\alpha+Y^j_\alpha C^i_{jk}B^k_\beta-Y^i_\gamma C^\gamma_{\alpha\beta}.$$

Then we have

$$(1.15)\qquad g_{ij|\beta}=g_{ij|k}B^k_\beta+g_{ij|k}H^k_\beta=0, \qquad g_{ij}|_\beta=g_{ij}|_kB^k_\beta=0.$$

From our definition of relative covariant differentiations and (1.11), (1.12), (1.15), we get the Gauss and Weingarten formulas as follows :

$$(1.16)\qquad B^i_{\alpha|\beta}=H_{\alpha\beta}B^i+T^\gamma_{\alpha\beta}B^i_\gamma, \quad B^i_\alpha|_\beta=M_{\alpha\beta}B^i, \quad B^i{}_{|\beta}=-H^\alpha{}_\beta B^i_\alpha, \quad B^i|_\beta=-M^\alpha_\beta B^i_\alpha,$$

where we put

$$(1{,}17)\qquad T^\alpha_{\beta\gamma}=H_0(C^\alpha_{\gamma\varepsilon}M^\varepsilon_\beta-C^\varepsilon_{\beta\gamma}M^\alpha_\varepsilon)-(M^\alpha_\gamma H_\beta-M_{\beta\gamma}H^\alpha),$$

$$(1.18)\qquad H_{\alpha\beta}=\overline{H}_{\alpha\beta}+M_\alpha H_\beta-H_0M_{\alpha\varepsilon}M^\varepsilon_\beta\,.$$

Then we can easily show that

$$(1.19)\qquad T^\alpha_{0\beta}=-H_0M^\alpha_\beta, \qquad T^\alpha_{\beta0}=0, \qquad T^0_{\alpha\beta}=H_0M_{\alpha\beta},$$

$$(1.20)\qquad H_{0\beta}=H_\beta, \quad H_{\beta0}=H_\beta+H_0M_\beta, \quad \dot\partial_\gamma H_\beta=H_{\gamma\beta}+Q_{\beta\gamma}+H_0M_{\beta\varepsilon}M^\varepsilon_\gamma,$$

where $Q_{\beta\gamma}=C_{ijk|0}B^i_\beta B^j_\gamma B^k$.

Now we are in a position to find so-called Gauss and Codazzi equations with respect to the Cartan connection. At first, we define the mixed h- and hv-curvature tensors $R_j{}^i{}_{\beta\gamma}$, $P_j{}^i{}_{\beta\gamma}$ by

$$(1.21)\qquad R_j{}^i{}_{\beta\gamma}=R_j{}^i{}_{kh}B^{kh}_{\beta\gamma}+P_j{}^i{}_{kh}[B^k_\beta H^h_\gamma-\beta|\gamma]+S_j{}^i{}_{kh}H^k_\beta H^h_\gamma,$$

$$(1.22)\qquad P_j{}^i{}_{\beta\gamma}=P_j{}^i{}_{kh}B^{kh}_{\beta\gamma}+S_j{}^i{}_{kh}H^k_\beta B^h_\gamma,$$

where the notation $\beta|\gamma$ denotes terms obtained by interchanging the indices β and γ in all the preceding terms. We obtain in the same way as Matsumoto [10]

$$(1.23)\qquad R_\alpha{}^\delta{}_{\beta\gamma}=B^\delta_iB^j_\alpha R_j{}^i{}_{\beta\gamma}-[T^\delta_{\alpha\beta|\gamma}+T^\varepsilon_{\alpha\beta}T^\delta_{\varepsilon\gamma}-H_{\alpha\beta}H^\delta{}_\gamma-\beta|\gamma],$$

$$(1.24)\qquad M_{\alpha\varepsilon}R^\varepsilon{}_{\beta\gamma}=B_iB^j_\alpha R_j{}^i{}_{\beta\gamma}-[H_{\alpha\beta|\gamma}+T^\varepsilon_{\alpha\beta}H_{\varepsilon\gamma}-\beta|\gamma],$$

$$(1.25)\qquad P_\alpha{}^\delta{}_{\beta\gamma}=B^\delta_iB^j_\alpha P_j{}^i{}_{\beta\gamma}-T^\delta_{\alpha\beta}|_\gamma-T^\delta_{\alpha\varepsilon}C^\varepsilon_{\beta\gamma}+H_{\alpha\beta}M^\delta_\gamma-M_{\alpha\gamma}H^\delta{}_\beta,$$

$$(1.26)\qquad M_{\alpha\varepsilon}P^\varepsilon_{\beta\gamma}=B_iB^j_\alpha P_j{}^i{}_{\beta\gamma}-H_{\alpha\beta}|_\gamma-H_{\alpha\varepsilon}C^\varepsilon_{\beta\gamma}+M_{\alpha\gamma|\beta}-T^\varepsilon_{\alpha\beta}M_{\varepsilon\gamma},$$

$$(1.27)\qquad S_\alpha{}^\delta{}_{\beta\gamma}=B^\delta_iB^j_\alpha S_j{}^i{}_{kh}B^{kh}_{\beta\gamma}+M_{\alpha\beta}M^\delta_\gamma-M_{\alpha\gamma}M^\delta_\beta,$$

$$(1.28)\qquad B_iB^j_\alpha S_j{}^i{}_{kh}B^{kh}_{\beta\gamma}=M_{\alpha\beta}|_\gamma-M_{\alpha\gamma}|_\beta.$$

These equations have already been obtained by Čomić [2].

§2. **Definition of a totally umbilic hypersurface.** From the Gauss and Weingarten formulas (1.16) the tensors $H_{\alpha\beta}$ and $M_{\alpha\beta}$ may be called the second fundamental h-tensor and v-tensor, respectively, of a hypersurface with respect to the Cartan connection. Hence we define a *totally h-umbilic* hypersurface and a *totally v-umbilic* hypersurface by

$$(2.1)\qquad H_{\alpha\beta}=cg_{\alpha\beta}, \qquad\qquad (2.2)\qquad M_{\alpha\beta}=L^{-1}kh_{\alpha\beta},$$

respectively, where the scalar function c is called *h-mean curvature* and k is *v-mean curvature*. In particular, if $c=0$ (resp. $k=0$), then the hypersurface is called *totally h-geodesic* (resp. *totally v-geodesic*). We can easily show by (1.20).

Proposition 1. *If $H_{\alpha\beta}=cg_{\alpha\beta}$, then we have $H_\beta=cv_\beta$ and $H_0=L^2c$.*

Proposition 2. *The condition $H_{\alpha\beta}=0$ is equivalent to $H_\beta=0$ and $Q_{\alpha\beta}=0$. Then the hypersurface is called a hyperplane of the second kind (Matsumoto [10]).*

Remark. The equation $H_\beta=0$ is a condition for a hypersurface to be totally geodesic.

For the later use we prepare

Lemma 1. *For a tensor of F^n defined on F^{n-1}, projection onto a hypersurface and indicatrization are commutative. That is, for a tensor T_{ij}, we have*

$$'(T_{ij}B^{ij}_{\alpha\beta})='(T_{ij})B^{ij}_{\alpha\beta}, \tag{2.3}$$

where the prime denotes indicatrization : $'(T_{ij})=T_{rs}h^r_i h^s_j$ *and* $'(T_{\alpha\beta})=T_{\gamma\delta}h^\gamma_\alpha h^\delta_\beta$.

Proof. It is evident from $B^i_\varepsilon h^\varepsilon_\alpha=h^i_m B^m_\alpha$.

§3. **Totally v-umbilic hypersurfaces.** A Finsler space is called $S4$-like [8] when there exists such a tensor E_{ij} as

$$S_{jikh}=h_{jk}E_{ih}+h_{ih}E_{jk}-k|h, \tag{3.1}$$

and particulary, the space is called $S3$-like if E_{ij} is proportional to h_{ij}. From the Gauss equation (1.27) for S we have

Theorem 1. *A v-umbilic hypersurface of a $S4$-like (resp. $S3$-like) Finsler space is also $S4$-like (resp. $S3$-like).*

Next we are concerned with T-tensor of a Finsler space which is defined by $T_{ijkh}='(C_{ijk}|_h)$. Differentiating the last equation of (1.4) and using (1.6), (1.14), Lemma 1, we have

$$T_{\alpha\beta\gamma\delta}=T_{ijkh}B^{ijkh}_{\alpha\beta\gamma\delta}+M_{\alpha\beta}M_{\gamma\delta}+M_{\alpha\gamma}M_{\beta\delta}+M_{\alpha\delta}M_{\beta\gamma}. \tag{3.2}$$

Many authors have studied Finsler spaces satisfying ([4], [7], [13])

$$T_{ijkh}=L^{-2}\lambda(h_{ij}h_{kh}+h_{ik}h_{jh}+h_{ih}h_{jk}). \tag{3.3}$$

(The equation (3.2) leads to

Theorem 2. *Let F^{n-1} be a v-umbilic hypersurface of F^n with v-mean curvature k. If F^n satisfies (3.3) with λ, then F^{n-1} also satisfies (3.3) witgh $\lambda+k^2$.*

Corollary. *If a Finsler space F^n has vanishing T-tensor, then a totally v-geodesic hypersurface F^{n-1} of F^n has vanishing T-tensor.*

§4. **Totally h-umbilic and totally v-geodesic hypersurfaces.** Indicatrizing (1.23) under the assumptions of $H_{\alpha\beta}=cg_{\alpha\beta}$ and $M_{\alpha\beta}=0$, we get from Lemma 1 and Proposition 1

$$'(R_\alpha{}^\delta{}_{\beta\gamma})=B^\delta_i B^j_\alpha{}'(R_j{}^i{}_{kh})B^{kh}_{\beta\gamma}+c^2(h_{\alpha\beta}h^\delta_\gamma-h_{\alpha\gamma}h^\delta_\beta). \tag{4.1}$$

A Finsler space of Rp-scalar curvature [6] is defined by

$$'(R_j{}^i{}_{kh})=R(h_{jk}h^i_h-h_{jh}h^i_k). \tag{4.2}$$

From (4.1) and (4.2), we have

Theorem 3. *Let F^{n-1} be a totally h-umbilic and v-geodesic hypersurface of F^n with h-mean curvature c. If F^n is a space of Rp-scalar curvature R, then F^{n-1} is also of Rp-scalar curvature $R+c^2$.*

As to a hypersurface of $H_{\alpha\beta}=0$, we can easily prove

Theorem 3'. *A totally h-geodesic hypersurface of a Finsler space of Rp-scalar curvature R is also of Rp-scalar curvature R.*

H. Izumi and M. Yoshida define a space of perpendicular-scalar curvature R in [6] by

$$'(R_j{}^i{}_{kh})=R(h_{jk}h^i_h-h_{jh}h^i_k)+1/2['(R^m{}^i{}_h)C_{mjk}+'(R^m{}_{jk})C^i_{mh}-k|h]. \tag{4.3}$$

Theorem 4. *Let F^{n-1} be a totally h-umbilic and totally v-geodesic (or only totally h-geodesic) hypersurface of F^n. If F^n is a space of perpendicular-scalar curvature, then F^{n-1} is also of perpendicular-scalar curvature.*

Proof. When F^{n-1} satisfies $h_{\alpha\beta}=cg_{\alpha\beta}$ and $M_{\alpha\beta}=0$, contraction of (1.23) with v^α and indicatrization yield

$$'(R^\delta{}_{\beta\gamma})=B^\delta_i{}'(R^i{}_{kh})B^{kh}_{\beta\gamma}. \tag{4.4}$$

From this and the last equation of (1.4), we have

$$'(R^{\varepsilon\delta}{}_\gamma)C_{\varepsilon\alpha\beta}='(R^{mi}{}_h)C_{mjk}B^\delta_i B^j_\alpha B^{kh}_{\beta\gamma}-M_{\alpha\beta}B_m{}'(R^{mi}{}_h)B^\delta_i B^h_\gamma. \tag{4.5}$$

The second term of the right side vanishes from $M_{\alpha\beta}=0$. Substituting (4.3) into (4.1), we complete the proof in case of $H_{\alpha\beta}=cg_{\alpha\beta}$, $M_{\alpha\beta}=0$.

In case of $H_{\alpha\beta}=0$, we have $B_i R^i{}_{kh}B^{kh}_{\beta\gamma}=0$ from the Codazzi equation (1.24). Therefore $B_m{}'(R^{mi}{}_h)B^\delta_i B^h_\gamma=B_m h^m_r R^{rs}{}_t h^i_s h^t_h B^\delta_i B^h_\gamma=B_m R^{mi}{}_h B^\rho_i B^h_\tau h^\delta_\rho h^\tau_\gamma=0$. From $H_{\alpha\beta}=0$, we have (4.4) and (4.5) similarly and the second term of (4.5) vanishes. The equation (4.1) is satisfied, too, with $c=0$. We have only to substitute (4.3) into (4.1). Q.E.D.

§ 5. **Totally h-umbilic and totally v-umbilic hypersurfaces.** By the definition of $T^\alpha_{\beta\gamma}$ we have

Proposition 3. *If $H_{\alpha\beta}=cg_{\alpha\beta}$ and $M_{\alpha\beta}=L^{-1}kh_{\alpha\beta}$, then we have $T^\alpha_{\beta\gamma}=ck(h_{\beta\gamma}l^\alpha-h^\alpha_\gamma l_\beta)$.*

Contracting (1.26) with v^α, we have

Proposition 4. *If $H_{\alpha\beta}=cg_{\alpha\beta}$, $M_{\alpha\beta}=L^{-1}kh_{\alpha\beta}$ and c, k are constant, then we have $Q_{\alpha\beta}=-ck^2h_{\alpha\beta}$.*

A Finsler space is called a $^*P$-space ([3], [5]) if

$$P_{ijk}=L\lambda C_{ijk}. \tag{5.1}$$

In general, we get from (1.25)

$$P^\alpha{}_{\beta\gamma}=B^\alpha_i P^i{}_{kh}B^{kh}_{\beta\gamma}-T^\alpha_{0\varepsilon}C^\varepsilon_{\beta\gamma}-T^\alpha_{\varepsilon\beta}{}_\gamma v^\varepsilon+H_\beta M^\alpha_\gamma. \tag{5.2}$$

Applying Proposition 4 and the identity $h^\alpha_{\beta\gamma}=-L^{-1}(h^\alpha_\gamma l_\beta+h_{\beta\gamma}l^\alpha)$ to (5.2), we have

$$P^\alpha{}_{\beta\gamma}=B^\alpha_i P^i{}_{kh}B^{kh}_{\beta\gamma}+LckC^\alpha{}_{\beta\gamma}. \tag{5.3}$$

From (5.1) and (5.3), we have

Theorem 5. *Let F^{n-1} be a totally h-umbilic and totally v-umbilic hypersurface of F^n with constant mean curvature c and k. If F^n is a $^*P$-space with λ, then F^{n-1} is also a $^*P$-space with $\lambda+ck$.*

If $H_\alpha=0$, the equation (5.2) is reduced to $P^\alpha{}_{\beta\gamma}=B^\alpha_i P^i{}_{kh}B^{kh}_{\beta\gamma}$. Hence

Theorem 5'. *A totally geodesic hypersurface of a $^*P$-space is also a $^*P$-space.*

Now we are concerned with the property $\mathscr{H}$ ([6], [12]), which is defined by

$$'(R^i{}_{jk})=h^i_k z_j-h^i_j z_k. \tag{5.4}$$

In general, contracting (1.23) with v we have

$$R^{\alpha}{}_{\beta\gamma}=B^{\alpha}_{i}R^{i}{}_{kh}B^{kh}_{\beta\gamma}+[Q^{\alpha}_{\beta}H_{\gamma}-H_{0}B^{\alpha}_{i}P^{i}{}_{kh}B^{kh}_{\beta\varepsilon}M^{\varepsilon}_{\gamma}-T^{\alpha}_{0\gamma}-T^{\varepsilon}_{0\beta}T^{\alpha}_{\varepsilon\gamma}+H^{\alpha}{}_{\gamma}H_{\beta}-\beta|\gamma], \tag{5.5}$$

$$R^{\alpha}{}_{0\gamma}=B^{\alpha}_{i}R^{i}{}_{0h}B^{h}_{\gamma}-H_{0}Q^{\alpha}_{\gamma}+T^{\alpha}_{0\gamma|0}+T^{\varepsilon}_{0\gamma}T^{\alpha}_{\varepsilon 0}+H_{0}H^{\alpha}{}_{\gamma}-H^{\alpha}H_{\gamma}. \tag{5.6}$$

Indicatrizing (5.5) and using Lemma 1, Proposition 4 and (5.3), we get $'(R^{\alpha}{}_{\beta\gamma})=B^{\alpha}_{i}\,'(R^{i}{}_{kh})B^{kh}_{\beta\gamma}$, so that we have

Theorem 6. *Let F^{n-1} be a totally h-umbilic and totally v-umbilic hypersurface of F^{n} with constant mean curvatures c and k. If F^{n} has the property $\mathscr{H}$, then F^{n-1} has the property $\mathscr{H}$.*

If $H_{\alpha}=0$, the equation (5.5) is reduced to $R^{\alpha}{}_{\beta\gamma}=B^{\alpha}_{i}R^{i}{}_{kh}B^{kh}_{\beta\gamma}$. Hence we have

Theorem 6'. *A totally geodesic hypersurface of a Finsler space with the property $\mathscr{H}$ has the property $\mathscr{H}$.*

A Finsler space of scalar curvature R is defined by $R^{i}{}_{0h}=L^{2}Rh^{i}_{h}$. Applying Proposition 4 and 5 to the equation (5.6), we have

Theorem 7. *Let F^{n-1} be a totally h-umbilic and totally v-umbilic hypersurface of F^{n} with constant mean curvatures c and k. If F^{n} is a space of scalar curvature R, then F^{n-1} is also of scalar curvature $R+c^{2}+c^{2}k^{2}$.*

Remark. A similar theorem is proved in [14] by means of Gauss and Codazzi equations for the Berwald connection.

§ 6. **Hypersurfaces of an isotropic Finsler space.** An isotropic Finsler space is defined by

$$R_{j}{}^{i}{}_{kh}=R(g_{jk}\delta^{i}_{h}-g_{jh}\delta^{i}_{k}). \tag{6.1}$$

As to such a space, the following theorem is well-known [1]:

Theorem A. *If a Finsler space is isotropic with nonzero R, the hv-curvature tensor P satisfies $P_{j}{}^{i}{}_{kh}=P_{j}{}^{i}{}_{hk}$ and the v-curvature tensor S vanishes identically.*

The property of $P_{j}{}^{i}{}_{kh}=P_{j}{}^{i}{}_{hk}$ is called P-symmetry [8]. We prepare

Lemma 2. *If a Finsler space is P-symmetric, then the hv-curvature tensor satisfies $P_{jikm}=P_{jmik}-P_{imjk}$.*

Proof. P is written in general as $P_{jikh}=C_{ikh|j}-C_{jkh|i}+C_{jkr}P^{r}{}_{ih}-C_{ikr}P^{r}{}_{jh}$. Therefore P-symmetry means $P_{jikh}-P_{jihk}=C_{jkr}P^{r}{}_{ih}-C_{ikr}P^{r}{}_{jh}-C_{jhr}P^{r}{}_{ik}+C_{ihr}P^{r}{}_{jk}=0$, and $P_{jikm}-P_{jmik}+P_{imjk}=0$ is easily obtained.

Theorem 8. *Let F^{n-1} be a totally h-umbilic and totally v-umbilic hypersurface of F^{n} with constant mean curvatures c and k. If F^{n} is isotropic with nonzero R, then F^{n-1} is also isotropic with $R+c^{2}+c^{2}k^{2}$.*

Proof. From Proposition 4 and Theorem A, the Gauss equation (1.23) is reduced to

$$\begin{aligned}R_{\alpha}{}^{\delta}{}_{\beta\gamma}=&B^{\delta}_{i}B^{j}_{\alpha}R_{j}{}^{i}{}_{kh}B^{kh}_{\beta\gamma}+B^{\delta}_{i}B^{j}_{\alpha}P_{j}{}^{i}{}_{kh}[B^{k}_{\beta}(B^{h}H_{\gamma}-H_{0}M^{\varepsilon}_{\gamma}B^{h}_{\varepsilon})-\beta|\gamma]+\\&+c^{2}(g_{\alpha\beta}\delta^{\delta}_{\gamma}-g_{\alpha\gamma}\delta^{\delta}_{\beta})+c^{2}k^{2}(h_{\alpha\beta}h^{\delta}_{\gamma}-h_{\alpha\gamma}h^{\delta}_{\beta}).\end{aligned} \tag{6.2}$$

And the Codazzi equation (1.26) for P is reduced to

$$B_{i}B^{j}_{\alpha}P_{j}{}^{i}{}_{kh}B^{kh}_{\beta\gamma}=L^{-1}kP_{\alpha\beta\gamma}+cC_{\alpha\beta\gamma}-L^{-1}ck^{2}h_{\beta\gamma}l_{\alpha}. \tag{6.3}$$

Applying Lemma 2 to this equation, we get

$$B^{\delta}_{i}B^{j}_{\alpha}P_{j}{}^{i}{}_{kh}(B^{k}_{\beta}B^{h}H_{\gamma}-\beta|\gamma)=c^{2}k^{2}(h_{\alpha\beta}l^{\delta}l_{\gamma}-h^{\delta}_{\beta}l_{\alpha}l_{\gamma}-\beta|\gamma). \tag{6.4}$$

On the other hand, because of $P_{j}{}^{i}{}_{h0}=0$ and P-symmetry we have

$$B^{\delta}_{i}B^{j}_{\alpha}P_{j}{}^{i}{}_{kh}B^{k}_{\beta}B^{h}_{\varepsilon}M^{\varepsilon}_{\gamma}-\beta|\gamma=0. \tag{6.5}$$

Substituting (6.4) and (6.5) into (6.2), we have

$$R_{\alpha}{}^{\delta}{}_{\beta\gamma}=(R+c^{2}+c^{2}k^{2})(g_{\alpha\beta}\delta^{\delta}_{\gamma}-g_{\alpha\gamma}\delta^{\delta}_{\beta}), \tag{6.6}$$

which shows the conclusion. Q.E.D.

The following theorem can be easily proved from the Gauss equation (1.23) without using Theorem A.

Theorem 8'. *If the ambient space F^{n} is isotropic with R, then a totally h-geodesic hypersurface F^{n-1} is also isotropic with R.*

In case $R=0$, we can prove in the same way as Theorem 8

Theorem 9. *Let F^{n-1} be a totally h-umbilic and totally v-umbilic hypersurface with constant mean curvatures c and k. If the ambient space F^{n} satisfies $R_{j}{}^{i}{}_{kh}=0$, P-symmetry and $S_{j}{}^{i}{}_{kh}=0$, then F^{n-1} is isotropic with $c^{2}+c^{2}k^{2}$.*

Theorem 10. *Let F^{n-1} be a totally h-umbilic and totally v-geodesic hypersurface of F^{n} with constant mean curvature c. If F^{n} satisfies $R_{j}{}^{i}{}_{kh}=0$ and P-symmetry, then F^{n-1} is an isotropic space with c^{2}.*

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