Oproiu, Elena Vamanu, Gheorghe Murărescu, Mircea Craioveanu, Liliana Maxim, Aurel Bejancu, Gheorghe Zet, Adrian Roman, Gheorghe Atanasiu, Dan Brânzei, Alexandru Neagu, Valeriu Sava, Ilie Burdujan, Neculai Papaghiuc, Profira Sandovici, Constantin Fetecău, Elena Macovei, Gheorghe Procopiuc, Francisc Klepp, Victor Blănuţă and Gheorghe Munteanu. Twenty-six doctorships got under the scientific authority of Professor Gheorghe Gheorghiev is a performance in the history of Mathematics in Romania.

Professor Gheorghe Gheorghiev has also permanently consecrated his skill, devotion, gift and energy to society, by carrying on a vast political and public activity.

Even being 80 years of age, Professor Gheorghiev is still the same tireless man of science, explorer of some abstract fields of splendid profoundness and beauty.

We have the opportunity to render our respectful homage to Professor Gheorghe Gheorghiev, and we are doing it with all our love and regard that his enormous work devoted to the Romanian education and science deserves, by wishing him many years of life, health and joy.

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A CHARACTERIZATION OF MORPHISMS OF *H*-CONES

BY

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§ 0. **Preliminaries.** We use the theory of H-cones, as presented in [1].

Let us recall [4] that, S being an H-cone, we denote by $\tilde{S}$ the set of all mappings $\alpha : S^* \to \overline{\mathbf{R}}_+$, such that :

1) $\alpha(\mu+\nu)=\alpha(\mu)+\alpha(\nu)$, $\forall \mu, \nu \in S^*$; 2) $\mu \leq \nu \Rightarrow \alpha(\mu) \leq \alpha(\nu)$;

3) $\forall F \subseteq S^*$ increasing and dominated : $\alpha(\vee F)=\sup\limits_{\mu \in F} \alpha(\mu)$;

4) there exists an increasing family $A \subseteq S^{**}$ such that :

$$\alpha(\mu)=\sup_{s \in A} s(\mu), \ \forall \mu \in S^*.$$

$\tilde{S}$ has a canonical structure of an ordered convex cone.

We denote by $H(S)$ the set of all parts $F \subseteq S$, which are : nonvoid increasing, solid and such that $A \subseteq F$ and A dominated in $S \Rightarrow \vee A \in F$ $H(S)$ was introduced by E r i k s s o n [3]. $H(S)$ and $\tilde{S}$ are compared in [5].

Let S and T be H-cones. We call morphism of H-cones (from S to T) any map :

$$\varphi : \tilde{S} \to \tilde{T}$$

which is :

additive : $\varphi(s+t)=\varphi(s)+\varphi(t)$, $\forall s, t \in \tilde{S}$,
monotone : $s \leq t \Rightarrow \varphi(s) \leq \varphi(t)$,
continuous : $F \subseteq \tilde{S}$ increasing $\Rightarrow \varphi(\vee F)=\sup\limits_{s \in F} \varphi(s)$.

The domain of the morphism φ is the set :

$$D(\varphi)=\{s \in S \mid \varphi(s) \in T^{**}\}.$$

A morphism φ is called semi-finite if $D(\varphi)$ is dense in S. The set of all morphisms from S to T is denoted by $\mathrm{Hom}(S, T)$. The set of all semi-finite morphisms is denoted by $H(S, T)$.

The main result of this paper is th. 4, which insures the existence of a bijection between the set of semi-finite morphisms between two standard H-cones and a class of functions, defined on the product of the spaces of representation. This characterisation is obtained using the following result, due to C o r n e a-H ö l l e i n [2] :

Theorem. *Let $A : S_0 \times T_0^* \to \mathbf{R}_+$ be such that:*

$$A(s+t, \mu) = A(s, \mu) + A(t, \mu), \quad A(s, \mu+\nu) = A(s, \mu) + A(s, \nu),$$

$$A(\alpha s, \mu) = A(s, \alpha\mu) = \alpha A(s, \mu), \quad \alpha \in \mathbf{R}_+$$

$$A(s, \mu) \leqslant A(s', \mu'), \quad s \leqslant s', \ \mu \leqslant \mu'.$$

If $T = T^{**}$, then there exists a uniquely defined $\varphi \in H(S, T)$ satisfying:

$$A(s, \mu) = \mu(\varphi(s)), \quad \forall s \in S_0, \ \mu \in T_0^*.$$

In § 1, we extend this result, obtaining thus characterisations for arbitrary morphisms.

§ 1. **Morphisms and Biliniar Forms. Theorem 1.** *Let S and T be H-cones. Suppose that S is such that:*

S^* separates S and S is dense in S^{**},

$\forall F \subseteq S$ increasing and $t \in S$:

$$(\vee F) \wedge t = \bigvee_{s \in F} (s \wedge t), \quad (\text{in } \tilde{S}).$$

Let $A : S \times T^* \to \overline{\mathbf{R}}_+$ be such that:

1) $\forall s, s' \in S, \ \mu, \mu' \in T$:

$$A(s+s', \mu+\mu') = A(s, \mu) + A(s', \mu) + A(s, \mu') + A(s', \mu');$$

2) $\forall F \subseteq S, F^* \subseteq T^*$ *increasing and dominated*: $A(\vee F, \vee F^*) = \sup\limits_{\substack{s \in F \\ \mu \in F^*}} A(s, \mu)$;

3) $\forall s \in S \ \exists F \subseteq T$ *increasing, such that* $\forall \mu \in T^*$,

$$A(s, \mu) = \sup_{t \in F} \mu(t).$$

Then, there exists a unique $\varphi \in \mathrm{Hom}(S, T)$, such that:

$$\mu(\varphi(s)) = A(s, \mu), \quad \forall s \in S, \ \mu \in T^*.$$

Proof. Since A has the properties 1) and 2), there exists a unique map:

$$\varphi : H(S) \to \mathrm{Hom}(T^*, \mathbf{R}_+),$$

such that:

$$\mu(\varphi(s)) = A(s, \mu), \quad \forall s \in S, \ \mu \in T^*,$$

φ is additive, monotone and continuous. By definition, $\tilde{T}$ is a part of $\mathrm{Hom}(T^*, \mathbf{R}_+)$: the property 3) assures that φ takes its values in $\tilde{T}$. Finally, the conditions imposed on S are sufficient (and necessary) in order that $\tilde{S} = H(S)$ (see [5]).

Remarks. a) Suppose that T has the following properties:

$\forall t \in T, \ t \neq 0$ is a weak unit,

$\forall t \in T, \ t \neq 0 \ \exists \mu \in T^*$ such that $\mu(t) \neq 0$.

Then the condition 3) on A can be ommited.

b) Conversely, if S^* separates S and $\varphi \in \mathrm{Hom}(S, T)$, then:

$$A(s, \mu) = \mu(\varphi(s))$$

defines a mapping $A : S \times T^* \to \mathbf{R}_+$, having the properties 1), 2), 3) of the theorem.

c) If we suppose that, moreover, T satisfies the same properties as S, and A has the property:

3') $\forall \mu \in T^* \ \exists A^* \subseteq S^*$ increasing, such that $\forall s \in S$:

$$A(s, \mu) = \sup_{\nu \in A^*} \nu(s)$$

then there exists a unique $\varphi \in \mathrm{Hom}(S, T)$ which has an adjoint, and satisfies:

$$A(s, \mu) = \mu(\varphi(s)) = \varphi^*(\mu)(s), \quad \forall s \in S, \ \forall \mu \in T^*.$$

The converse also holds.

Proposition 2. *Let S and T be H-cones, satisfying the conditions from th. 1. Let $S' \subseteq S$ and $T' \subseteq T^*$ be dense, convex subcones. Let: $A : S' \times T' \to \overline{\mathbf{R}}_+$ be a mapping with the properties 1), 2) from th. 1 and:*

3') *$\forall s \in S'$ the set:*

$$\{\mu \in T' \mid A(s, \mu) < \infty\}$$

is dense in T^*.

Then there exists a unique $\varphi \in H(S, T)$ such that:

$$\mu(\varphi(s)) = A(s, \mu), \quad \forall s \in S', \ \forall \mu \in T'.$$

Remarks. a) Supposing also the symmetric of 3'), one gets a characterisation for the fact that φ^* exists ans is semi-finite.

b) If we ask, moreover, that $S' = S$ and $T' = T^*$, we obtain a characterisation for finite morphisms (i.e. such that $D(\varphi) = S$).

§ 2. **Morphisms and Separately Hyperharmonic Mappings.** We begin with some notations. Let A, B, C be sets, and $f : A \times B \to C$ be a map. For $a \in A$, resp. $b \in B$, we denote:

$$f_a : B \to C, \ \text{resp.} \ f_b : A \to C$$

the maps defined by:

$$\forall b \in B : f_a(b) = f(a, b) \ \text{and} \ \forall a \in A : f_b(a) = f(a, b).$$

The map $f^\nu : B \times A \to C$ is defined by: $f^\nu(b, a) = f(a, b)$.

Let S be a standard H-cone of functions on the nearly saturated space X. Let $\mathcal{B}$ be the natural Borel σ-algebra of X. $\mathcal{B}_0$ will denote the universal σ-algebra:

$$\bigcap_{\mu \in S_0^*} \overline{\mathcal{B}}^\mu.$$

Each $\mu \in S_0^*$ extends to a measure on $\mathcal{B}_0$. $A \subseteq X$ will be called exterior negligible if: $A \in \mathcal{B}_0$ and $\bar{\mu}(A) = 0$, $\forall \mu \in S_0^*$. Hence, for each $\bar{\mu} \in S_0^*$, there exists a Borel set $A^{\mu} \in \mathcal{B}$, such that : $A \subseteq A_{\mu}$ and $\mu(A^{\mu}) = 0$. Any exterior negligible set is negligible [1], hence, if $A \subseteq X$ is exterior negligible, then $X \setminus A$ is still nearly saturated.

Let S be a standard H-cone and S^* be represented on the nearly saturated space X^*. Let T be a standard H-cone of functions on the nearly saturated space Y. Let $\mathcal{B}_0^*$, resp. $\mathcal{B}_n$ be the universal σ-algebras on X^*, resp. Y. We shall consider $Y \times X^*$ endowed with the universal σ-algebra :

$$\bigcap_{\substack{\mu \in T_0^* \\ s \in S_0}} \overline{\mathcal{B}_0 \otimes \mathcal{B}_0^*}^{\mu \otimes s}$$

Proposition 3. *Let $f : Y \times X^* \to \overline{\mathbf{R}}_+$ be a universally measurable map, such that :*

$$\forall x^* \in X^* : f_x \in \tilde{T} \; ; \; \forall y \in Y : f_y \in \tilde{S}^*.$$

Then the following statements are equivalent :

1) There exists weak units $u \in S^{**}$, resp. $v^* \in T^*$ which are H-measures on X^*, resp. Y, such that f is $v^* \otimes u$-integrable.

2) There exists exterior negligible sets $A^* \subseteq X^*$, $A \subseteq Y$ such that :

$$\forall x^* \in X^* \setminus A^* : f_{x^*} \in T \; ; \; \forall y \in Y \setminus A : f_y \in S^*.$$

Proof. 1)$\Rightarrow$2). Let $\{\mu_m\}_{m \in N}$ be a countable, dense part of T_0^*. We denote :

$$A = \{x^* \in X^* \mid f_{x^*} \in T\}$$

$A_m = \{x^* \in X^* \mid f_{x^*}$ is not μ_m-integrable$\}$.

Since there exists $\alpha_m > 0$ such that : $\mu_0 = \sum_{m=1}^{\infty} \alpha_m \mu_m \in T^*$, it follows that $A \subseteq \bigcup_{m=1}^{\infty} A_m$. Fubini's theorem shows that each $A_m \in \mathcal{B}$ is s-negligible, $\forall s \in S_0$. Hence, A is exterior negligible. Analogously: $\{y \in Y \mid f_y \in S^*\}$ is exterior negligible.

2)$\Rightarrow$1). $\forall x^* \in X^* \setminus A^*$, f_{x^*} is s-integrable, $\forall s \in S_0$; $\forall y \in Y \setminus A$, f_y is μ-integrable, $\forall \mu \in T_0^*$. Hence, we can find $\alpha_m > 0$, such that $v^* = \sum_{m=1}^{\infty} \alpha_m \mu_m \in T^*$ is a weak unit. In the same way, there exists $u \in S$ weak unit, such that, by Fubini's theorem, f be $v^* \otimes u$-integrable.

Theorem 4. *Let S be a standard H-cone, with S^* represented as a standard H-cone functions on the nearly saturated space X^* ; let T be a standard H-cone of functions on the nearly saturated space Y.*

Then there exists a bijection between : $H(S, T)$ (the set of semi-finite morphisms from S to T) and the set of all functions $f : Y \times X^* \to \overline{\mathbf{R}}_+$, which are universally measurable, $\forall x^* \in X^*$, $f_{x^*} \in \tilde{T}$, $\forall y \in Y$, $f_y \in \tilde{S}^*$, and such that there exists exterior negligible sets $A^* \subseteq X^*$ and $A \subseteq Y$ such that: $\forall x^* \in X^* \setminus A^*$, $f_{x^*} \in T$ and $\forall y \in Y \setminus A$, $f_y \in S^*$.

The morphism φ and the function f are related by :

$$\mu(\varphi(s)) = \int_{Y \times X^*} f(y, x^*) d(\mu \otimes s)$$

$\forall \mu \in T^*$, $\forall s \in S$ which are H-measures.

Proof. a) If we identify each $x^* \in X^*$ with the corresponding element form S^{**} ; and each $y \in Y$ with the corresponding element from T^*, it follows that : $f_\varphi(y, x^*) = y(\varphi(x^*))$ defines a function : $f_\varphi : Y \times X^* \to \mathbf{R}_+$.

Since $(f_\varphi)_y = \varphi^*(y)$ and $(f_\varphi)x^* = \varphi(x^*)$, it follows that: $\forall y \in Y$, $(f_\varphi)_y \in \tilde{S}^*$; $\forall x^* \in X^*$, $(f_\varphi)_{x^*} \in \tilde{T}$. We prove now that f_φ is universally measurable. Let us choose [2, lemma 1.3] $u \in S^{**}$ weak unit, such that $\varphi(u) \in T^{**}$. If we denote :

$$Y_m = \{y \in Y \mid \varphi(u)(y) \leq m\}$$

then: each $Y_m \subseteq Y$ is naturally measurable, and $Y \setminus \bigcup_{m=1}^{\infty} Y_m$ is a polar set. Hence, it is enough to prove :

b) **Lemma.** *$f_\varphi / Y_m \times X^*$ is naturally l.s.c.*

Proof. This is an adaptation of [1, cor. 4.5.3.]. Let (y_n) resp. (x_n^*) be naturally convergent sequences from Y_m, resp. X^*, to $y \in Y_m$ resp. $x^* \in X^*$. For each $p \in S_0$ such that $p \leq x^*$, and for each $\varepsilon > 0$ there exists n_ε such that :

$$\forall n \geq n_\varepsilon : \quad p \leq x_n^* + \varepsilon u.$$

We get :

$$\forall n \geq n_\varepsilon : \; y_n(\varphi(p)) \leq y_n(\varphi(x_n^*)) + \varepsilon y_n(\varphi(u))$$

$$\forall n \geq n_\varepsilon : \; y_n(\varphi(p)) \leq y_n(\varphi(x_n^*)) + \varepsilon m$$

$$\liminf_{n \to \infty} y_n(\varphi(p)) \leq \liminf_{n \to \infty} y_n(\varphi(x_n^*)) + \varepsilon m.$$

Since $\varphi(p) \in T^{**}$, it results :

$$y(\varphi(p)) \leq \liminf_{n \to \infty} y_n(\varphi(p)).$$

Since ε and p are arbitrary, it follows that :

$$y(\varphi(x^*)) \leq \liminf_{n \to \infty} y_n(\varphi(x^*)).$$

Let now $\mu \in T^*$ and $s \in S$ be H-measures on Y, resp. X^*. Since

$$\int_{Y \times X^*} f_\varphi(y, x^*) d(\mu \otimes s) = \int_{X^*} s(dx^*) \int_Y f_\varphi(y, x^*) \mu(dy)$$

and :

$$\int_Y f_\varphi(y, x^*) \mu(dy) = \int_Y (f_\varphi)_{x^*} d\mu = \mu((f_\varphi)_{x^*}) = \mu(\varphi(x^*)) = \varphi^*(\mu)(x^*).$$

We obtain finally :

$$\int_{Y \times X^*} f_\varphi(y, x^*) d(\mu \otimes s) = \int_{X^*} \varphi^*(\mu)(x^*) s(dx^*) = \tilde{s}(\varphi^*)(\mu) = \mu(\varphi(s))$$

Particularly, using again [2, lemma 1.3] we can construct a weak unit $u \in S^{**}$, which is an H-measure and $u \in D(\varphi)$; and a weak unit $v^* \in T^*$, which is an H-measure and: $v^*(\varphi(u)) < \infty$. Hence, f_φ is $v^* \otimes u$-integrable.

c) We define now the inverse correspondence $f \to \varphi_f$. The hypotheses on f show that by:

$$A(s, \mu) = \int_{Y \times X^*} f(y, x^*) d(\mu \otimes s), \quad \forall \mu \in T_0^*, \quad \forall s \in S_0$$

is defined a mapping: $A : S_0 \times T_0^* \to \mathbf{R}_+$, having the properties required in the theorem of C o r n e a-H ö l l e i n [2]. Hence, there exists a semi-finite morphism $\varphi \in H(S, T)$, uniquely defined by:

$$\mu(\varphi_f(s)) = \int_{Y \times X^*} f(y, x^*) d(\mu \otimes s), \quad \forall \mu \in T_0^*, \ s \in S_0.$$

This equality extends to any H-measure, by Fubini's theorem. Indeed, if $\mu \in T_0^*$ and $s \in S$ is a H-measure, and (s_n) is an increasing sequence from S_0, such that: $s = \bigvee_{n \in N} s_n$, then:

$$\int_{Y \times X^*} f(y, x^*) d(\mu \otimes s) = \int_Y \mu(dy) \int_X f_y(x^*) s(dx^*) = \int_Y s(f_y) \cdot \mu(dy) = \int_Y \sup_n s_n(f_y) \cdot$$
$$\cdot \mu(dy) = \sup_n \int_Y s_n(f_y) \cdot \mu(dy) = \sup_n \int_{Y \times X^*} f(y, x^*) d(\mu \otimes s_n).$$

d) Finally, let us prove that the two correspondences: $\varphi \to f_\varphi$ and $f \to \varphi_f$ are inverse one to another. If $\varphi_f \leqslant \varphi_g$, then $f \leqslant g$, $\mu \otimes s$-a.e. With Fubini's theorem, we obtain succesively: there exists $A' \subseteq Y$, μ-negligible, such that: $f_y \leqslant g_y$, $\forall y \in Y \setminus A'$, s-a.e., $\forall s \in S_0$. Hence: $f_y \leqslant g_y$ on X^*, $\forall y \in Y \setminus A'$. This means that: $f_{x^*} \leqslant g_{x^*}$, $\forall x^* \in X^*$ μ-a.e., $\forall \mu \in T_0^*$. We get now: $f_{x^*} \leqslant g_{x^*}$ on Y, $\forall x^* \in X^*$, hence $f \leqslant g$ on $Y \times X^*$.

Remarks. a) The bijections $\varphi \to f_\varphi$ and $f \to \varphi_f$ are additive, positively homogeneous and bimonotone. Moreover, the following formulas hold:

$$f_{\varphi^*} = f_\varphi^v \text{ and } (\varphi_f)^* = \varphi_{f^v}.$$

b) If $\varphi : Y \to X$ is a H-map (particularly a harmonic morphism), then f_φ is defined by:

$$f_\varphi(y, x^*) = G(\varphi(y), x^*)$$

where $G : X \times X^* \to \overline{\mathbf{R}}_+$ is the Green function.

§ 3. **Complementary Results.** a) From part a) of the proof of th. 4 we get:

Corollary. *Let S and T be standard H-cones and $\varphi \in H(S, T)$. There exists then:*

(i) *nearly saturated spaces Y and X^* for T, resp. S^*.*

(ii) *A universally measurable function $f : Y \times X^* \to \overline{\mathbf{R}}_+$, which satisfies:*

$$\forall y \in Y : f_y \in S^* ; \ \forall x^* \in X^* : f_{x^*} \in T^{**} ;$$

f is related to φ by the relation:

$$\mu(\varphi(s)) = \int_{Y \times X^*} f(y, x^*) d(\mu \otimes s),$$

which holds for any H-measures $\mu \in T^*$, $s \in S^{**}$.

b) The proof of lemma b) gives the following extension of [1, cor. 4.5.3].

Proposition 5. *Let S and T be standard H-cones, and $\varphi \in \mathrm{Hom}(S, T)$ be a morphism having adjoint and such that:*

there exists a weak unit $u \in S$, such that $\varphi(u)$ is dominated by a universally bounded, naturally continuous element from T^{**}.

Then the map:

$D(\varphi) \times D(\varphi^*) \to \overline{\mathbf{R}}_+$ $(s, \mu) \to \mu(\varphi(s))$. is naturally l.s.c.

c) Using prop. 2. we have the following extension of th. 4.

Proposition 6. *Let S and T be H-cones; $S' \subseteq S$ and $T' \subseteq T^*$ be dense, convex subcones. Suppose the following hold:*

S^* separates S and S is dense in S^{**},

T^* separates T and T is dense in T^{**},

S^* is an H-cone of functions on X^*,

T is an H-cone of functions on Y

the Stonian topology on Y (cf. [1, ex. 2.8]) is metrisable and separable; any $\mu \in T'$ is representable as a measure on $(Y, \mathcal{B}_y)$.

any $s \in S'$ is representable as a measure on $(X^*, \mathcal{B}^*)$, where B^* is the smallest σ-algebra for which any $\mu \in S^*$ is measurable.

Then there exists a bijection (which is biminotone, additive and positively homogeneous) between:

a) *the set of all morphisms φ from S to T, which has adjoint and:*

$$S' \subseteq D(\varphi), \quad T' \subseteq D(\varphi^*) ;$$

b) *The set of all $B_y \otimes B^*$-measurable functions $f : Y \times X^* \to \overline{\mathbf{R}}_+$, which are such that:*

$$\forall x^* \in X^* : f_{x^*} \in \tilde{T} ; \ \forall y \in Y : f_y \in \tilde{S^*},$$

$\forall s \in S'$ the set: $\{\mu \in T' \mid f$ is $\mu \otimes s$-integrable$\}$ is dense in T^*,

$\forall \mu \in T'$ the set: $\{s \in S' \mid f$ is $\mu \otimes s$-integrable$\}$ is dense in S.

This bijection is uniquely defined by the relation:

$$\mu(\varphi(s)) = \int_{Y \times X^*} f(y, x^*) d(\mu \otimes s), \quad \forall \mu \in T', \ \forall s \in S'.$$

Moreover:

$$f_{\varphi^*} = f_\varphi^v ; \ (\varphi_f)^* = \varphi_{f^v}.$$

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