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# ITERATIVE PROCESSES OF MAXIMAL TYPE IN METRIC AND BANACH SPACES

BY

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**0. Introduction.** Let $X$ be a Banach space and let $T$ be a continuous mapping from the open sphere $B_0=B(x_0, \rho_0)\subset X$ into $X$. Concerning the solvability by numerical procedures of the operator equation $Tx=0$, the following result established by Kantorovich [13] may be considered as the start point of our developments.

**Theorem 1.** *Suppose that*

(i) $S_0=T'(x_0)^{-1}$ *exists as an element of* $L(X)$

*and also that, a couple of points* $t_0\in R$, $\sigma_0\in(0, \rho_0)$ *and a function* $f:[t_0, t_0+\rho_0)\to R$ *may be found with the properties*:

(ii) $\|S_0Tx_0\|\leqslant c_0f(t_0)$, *where* $c_0=-1/f'(t_0)>0$,

(iii) $\|S_0T''(x)\|\leqslant c_0f''(t)$, *provided* $\|x-x_0\|\leqslant t-t_0\leqslant\sigma_0$, *with* $f''$ *continuous on* $[t_0, t_0+\sigma_0]$,

(iv) $f(t)=0$, *for at least one* $t\in[t_0, t_0+\sigma_0]$.

*Then, the* $x_0$*-generated Newton process*

$$x_{n+1}=x_n-T'(x_n)^{-1}Tx_n,\ n\in N, \tag{1}$$

*exists, remains in the closed sphere* $D_0=B^-(x_0, \sigma_0)$ *and converges to a solution in* $D_0$ *of the operator equation* $Tx=0$.

As already Kantorovich indicated in his paper, Theorem 1 represents — through its immediate consequences — a useful method for the numerical treatment of non-linear operator equations (cf. the 1970 Ortega-Rheinboldt's monograph [17] or the 1981 Kantorovich-Akilov's treatise [14, ch. XVIII]) and, for this reason, the problem of extending it appeared as a quite natural one. Aside from the technical developments of Theorem 1 performed in the above quoted works, an essentially different way of solving this problem, due to Altman [3], may be formulated along the following lines. Let $(X, d)$ be a complete metric space. Given the *closed* mapping $T$ from $D\subset X$ into the Banach space $(Z, \|.\|)$ (that is, $(x_n; n\in N)$ in $D$, $x_n\to x$ and $Tx_n\to z$ imply $x\in D$ and $Tx=z$), the following local result about the solvability of the equation $Tx=0$ can be stated.

**Theorem 2.** *Suppose that the continuous increasing function* $b:[0, \infty)\to[0, \infty)$, *with*

(i)' $b^*(t)=\int_0^t(b(s)/s)ds<\infty$, $t>0$, *the point* $x_0 \in D$, *and the constants* $q \in (0, 1)$, $\alpha > 0$, $M > 0$, *with*

(ii)' $b^*(M \|Tx_0\|^{1-q}) \leq \alpha(1-q)$,

*are such that, for each* $x$ *in* $D_0 = D \cap B(x_0, \alpha)$ *there exist* $x'$ *in* $D$ *and* $\varepsilon$ *in* $(0, 1]$ *in such a way that*

$$\|Tx' - (1-\varepsilon)Tx\| \leq \varepsilon q \|Tx\|, \quad d(x, x') \leq \varepsilon b(M \|Tx\|). \tag{2}$$

*Then,* $Tx = 0$ *for at least one* $x$ *in* $D_0$.

Roughly speaking, the basic feature of Theorem 2 is the replacement of the *sequential* Newton-Kantorovich process indicated in (1) by a *non-sequential* one (expressed by the generic transformation $x \vdash x'$) constructed under (2) and treatable by transfinite induction (see in this direction Altman [1], as well as the above references); an equivalent argument, based upon the use of the Brézis-Browder ordering principle [5] may be found in Cramer and Ray [8]. As established by these authors, Theorem 2 may be viewed as a common extension of some related contributions in this area due to Pokhozhaev [19], Browder [6], Zabreiko and Krasnoselskii [29], Kirk and Caristi [15], Downing and Kirk [10]; on the other hand, it was observed by Altman [3] that, under further restrictions on the function $f$ (including the standard quadratic case) Theorem 1 may be identified with a certain constructive portion of Theorem 2 under some appropriate hypotheses involving the Fréchet derivative of the ambient map. However, a close analysis of this last result shows it does not cover all situations described by the first one so that, a unified treatment of them — having as basic instrument a maximality principle established by the author [25] — would be of interest. It is exactly our main aim to show that, as a candidate for such a unifying device, a mapping theorem involving (real-valued) operator equations of the form $Fx = 0$ may be taken, the special feature of it, being the use of a "comparative" maximization procedure in a related ordered metric structure. As immediate consequences, two results (one of them providing us with a direct extension of Theorem 1) about the numerical solvability of the (vector-valued) operator equation $Tx = 0$ via functional super-additive and, respectively, sub-additive Lipschitz type assumptions upon the Fréchet derivative $T'$ are stated and proved. As another consequence of the main result, a related mapping theorem (of a „functional" type with respect to (2)) is given, to conclude that Theorem 2 is also reductible to our statements. A further discussion about the relationships between these two strategies of solving operator equations will be performed elsewhere.

1. **Preliminaries about maximality principles.** Let $X$ be a nonempty set and let $\leq$ be a *quasi-ordering* (i.e., a reflexive and transitive relation) over $X$. Given the function $\varphi : X \to R$, call the point $z \in X$, *$\varphi$-maximal* when $z \leq w$ implies $\varphi(z) = \varphi(w)$ (that is, $\varphi$ is constant on $X(z, \leq) = \{x \in X ; z \leq x\}$). An important problem involving this notion is that of determining under what conditions each point of $X$ is bounded above by a $\varphi$-maximal element. To this end, a basic assumption (used throughout this section) about the ambient space must be made, namely

($C_1$) *each increasing sequence in* $X$ *has an upper bound.*



Under this requirement, a fundamental answer to the above posed problem is that contained the 1976 Brézis-Browder's ordering principle [5] which, in a convenient manner, can be stated as follows.

**Theorem A.** *Let the function* $\varphi$ *be decreasing and bounded from below. Then, to each* $x \in X$ *there corresponds a* $\varphi$*-maximal element* $z \in X$ *with* $x \leq z$.

In view of its increasing importance to convex as well as nonconvex analysis (cf. the above reference) this principle has been extended in many ways until present. To state the most general of them, obtained in 1985 by Kang [12], let $d : X^2 \to [0, \infty)$ be a *pseudometric* on $X$ (in the sense that $d(x, x) = 0$, $x \in X$) and call an element $z \in X$, *d-maximal* provided $d(u, v) = 0$ for all $u, v \in X$ with $z \leq u \leq v$. As before, the point is to determine under what conditions each element of $X$ is bounded above by a certain $d$-maximal one. In this direction, the announced Kang's maximality result is

**Theorem B.** *Suppose that the pseudometric* $d$ *on* $X$ *fulfils*

$$(C_2) \quad \begin{cases} \text{for each } x \in X,\ \varepsilon > 0, \text{ there exists } y = y(x, \varepsilon) \geq x \text{ such that,} \\ d(u, v) < \varepsilon, \text{ whenever } y \leq u \leq v. \end{cases}$$

*Then, to each* $x \in X$ *there corresponds a d-maximal element* $z \in X$ *with* $x \leq z$.

As already precised in that paper, an extra condition like

$$(C_3) \quad \begin{cases} \text{for each } \varepsilon > 0 \text{ there exists } \delta > 0 \text{ such that } x \leq y \leq z \text{ and} \\ d(x, y),\ d(x, z) < \delta \text{ imply } d(y, z) < \varepsilon \end{cases}$$

makes Theorem $B$ reductible to the 1984 author's result [28] which in turn extends the 1982 Altman's ordering principle [4]. The obtained Kang's contribution, as well as the above quoted ones are, of course, interesting from a practical viewpoint but we must say that, in all concrete cases when a maximality principle of this type is to be applied, a substitution of it by the Brézis-Browder's one is always possible. This fact raises the question of whether or not this generalization of Theorem $A$ is effective. The answer is negative as it results from the argument below.

*Proof of Theorem B.* Without any loss one may assume $d(x, y) \leq 1$, $x, y \in X$ since, otherwise, passing to the pseudometric

$$e(x, y) = \min(1, d(x, y)), \quad x, y \in X$$

(fulfiling this property) condition ($C_2$) will remain valid. Let the (decreasing and bounded below) function $\varphi : X \to [0, \infty)$ be defined as

$$\varphi(x) = \sup\{d(u, v) ;\ x \leq u \leq v\}, \quad x \in X.$$

By Theorem $A$ it follows that, given $x \in X$, a point $z \in X$ may be determined with (a) $x \leq z$, (b) $z \leq w$ implies $\varphi(z) = \varphi(w)$. On the other hand, $\varphi(z) > 0$ is impossible by ($C_2$). So, $\varphi(z) = 0$ and the result follows. q.e.d.

The obtained equivalence between these two results raises the question of whether or not extensions of Theorem $A$ exist without being (technically) reductible to it. This seems to be a very delicate question; as an attempt of solving it, the following statement may be taken into consideration. Let again $d$ stand for a pseudometric over $X$. The convergence property of a sequence $(x_n ; n \in N)$ in $X$ towards an element $x \in X$ (written

$x_n \to x$) being introduced as $\lim_n d(x_n, x) = 0$ denote $\lim(x_n) = \{x \in X;\ x_n \to x,\ x_n \leq x,\ n \in N\}$ and call the point $z \in X$, *sequentially d-maximal* when $X(z, \leq) \subset \operatorname{olim}(x_n)$ for some increasing sequence $(x_n\,;\ n \in N)$ in $X$. The basic problem involving this notion being the same (under what conditions we have that each point in $X$ is bounded above by a sequentially $d$-maximal one) an appropriate answer is contained in

**Theorem C.** *Let the pseudometric $d$ on $X$ be such that*

$$(C_4)\quad \begin{cases} \text{for each } x \in X,\ \varepsilon > 0 \text{ there exists } y = y(x, \varepsilon) \geq x \text{ such} \\ \text{that } d(y, u) < \varepsilon \text{ whenever } u \geq y. \end{cases}$$

*Then, to each $x \in X$ there corresponds a sequentially d-maximal element $z \in X$ with $x \leq z$.*

*Proof.* Let $x \in X$ be arbitrary, but fixed. It immediately follows by $(C_4)$ that an increasing sequence $(x_n\,;\ n \in N)$ in $X$ may be constructed with $x \leq x_n$, $n \in N$, and

$$d(x_n, y) < 2^{-n},\quad y \geq x_n,\quad n \in N.$$

Let $z$ be an upper bound of this sequence (existing by $(C_1)$) ; it is almost evident, in view of the above inequality, that $z$ is our desired element. This ends the argument. q.e.d.

Now, clearly, an extra condition like $(C_3)$ reduces this result to Theorem 2 in the above quoted author's paper which, as precised there, contains in particular the Brézis-Browder's ordering principle subsumed to Theorem $A$. The converse of this assertion remains an open problem in the absence of $(C_3)$ ; we conjecture that the answer is negative.

2. **An existence result.** Let $(X, d)$ be a metric space and let $\leq$ be a quasi-ordering on $X$. Given the sequence $(x_n\,;\ n \in N)$ in $X$, let us term it *asymptotic* when $\liminf_n d(x_n, x_{n+1}) = 0$ ; correspondingly, the ambient space will be said to be *$\leq$-asymptotic* provided each increasing sequence in $X$ is asymptotic. At the same time, calling the subset $D$ of $X$, *$\leq$-closed* when the limit of an increasing convergent sequence in $D$ belongs to $D$, we shall say the quasi-ordering $\leq$ is *self-closed* provided $X(x, \leq)$ is $\leq$-closed for each $x \in X$. Also, given the function $G$ from $D \subset X$ into $R$, let us term it *left-lsc(usc)* provided $(x_n\,;\ n \in N)$ increasing in $D$, $x_n \to x \in D$, $x_n \leq x$, $n \in N$, and $Gx_n \leq \lambda(Gx_n \geq \lambda)$, $n \in N$, imply $Gx \leq \lambda(Gx \geq \lambda)$ ; for example, if $G$ is *decreasing* (*increasing*) in the usual sense, it is necessarily left-lsc(usc). Finally, given another metric space $(Z, e)$ and a mapping $H$ from $D$ into $Z$, call it *$\leq$-closed* provided $(x_n\,;\ n \in N)$ increasing in $D$, $x_n \to x$ and $Hx_n \to z$ imply $x \in D$ and $Hx = z$ ; in particular, if $D$ is $\leq$-closed and $H$ is *left -continuous* (that is, $(x_n\,;\ n \in N)$ increasing in $D$ and $x_n \to x$ imply $Hx_n \to Hx$) it is necessarily $\leq$-closed.

Suppose further $X$ is a *$\leq$-complete* metric space (in the sense of [25]) under the metric $d$ and the self-closed quasi-ordering $\leq$, and $F$ is a function from $D \subset X$ into $[0, \infty)$. Given the couple of ordered metric spaces $(M, d, \leq)$, $(\Omega, d, \leq)$, with $\leq$ self-closed in both $M$ and $\Omega$ which, in addition are $\leq$-asymptotic and $\leq$-complete, let again $d$ stand for the "product" metric on $M \times \Omega$ (that is,

$$d((x, \tau), (y, \sigma)) = \max(d(x, y), d(\tau, \sigma)),\ (x, \tau), (y, \sigma) \in M \times \Omega)$$



and $\leq$ the „product" ordering on $M \times \Omega$ (introduced by

$$(x, \tau) \leq (y, \sigma) \text{ if and only if } x \leq y \text{ and } \tau \leq \sigma).$$

Our main intention, concerning the (real-valued) operator equation $Fx = 0$ is that, given $y_0 \in D$ (which, without any loss may be taken so as to satisfy $Fy_0 \neq 0$) a *comparative procedure* (imagined as a maximization process in a related ($\leq$-closed) subset of the metric structure $(M \times \Omega, d, \leq)$) be determined in order that the underlying equation be solvable in a certain $D$-sphere around $y_0$. To this end, for a number of reasons we shall explain in the sequel we have to introduce a „derived" function $F^{\sim}: D \to [0, \infty)$ and (with $(Z, e)$ being a complete metric space ) an "associated" mapping $G : D \to Z$ which is supposed to be $\leq$-closed in the above sense. Under these preliminaries, the main result of the present paper is

**Theorem 3.** *Assume that a $\leq$-closed subset $A$ of $M \times \Omega$ with*

$$(v)\quad \begin{cases} (t, \tau) \in A \text{ and } \tau = \text{maximal element in } \Omega, \text{ imply } t = \text{maximal element} \\ \text{in } M \end{cases}$$

*a couple of non-maximal elements $s_0 \in M$, $\sigma_0 \in \Omega$ with $(s_0, \sigma_0) \in A$, a couple of decreasing functions $g, h$ from $A$ into $[0, \infty)$, and a couple of functions $f : M \to [0, \infty)$, $f^{\sim} : \Omega \to [0, \infty)$ with $Fy_0 \leq f(s_0)$, $F^{\sim} y_0 \geq f^{\sim}(\sigma_0)$, as well as*

$$(vi)\quad \begin{cases} f - F \text{ is left-usc on } D_0 \times M \text{ where } D_0 = \{x \in D\,;\ y_0 \leq x, \\ d(y_0, x) \leq g(s_0, \sigma_0),\ e(Gy_0, Gx) \leq h(s_0, \sigma_0)\}, \text{ and } f^{\sim} - F^{\sim} \\ \text{is left-lsc on } D_0^* \times \Omega, \text{ where } D_0^* = \{x \in D\,:\ y_0 \leq x, \\ d(y_0, x) < g(s_0, \sigma_0),\ e(Gy_0, Gx) < h(s_0, \sigma_0)\}, \end{cases}$$

$$(vii)\quad \begin{cases} f(s^*) = 0 \text{ at each maximal element } s^* \in M, \text{ and } g(t, \tau) > 0, \\ h(t, \tau) > 0, \text{ at each } (t, \tau) \in M \times \Omega \text{ with } t = \text{non-maximal in} \\ M,\ \tau = \text{non-maximal in } \Omega, \end{cases}$$

*may be found so that, for each $x$ in $D_0^*$ and each $(t, \tau)$ in $A_0 = A((s_0, \sigma_0), \leq)$ with $Fx \neq 0$ and*

$$(3)\quad \begin{cases} Fx \leq f(t),\ F^{\sim} x \geq f^{\sim}(\tau),\ d(y_0, x) \leq g(s_0, \sigma_0) - g(t, \tau),\ e(Gy_0, Gx) \leq \\ h(s_0, \sigma_0) - h(t, \tau) \end{cases}$$

*there exist $x' \geq x$ in $D$ and $(t', \tau') > (t, \tau)$ in $A$ with*

$$(4)\quad \begin{cases} Fx' \leq f(t'),\ F^{\sim} x' \geq f^{\sim}(\tau'),\ d(x, x') \leq g(t, \tau) - g(t', \tau'), \\ e(Gx, Gx') \leq h(t, \tau) - h(t', \tau'). \end{cases}$$

*Then, $Fx = 0$ has at least a solution in $D_0$.*

*Proof.* Assume by contradiction that $Fx \neq 0$ for all $x \in D_0$ and let $E = E(D_0, A_0)$ denote the subset of all $(x, t, \tau)$ in $D_0 \times A_0$ for which (3) takes place, metrized by the standard product metric and ordered by the convention (extended over $D_0 \times A_0$)

$$\begin{cases} (x, t, \tau) \leqq (x', t', \tau') \text{ if and only if } x \leq x',\ (t, \tau) \leq (t', \tau') \\ \text{and the second half of (4) holds.} \end{cases}$$

Let $((x_n, t_n, \tau_n);\ n\in N)$ be an $\leqslant$-increasing sequence in $E$, namely, $x_n \leqslant x_m$, $(t_n, \tau_n)\leqslant(t_m, \tau_m)$, and

(5) $\quad d(x_n, x_m)\leqslant g(t_n, \tau_n)-g(t_m, \tau_m),\ e(Gx_n, Gx_m)\leqslant h(t_n, \tau_n)-h(t_m, \tau_m)$, if $n\leqslant m$.

As $(g(t_n, \tau_n);\ n\in N)$ and $(h(t_n, \tau_n);\ n\in N)$ are decreasing (hence Cauchy) sequences in $[0, \infty)$ it is clear that $(x_n;\ n\in N)$ is an increasing Cauchy sequence in $D_0$ and $(Gx_n;\ n\in N)$ a Cauchy sequence in $Z$ so, by our completeness/closedness hypotheses, $x_n\to x$ (hence $x_n\leqslant x$, $x\in N$) and $Gx_n\to Gx$ for some $x\in D_0$. At the same time, it is clear (by the hypotheses about $M$, $\Omega$ and $A$) that $(t_n, \tau_n)\to(t, \tau)$ (hence $(t_n, \tau_n)\leqslant(t, \tau)$, $n\in N$) for some $(t, \tau)\in A_0$. As an immediate consequence of these facts, we get, by (5),

$$d(x_n, x)\leqslant g(t_n, \tau_n)-g(t, \tau),\ e(Gx_n, Gx)\leqslant h(t_n, \tau_n)-h(t, \tau),\ n\in N$$

that is, $(x_n, t_n, \tau_n)\leqslant(x, t, \tau)$, $n\in N$, with $(x, t, \tau)\in D_0\times A_0$. We claim that $(x, t, \tau)\in E$. Indeed, in view of the inequalities (deduced from the second half of (3))

$$d(y_0, x_n)\leqslant g(s_0, \sigma_0)-g(t_n, \tau_n),\ e(Gy_0, Gx_n)\leqslant h(s_0, \sigma_0)-h(t_n, \tau_n),\ n\in N,$$

combined with the above one, it is transparent that

$$d(y_0, x)\leqslant g(s_0, \sigma_0)-g(t, \tau),\ e(Gy_0, Gx)\leqslant h(s_0, \sigma_0)-h(t, \tau);$$

on the other hand, as $\leqq$ is finer than $\leqslant$ (the product quasi-ordering on $D_0\times A_0$) it is also clear that the relations (obtained by the first half of (3))

(6) $$f(t_n)-Fx_n\geqslant 0,\ f^{\sim}(\tau_n)-F^{\sim}x_n\leqslant 0,\ n\in N,$$

give, by (vi) (the first half) $f(t)-Fx\geqslant 0$ (whence, by (vii) plus (v), $x\in D_0^{*}$ (and, consequently, $x_n\in D_0^{*}$, $n\in N$)) and by (vi) (the second half) $f^{\sim}(\tau)-F^{\sim}x\leqslant 0$, proving our assertion. This shows all conditions involved in author's maximality principle [25] are fulfilled so, given $(y_0, s_0, \sigma_0)$ in $E$ there exists a maximal (modulo $\leqq$) element $(x, t, \tau)$ in $E$ with $(y_0, s_0, \sigma_0)\leqslant(x, t, \tau)$. However, as $Fx\neq 0$ and (3) takes place, $x\in D_0^{*}$ so, there exist by hypothesis $x'\geqslant x$ in $D$ and $(t', \tau')>(t, \tau)$ in $A$ (hence in $A_0$) such that (4) holds and this immediately gives $y_0\leqslant x'$ and

$$\begin{cases} d(y_0, x')\leqslant d(y_0, x)+d(x, x')\leqslant g(s_0, \sigma_0)-g(t, \tau)+g(t, \tau)-\\ g(t', \tau')=g(s_0, \sigma_0)-g(t', \tau'),\end{cases}$$

$$\begin{cases} e(Gy_0, Gx')\leqslant e(Gy_0, Gx)+e(Gx, Gx')\leqslant h(s_0, \sigma_0)-h(t, \tau)+\\ h(t, \tau)-h(t', \tau')=h(s_0, \sigma_0)-h(t', \tau')\end{cases}$$

Therefore, $(x', t', \tau')\in E$ and moreover (again invoking (4) (the second half)) $(x, t, \tau)\leqq(x', t', \tau')$; since, on the other hand $(t, \tau)\neq(t', \tau')$ we get $(x, t, \tau)\neq(x', t', \tau')$, contradicting the maximality of $(x, t, \tau)$ in $(E, \leqq)$. Consequently, the assumption $Fx\neq 0$ for each $x\in D_0$ is false and the result follows. q.e.d.

As a visible particular case of Theorem 3, let $(Z, e)$ be identical with $(X, d)$ and $G$ be the identity mapping from $D$ to itself (note that in such a circumstance $G$ is $\leqslant$-closed if and only if $D$ is $\leqslant$-closed); then, the following practical consequence of the main result may be formulated.



**Corollary 1.** *Assume there exist a $\leqslant$-closed subset $A$ of $M$ with the property* (v), *a couple of non-maximal elements $s_0\in M$, $\sigma_0\in\Omega$ with $(s_0, \sigma_0)\in A$, a decreasing function $g: A\to[0, \infty)$ and a couple $f: M\to[0, \infty)$, $f^{\sim}:\Omega\to[0, \infty)$ with* (vi) + (vii) *as well as $Fy_0\leqslant f(s_0)$, $F^{\sim}y_0\geqslant f^{\sim}(\sigma_0)$, such that, for each $x$ in*

$$D_0^{*}=D(y_0, s_0, \sigma_0)=\{x\in D;\ y_0\leqslant x,\ d(y_0, x)<g(s_0, \sigma_0)\}$$

*and each $(t, \tau)$ in $A_0$ with $Fx\neq 0$ and*

(3)′ $$Fx\leqslant f(t),\ F^{\sim}x\geqslant f^{\sim}(\tau),\ d(y_0, x)\leqslant g(s_0, \sigma_0)-g(t, \tau)$$

*there exist $x'\geqslant x$ in $D$ and $(t', \tau')>(t, \tau)$ in $A$ with*

(4)′ $$Fx'\leqslant f(t'),\ F^{\sim}x'\geqslant f^{\sim}(\tau'),\ d(x, x')\leqslant g(t, \tau)-g(t', \tau').$$

*Then, $Fx=0$, has at least a solution in $D_0=\{x\in D:\ y_0\leqslant x,\ d(y_0, x)\leqslant g(s_0, \sigma_0)\}$*

Now, returning to the general case, suppose the whole information about the variables lying in $\Omega$ would be eliminated then, as another practical consequence of Theorem 3 we have

**Corollary 2.** *Assume there exist a non-maximal element $s_0\in M$, a couple of decreasing functions $g$, $h$ from $M$ to $[0, \infty)$ and a function $f: M\to[0, \infty)$ for which* (vi) *(the first half) and* (vii) *hold, as well as $Fy_0\leqslant f(s_0)$, such that, for each $x$ in*

$$D_0^{*}=D(y_0, s_0)=\{x\in D;\ d(y_0, x)<g(s_0),\ e(Gy_0, Gx)<h(s_0)\}$$

*and each $t$ in $M_0=M(s_0, \leqslant)$, with $Fx\neq 0$ and*

(3)″ $$Fx\leqslant f(t),\ d(y_0, x)\leqslant g(s_0)-g(t),\ e(Gy_0, Gx)\leqslant h(s_0)-h(t),$$

*there exist $x'\geqslant x$ in $D$ and $t'>t$ in $M$ with*

(4)″ $$Fx'\leqslant f(t'),\ d(x, x')\leqslant g(t)-g(t'),\ e(Gx, Gx')\leqslant h(t)-h(t').$$

*Then, $Fx=0$ has at least a solution in $D_0=\{x\in D;\ y_0\leqslant x,\ d(y_0, x)\leqslant g(s_0),\ e(Gy_0, Gx)\leqslant h(s_0)\}$.*

A basic problem concerning the (transfinite) process $x\vdash x'$ upon which we based our argument is that of determine under what circumstances it becomes a constructive one in the sense of A l t m a n [3]. To this end, given $x_0\in D_0^{*}$ and $(t_0, \tau_0)$ in $A_0$ for which $Fx_0\neq 0$ and (3) holds with $(x, t, \tau)$ replaced by $(x_0, t_0, \tau_0)$, let $x_1\geqslant x_0$ in $D_0$ and $(t_1, \tau_1)>(t_0, \tau_0)$ in $A_0$ be such that (4) is fulfilled with $(x_1, t_1, \tau_1)$ in place of $(x', t', \tau')$. Assuming $Fx_1\neq 0$ (hence $x_1\in D_0^{*}$) let $x_2\geqslant x_1$ in $D_0$ and $(t_2, \tau_2)>(t_1, \tau_1)$ in $A_0$ be given by (4) with $(x_1, t_1, \tau_1)$ in place of $(x, t, \tau)$ and $(x_2, t_2, \tau_2)$ in place of $(x', t', \tau')$ and so on. By an ordinary induction argument we obtain (if one eliminates at each step the possibility of accidentally deriving by this procedure a solution of the equation $Fx=0$) an increasing sequence $(x_n;\ n\in N)$ in $D_0^{*}$ and an increasing sequence $((t_n, \tau_n);\ n\in N)$ in $A_0$ satisfying (5) + (6). Now, by the same reasoning as in Theorem 3 we get $x_n\to x$ (hence $x_n\leqslant x$, $n\in N$) and $(t_n, \tau_n)\to(t, \tau)$ (hence $(t_n, \tau_n)\leqslant(t, \tau)$, $n\in N$) for some $x\in D_0$, $(t, \tau)\in A_0$, with the properties (deduced by (5) + (6) through a limit process)

$$\begin{cases} Fx\leqslant f(t),\ F^{\sim}x\geqslant f^{\sim}(\tau),\ d(x_n, x)\leqslant g(t_n, \tau_n)-\lim_n g(t_n, \tau_n),\\ e(Gx_n, Gx)\leqslant h(t_n, \tau_n)-\lim_n h(t_n, \tau_n),\ n\in N.\end{cases}$$

This relation tells us that, if either $t \in M$ or $\tau \in \Omega$ is maximal, the point $x \in D_0$ is necessarily a solution of the underlying equation — in view of $(v)+(vii)$ — approximated by the sequence $(x_n;\ n \in N)$ (referred to in the sequel as an abstract Newton-Kantorovich process) under an error evaluation formula given by its second half. If however, $t$ is not maximal in $M$ (hence, by $(v)$, $\tau$ is not maximal in $\Omega$) but

$$\liminf_n f(t_n)=0 \tag{7}$$

then, under the requirement $F$ is left-continuous it is clear that passing to lim inf in (6), the point $x \in D_0$ is again a solution of the considered equation approximated in the same way as above. (Note that $(vi)$ as well as the $\leqslant$-closedness of $A$ were not used here). Some partial extensions to the metrizable uniform case of these developments were performed in author's papers [24] and [27].

3. **Newton-Kantorovich Processes.** Let $X$ be a Banach space, $B_0=B(y_0, \rho_0)$ an open sphere in $X$ and $T: B_0 \to X$ a mapping. We want to determine under what conditions (comprising those of Theorem 1) the process (1) exists and converges to a solution in $B_0$ of the equation $Tx=0$. Since one intends to realize this via the preceding results, we have to introduce the model of the underlying process which, for the first part of our developments is to be taken as follows. Let $s_0 \in R$ and the function $f: [s_0, s_0+\rho_0) \to R$ be such that a point $s^*$ in $(s_0, s_0+\rho_0)$ may be determined with

(viii) $\left\{\text{\textit{f is strictly positive on} } [s_0, s^*) \text{ \textit{and continuous on} } [s_0, s^*]\right.$

(ix) $f'(t)<f(t)/(t-s^*),\ s_0 \leqslant t<s^*$

(x) $\left\{ g(t)=\sup\{|f'(s)|;\ s_0 \leqslant s \leqslant t\} \text{ \textit{exists for each} } t \text{ \textit{in} } [s_0, s^*) \text{ \textit{and} } G(t)=\int_t^{s^*} g(s)ds<\infty,\ t \in [s_0, s^*).\right.$

The following simple fact will be essential to the considerations bellow.

**Lemma 1.** *Under the above hypotheses, the Newton process generated by any $t_0$ in $[s_0, s^*)$*

$$t_{n+1}=t_n-f(t_n)/f'(t_n),\ n \in N \tag{1)'}$$

*exists, remains in $[s_0, s^*)$ and converges increasingly to $s^*$, the unique solution in $[s_0, s^*]$ of the equation $f(t)=0$.*

*Proof.* The fact that (1)' can be effectively constructed follows immediately from the inequality

$$t-f(t)/f'(t)<s^*,\ s_0 \leqslant t \leqslant s^*$$

obtained by $(viii)$ (the first half) plus $(ix)$. As an easy consequence of this construction plus $(x)$ (the first half)

$$0<f(t_n)=|f'(t_n)|(t_{n+1}-t_n) \leqslant g(t_n)(t_{n+1}-t_n) \leqslant G(t_n)-G(t_{n+1}),\ n \in N$$

so, by $(x)$ (the second half) $f(t_n) \to 0$. On the other hand, if we denote $\tilde{t}=\lim_n t_n$ one has $f(t_n) \to f(\tilde{t})$ in view of $(viii)$ (the second half). Hence $f(\tilde{t})=0$ that is, $\tilde{t}=s^*$ and the proof is complete q.e.d.



*Remarks.* 1) It is evident that the multiplication by a positive factor of the function $f$ is of no effect for the statement of $(viii)-(x)$ or the construction of (1)' so, without any loss one may assume $f'(s_0)=-1$ (hence $f(s_0)<s^*-s_0$, by $(ix)$).

2) It would be interesting to find out whether the conclusion of the lemma remains true in the absence of $(x)$; we conjecture that the answer is negative.

3) A sufficient condition for $(ix)$ to be valid is that $f'$ be strictly increasing on $[s_0, s^*)$; this follows at once from the Lagrange mean value theorem.

4) A slight extension of the above lemma may be obtained if one replaces in $(ix)+(x)$ as well as in the construction of (1)' the bilateral derivatives by right derivatives.

Now, returning to the problem under consideration, assume that an element $s_0 \in R$, a function $f: [s_0, s_0+\rho_0) \to R$ and a point $s^* \in (s_0, s_0+\rho_0)$ satisfying (viii)$-(x)$ plus $f'(s_0)=-1$ (see the above remarks) may be found with the properties: $T$ is continuous on $D=B^-(y_0, s^*-s_0)$, $\|Ty_0\| \leqslant f(s_0)$, $T$ has a Fréchet derivative $T'(x) \in L(X)$ at each $x \in D^0=B(y_0, s^*-s_0)$, boundedly invertible at $y_0$ in the sense that $T'(y_0)^{-1}$ exists as an element of $L(X)$ (of course, without any loss — passing to the map $U=T'(y_0)^{-1}T$ if necessary — one may suppose $T'(y_0)=I$, the identity) and satisfying in addition

(xi) $\left\{ \|T'(x)-T'(y)\| \leqslant f'(t)-f'(s) \text{ \textit{whenever} } x, y \in D^0,\ \|x-y\| \leqslant t-s,\ \|x-y_0\| \leqslant t-s_0,\ \|y-y_0\| \leqslant s-s_0,\ t, s \in [s_0, s^*).\right.$

In this case, as a first answer to the above posed problem, we have

**Theorem 4.** *Under the assumptions we just formulated, the Newton-Kantorovich process (1) generated by $y_0$ exists, remains in $D^0$ and converges to a solution in $D$ of the equation $Tx=0$.*

*Proof.* Letting $\leqslant$ and $d$ stand for the trivial quasi-ordering on $X$ and the standard metric on $X$ induced by the norm, respectively put $Fx=\|Tx\|$, $x \in D$, and, indicating by $M$ the closed segment $[s_0, s^*]$ endowed with the usual metric and ordering, put $g(t)=s^*-t$, $t \in M$. Before effectively developing the argument, denote, after Potra and Pták [20].

$$\gamma(A)=\inf\{\|Ax\|;\ \|x\|=1\},\ A \in L(X).$$

As simple facts about this number observe that if $A=L(X)$ is boundedly invertible then $\gamma(A)>0$ and $\|A^{-1}\|=1/\gamma(A)$ in which case, after the well-known Banach inversion theorem (see, e.g., Kantorovich and Akilov [14, ch. V]) each $B \in L(X)$ for which $\|B-A\|<\gamma(A)$ is boundedly invertible too and $\gamma(B) \geqslant \gamma(A)-\|B-A\|$. With these preliminaries assume $x \in D^0$ and $t \in M$ are such that

$$0<\|Tx\| \leqslant f(t),\ \|x-y_0\| \leqslant t-s_0=g(s_0)-g(t).$$

Firstly, it is manifest that $t \neq s^*$, hence $t'=t-f(t)/f'(t)$ exists and $t<t'<s^*$; secondly, as $T'(y_0)=I$ we have by $(xi)$

$$\|T'(x)-T'(y_0)\| \leqslant f'(t)+1<1=\gamma(T'(y_0))$$

so that $T'(x)$ is boundedly invertible and

$$\gamma(T'(x)) \geqslant \gamma(T'(y_0)) - \|T'(x) - T'(y_0)\| \geqslant 1 - (f'(t)+1) = -f'(t).$$

Denoting $x' = x - T'(x)^{-1}Tx$, it is clear that

$$\|x - x'\| \leqslant \|Tx\|/\gamma(T'(x)) \leqslant -f(t)/f'(t) = t' - t = g(t) - g(t')$$

hence $x' \in D^0$; at the same time, if we put

$$U(\tau) = T(x + \tau(x' - x)) - \tau T'(x)(x' - x), \quad 0 \leqslant \tau \leqslant 1,$$

one has

$$U'(\tau) = (T'(x + \tau(x' - x)) - T'(x))(x' - x), \quad 0 \leqslant \tau \leqslant 1,$$

which gives (by $(xi)$ plus the above relation)

$$\|U'(\tau)\| \leqslant (f'(t + \tau(t' - t)) - f'(t))(t' - t), \quad 0 \leqslant \tau \leqslant 1$$

so, by the Dieudonné mean value theorem [9, ch. VIII] (see also Turinici [26])

$$\|Tx'\| = \|U(1) - U(0)\| \leqslant f(t') - f(t) - (t' - t)f'(t) = f(t').$$

This shows all conditions of Corollary 1 (under its variant with no information about the variables lying in $\Omega$) are fulfilled and therefore the operator equation $Tx=0$ has at least a solution in $D$. To complete the proof let $(y_n\,;\, n \in N)$ and $(s_n\,;\, n \in N)$ be the Newton-Kantorovich process (1) and (1)' respectively, starting from $y_0$ and $s_0$ respectively. By the above lemma, $s_n \to s^*$ (hence $f(s_n) \to 0$) so that, combining with

$$\|Ty_n\| \leqslant f(s_n), \quad n < N, \quad \|y_n - y_m\| \leqslant s_m - s_n, \quad n \leqslant m$$

it is clear that $y^* = \lim_n y_n$ is a solution in $D$ of the operator equation $Tx=0$ (approximated by this sequence under an error formula expressed as $\|y_n - y^*\| \leqslant s^* - s_n$, $n \in N$) and the result follows. q.e.d.

A simple inspection of these reasonings shows in fact that, for each $x_0 \in D^0$ satisfying

$$\|Tx_0\| \leqslant f(t_0), \quad \|x_0 - y_0\| \leqslant t_0 - s_0 \text{ for some } t_0 \in [s_0, s^*)$$

the Newton-Kantorovich process (1) generated by $x_0$ exists, remains in $D^0$ and converges to a solution in $D$ of the equation $Tx=0$. It is natural to ask at this moment whether the set of all solutions in $D$ of the underlying equation reduces to a single element; the answer is positive and may be obtained as follows. Letting $z \in D$ be a solution of this equation suppose $x \in D^0$ is such that

$$\begin{cases} \|z - x\| \leqslant r - t, \quad \|z - y_0\| \leqslant r - s_0, \quad \|x - y_0\| \leqslant t - s_0, \text{ where} \\ r \in [s_0, s^*], \; t \in [s_0, s^*). \end{cases}$$

Denote

$$V(\tau) = T(x + \tau(z - x)) - \tau T'(x)(z - x), \quad 0 \leqslant \tau \leqslant 1$$

then (remember that $T'(x)$ is existing only when $x \in D^0$)

$$V'(\tau) = (T'(x + \tau(z - x)) - T'(x))(z - x), \quad 0 \leqslant \tau \leqslant 1$$



combined with $(xi)$ gives at once

$$\|V'(\tau)\| \leqslant (f'(t + \tau(r - t)) - f'(t))(r - t), \quad 0 \leqslant \tau < 1$$

so, by the mean value theorem of Dieudonné we already quoted, one has (in view of $Tz=0$)

$$\begin{cases} \|T'(x)(z - x + T'(x)^{-1}Tx)\| = \|Tx + T'(x)(z - x)\| = \|V(1) - V(0)\| \leqslant \\ f(r) - f(t) - f'(t)(r - t) \end{cases}$$

in which case (adding the fact that $\gamma(T'(x)) \geqslant -f'(t)$),

$$\|z - x'\| \leqslant r - f(r)/f'(t) - (t - f(t)/f'(t)) \leqslant r' - t'$$

where $x' = x - T'(x)^{-1}Tx$, $t' = t - f(t)/f'(t)$, and

$$\begin{cases} r' = r - f(r)/f'(r), & s_0 \leqslant r < s^* \\ \phantom{r'} = r & , \; r = s^*. \end{cases}$$

As an immediate consequence of this evaluation we have (see the notations of Theorem 4)

$$\|z - y_n\| \leqslant r_n - s_n, \quad n \in N$$

where $(r_n\,;\, n \in N)$ is the Newton process (1)' generated by $r_0 = r$ (under the above convention about the case $r = s^*$) and therefore as $s_n \to s^*$, $r_n \to s^*$, one clearly has $z = y^*$, and our claim is proved.

Concerning the key hypothesis $(xi)$ it must be noted first at all that a particular case in which it holds is that characterized by the situations: $T$ has a second order derivative $T''(x)$ at each $x \in D^0$, $f$ has a second order derivative $f''(t)$ at each $t \in [s_0, s^*)$ and

$$\text{(xi)}' \quad \begin{cases} \|T''(x)\| \leqslant f''(t) \textit{ whenever } x \in D^0, \; \|x - y_0\| \leqslant t - s_0, \\ \textit{for some } t \textit{ in } [s_0, s^*)\,; \end{cases}$$

indeed, $x$ and $y$ being as in (xi), put $U(\tau) = T(y + \tau(x - y))$, $0 \leqslant \tau \leqslant 1$ and observe that, by virtue of the above assumption

$$\|U'(\tau)\| = \|T''(y + \tau(x - y))(x - y)\| \leqslant f''(s + \tau(t - s))(t - s), \quad 0 \leqslant \tau \leqslant 1$$

so, by a mean value argument

$$\|T'(x) - T'(y)\| = \|U(1) - U(0)\| \leqslant f'(t) - f'(s)$$

proving our assertion. It then follows Theorem 4 extends directly the basic result of Kantorovich [13] subsumed to Theorem 1 (cf. also Kantorovich and Akilov [14, ch. XVIII]). Secondly, hypothesis (xi) turns out to include a rather large variety of Lipschitz conditions involving the derivative $T'$. More exactly, assume that a continuous strictly increasing function $c : [0, \infty) \to [0, \infty)$ satisfying $c(0) = 0$ and $c(m_0) = 1$ for some $m_0$ in $(0, \rho_0)$ may be found so as, denoting $D^0 = B(y_0, m_0)$, the Fréchet derivative $T'(x)$ exists at each $x \in D^0$ and satisfies the (functional) Lipschitz condition

$$\text{(xii)} \quad \|T'(x) - T'(y)\| \leqslant c(t) \textit{ whenever } x, y \in D^0, \; \|x - y\| \leqslant t < m_0$$

then, if in addition $c$ is *super-additive*, that is

$$c(t-s) \leqslant c(t) - c(s), \quad 0 \leqslant s \leqslant t < \infty \tag{8}$$

condition (xi) holds with $s_0 = 0$, $s^* = m_0$, and $f(t) = C(m_0 - t)$, $0 \leqslant t \leqslant m_0$, where $C(t) = \int_0^t c(s)ds$, $t \geqslant 0$. In fact, let $x, y \in D^0$ be as in (xi) then, by (xii) plus (8)

$$\begin{cases} \|T'(x) - T'(y)\| \leqslant c(t-s) = c(m_0 - s - (m_0 - t)) \leqslant c(m_0 - s) - c(m_0 - t) = \\ f'(t) - f'(s) \end{cases}$$

and our assertion follows; moreover, under this choice, (viii)—(x) will be satisfied too, as it can be readily verified. For example, the case $c(t) = t^\alpha$, $t \geqslant 0$, for some $\alpha \geqslant 1$ is contained here (and, in particular, the ordinary Lipschitz case $\alpha = 1$, extensively treated in the 1970 *Ortega-Rheinboldt* monograph [17, ch. XII] (cf. also *Potra* and *Ptak* [20])). However, when $0 < \alpha < 1$, this procedure does not work because (8) is not fulfilled. To get a satisfactory answer to this case — which brings us into the second part of our developments — assume that, instead of satisfying (xi), the mapping $T$ is continuous on $D = B^-(y_0, m_0)$ and its derivative $T'$ (existing on $D^0$) satisfies (xii) with $c$ being *sub-additive*, that is

$$c(t-s) \geqslant c(t) - c(s), \quad 0 \leqslant s \leqslant t < \infty. \tag{8'}$$

Denote by $S$ the transformation $(t, \tau) \to (t', \tau')$ from $(0, m_0] \times (0, 1]$ into $(0, \infty) \times R$ defined as

$$t' = C(t/\tau), \quad \tau' = \tau - c(t/\tau)$$

and assume there exists a subset $A$ of $(0, m_0] \times (0, 1]$ *invariant* under $S$, that is

(xiii) $(t', \tau') \in A$ *whenever* $(t, \tau) \in A$.

The basic facts about the couple $(S, A)$, used in the sequel, are collected in

**Lemma 2.** *The following properties hold:*

(a) $(t, \tau) \in A$ *implies* $t' < t$, $\tau' < \tau$; *moreover, indicating by* $((t_n, \tau_n);\ n \in N)$ *the (decreasing) sequence of iterates generated by* $(t_0, \tau_0) = (t, \tau)$ *we necessarily have* $t_n/\tau_n \to 0$ *(hence* $t_n \to 0$*)*

(b) $(t_0, \tau_0) \in A$ *implies* $t_0/\tau_0 < m_0$; *in addition, given* $r_0 \in [0, t_0/\tau_0]$ *and putting* $r_{n+1} = C(r_n)/\tau_n$, $n \in N$, *where* $((t_n, \tau_n);\ n \in N)$ *is the sequence of* $S$ — *iterates generated by* $(t_0, \tau_0)$ *then* $(r_n;\ n \in N)$ *converges decreasingly to zero.*

*Proof.* (a). By (xiii) plus the choice of $A$, we have $t' > 0$, $\tau' > 0$, which gives $0 < c(t/\tau) < \tau$ (hence $\tau' < \tau$) and $t' = C(t/\tau) \leqslant t(c(t/\tau)/\tau) < t$. At the same time

$$t_n \leqslant t_n/\tau_n = c^{-1}(\tau_n - \tau_{n+1}), \quad n \in N$$

and this proves our claim because $\tau_n - \tau_{n+1} \to 0$.

(b). In view of (a) (the first half) $c(t_0/\tau_0) < 1 = c(m_0)$ so, $t_0/\tau_0 < m_0$. Now, letting $(r_n;\ n \in N)$ be introduced as before we have $r_0 \leqslant t_0/\tau_0$ and, successively,

$$r_1 = C(r_0)/\tau_0 \leqslant C(t_0/\tau_0)/\tau_1 = t_1/\tau_1,$$

$$r_2 = C(r_1)/\tau_1 \leqslant C(t_1/\tau_1)\tau_2 = t_2/\tau_2, \ldots$$



from which, by (a) (the second half) it is clear that $r_n \to 0$. Moreover in view of these relations

$$r_1 = C(r_0)/\tau_0 \leqslant r_0(c(r_0)/\tau_0) \leqslant r_0(c(t_0/\tau_0)/\tau_0) < r_0,$$

$$r_2 = C(r_1)/\tau_1 \leqslant r_1(c(r_1)/\tau_1) \leqslant r_1(c(t_1/\tau_1)/\tau_1) < r_1, \ldots$$

and the assertion follows. q.e.d.

Now, returning to the problem under consideration, assume there exists $s_0$ in $(0, m_0)$ satisfying $(s_0, 1) \in A$, for which the inequality $0 < \|Ty_0\| \leqslant \leqslant s_0$ holds. In this case, as a second answer to the problem we formulated at the beginning of this section, we have

**Theorem 5.** *Under the hypotheses we just indicated, conclusions of Theorem 4 remain valid, with $D^0$ and $D$ introduced as before.*

*Proof.* Define $Fx = \|Tx\|$, $x \in D$, $F^\sim x = \gamma(T'(x))$, $x \in D^0$ and, letting $M = [0, m_0]$, $\Omega = [0, 1]$, endowed with the usual metric and the usual dual ordering, put $g(t, \tau) = c^{-1}(\tau)$, $f(t) = t$, $f^\sim(\tau) = \tau$, $t \in M$, $\tau \in \Omega$. Taking $y_0 \in D^0$ and $(s_0, 1) \in A$ as starting elements let $x \in D^0$ and $(t, \tau)$ in $A$ be such that $t \leqslant s_0$, $\tau \leqslant 1$ and

$$0 < \|Tx\| \leqslant t, \quad \gamma(T'(x)) \geqslant \tau, \quad \|x - y_0\| \leqslant c^{-1}(1) - c^{-1}(\tau).$$

By the Lipschitz property (xii),

$$\|T'(x) - T'(y_0)\| \leqslant c(c^{-1}(1) - c^{-1}(\tau)) < 1 = \gamma(T'(y_0))$$

so $T'(x)$ is boundedly invertible; hence $x' = x - T'(x)^{-1}Tx$ exists and moreover (since (8)' implies $c^{-1}$ is super-additive)

$$\|x - x'\| \leqslant \|T'(x)^{-1}\| \, \|Tx\| \leqslant t/\tau = c^{-1}(\tau - \tau_1) \leqslant c^{-1}(\tau) - c^{-1}(\tau').$$

As an immediate consequence of this evaluation, we get by a mean value technique (see the corresponding arguments used in Theorem 4) plus Lemma 2, (a),

$$\|Tx'\| = \|Tx' - Tx - T'(x)(x' - x)\| \leqslant C(t/\tau) = t' < t;$$

at the same time, again by (xii) plus (a) of Lemma 2, $\|T'(x) - T'(x')\| \leqslant \leqslant c(t/\tau) < \tau$, in which case

$$\gamma(T'(x')) \geqslant \gamma(T'(x)) - \|T'(x) - T'(x')\| \geqslant \tau - c(t/\tau) = \tau'.$$

This shows the Newton-Kantorovich process $(y_n;\ n \in N)$ constructed by (1) from the starting point $y_0$, exists and remains in $D^0$; moreover if we denote by $((s_n, \tau_n);\ n \in N)$ the sequence of iterates modulo $S$ having as starting element $(s_0, 1) \in A$ one has by the above relations

$$\begin{cases} \|y_n - y_m\| \leqslant c^{-1}(\sigma_n) - c^{-1}(\sigma_n), & n \leqslant m, \\ \|Ty_n\| \leqslant s_n, \ \gamma(T'(y_n)) \geqslant \sigma_m, & n \in N, \end{cases}$$

and therefore, in view of Lemma 2, (a), the element $y^* = \lim_n y_n$ is a solution in $D$ of the underlying equation, approximated by the elements of this sequence under an error formula expressed as

$$\|y_n - y^*\| \leqslant c^{-1}(\sigma_n) - c^{-1}(\lim_n \sigma_n), \quad n \in N.$$

Our theorem is completely proved. q.e.d.

As already noted in the first part of the section, the above result says in fact that for each $x_0 \in D^0$ satisfying

$$0 < \|Tx_0\| \leqslant t_0,\ \gamma(T'(x_0)) \geqslant \tau_0,\ \|x_0 - y_0\| \leqslant c^{-1}(1) - c^{-1}(\tau_0)$$

with $(t_0, \tau_0) \in A$, the Newton-Kantorovich process (1) generated by $x_0$ exists, remains in $D^0$ and converges to a solution in $D$ of the equation $Tx=0$. Concerning the uniqueness problem, assume $z \in D$ is an arbitrary solution and let $x \in D^0$ satisfy the requirements written for $x_0$ above with $(t, \tau)$ in place of $(t_0, \tau_0)$, as well as $\|z-x\| \leqslant r \leqslant m_0$. Defining $V(\tau) = T(x + \tau(z - x)) - \tau T'(x)(z-x)$, $0 \leqslant \tau \leqslant 1$ and observing that

$$V'(\tau) = (T'(x + \tau(z-x)) - T'(x))(z-x),\ 0 \leqslant \tau < 1$$

gives at once (by an argument we already used)

$$\|T'(x)(z - x + T'(x)^{-1}Tx)\| = \|Tx + T'(x)(z-x)\| = \|V(1) - V(0)\| \leqslant C(r)$$

we have (in view of the evaluation $\gamma(T'(x)) \geqslant \tau$)

$$(9) \qquad \|z - x'\| \leqslant r' = C(r)/\tau, \text{ where } x' = x - T'(x)^{-1}Tx.$$

This relation tells us that, if $\|z - y_0\| \leqslant s_0$ then (see Lemma 2, (b))

$$\|z - y_n\| \leqslant r_n,\ n \in N,\ \text{with } r_n \to 0$$

so, necessarily $z = y^*$, proving the underlying equation has a unique solution in $D_0 = B^-(y_0, s_0) < D$, but when $s_0 < \|z - y_0\| \leqslant m_0$, this conclusion is no longer true, in general : however, under some reasonable hypotheses, an appropriate uniqueness property may be obtained along the following lines. Denote by $P$ the transformation $(t, \tau, r) \to (t', \tau', r')$ from $(0, m_0] \times (0, 1] \times (0, m_0]$ into $(0, \infty) \times R \times (0, \infty)$ defined as

$$t' = C(t/\tau),\ \tau' = \tau - c(t/\tau),\ \ r' = C(r)/\tau$$

and assume there exists a subset $B$ of $(0, m_0] \times (0, 1] \times (0, m_0]$ invariant under $P$ in the sense

(xiii)' $(t', \tau', r') \in B$ (with $r' < r$) *whenever* $(t, \tau, r) \in B$

and satisfying also

$$\text{(xiv)} \begin{cases} \textit{if } ((t_n, \tau_n, r_n) : n \in N) \textit{ is the sequence of } P\textit{-iterates generated by} \\ (t_0, \tau_0, r_0) \in B \textit{ then } r_n \to 0 \end{cases}$$

(note at this moment that, in view of Lemma 2, (a), we also have $t' < t$, $\tau' < \tau$, in addition to (xiii)', and $t_n/\tau_n \to 0$, in addition to (xiv)). Also, let us admit that an element $x$ in $(0, m_0)$ with $(s_0^*, 1, m_0) \in B$ may be found so as to satisfy $\|Ty_0\| \leqslant s_0^*$. Since the reasonings of Theorem 5 remain entirely valid when $A$ is replaced by $B$ it firstly follows the Newton-Kantorovich process (1) generated by $y_0$ exists, remains in $D^0$ and converges to a solution $y^* \in D$ of the equation $Tx=0$. Secondly, $z \in D$ being another solution of this equation, in view of (9) one clearly gets

$$\|z - y_0\| \leqslant r_0^* = m_0,\ \ \|z - y_n\| \leqslant r_n^*,\ n \in N$$

where $((s_n^*, \sigma_n^*, r_n^*) : n \in N)$ is the sequence of $P$-iterates having as starting element $(s_n^*, 1, m_0)$ and therefore, by (xiii)' +(xiv), $z = y^*$ follows at once, proving the equation $Tx=0$ has a unique solution in $D$.



Since the whole discussion contained in this part of the section was motivated by the hölderian case $c(t) = t^\alpha$, $t \geqslant 0$, for some $\alpha$ in $(0, 1]$ (observe at this moment the case $c(t) = Lt^\alpha$, $t \geqslant 0$, for some $L > 0$, $\alpha$ in $(0, 1]$ reduces to this one by suitably changing the ambient norm) it is natural to ask what happens with our specific hypotheses under this particular choice. Firstly, the transformation $S$ being

$$t' = (t/\tau)^{\alpha+1}/(\alpha+1),\ \ \tau' = \tau - (t/\tau)^\alpha$$

as a candidate for (xiii) one may take the subset $A$ of all couples $(t, \tau)$ in $(0, 1] \times (0, 1]$ satisfying $t^\alpha \leqslant H\tau^{\alpha+1}$ where $H$ is any solution in $(0, 1)$ of the inequality

$$(10) \qquad \Delta(H) = (H/(\alpha+1))^{\alpha/(\alpha+1)} + H \leqslant 1.$$

Noting that the function $\Delta(H)$ defined by the left member of (10) is strictly increasing in $(0, 1)$ and $\Delta(0) < 1$, $\Delta(1) > 1$, it follows the above inequality admits a maximal solution $H_0$ in $(0, 1)$ (for which it becomes equality) and therefore the best choice of $s_0$ appearing in Theorem 5 is, of course, $s_0 = H_0^{1/\alpha}$. Observing in addition that $\alpha/(\alpha+1) \leqslant H_0$ it follows our corresponding statement extends directly some related contributions in this area due to K e l l e r (unpublished result, quoted in R o k n e [22]) and J a n k o [11, ch. II] for $\alpha$ in $(0, 1]$ and, respectively, R h e i n b o l d t [21], C o l l a t z [7, ch. II], O s t r o w s k i [18] for $\alpha = 1$. It is worth mentioning these procedures may be also applied — without major changes — to the study of „perturbed" Newton-Kantorovich processes

$$x_{n+1} = x_n - U(x_n)^{-1}Tx_n,\ n \in N$$

(where $x \mapsto U(x)$ is a mapping from $D^0$ to $L(X)$ approximating in a certain sense the mapping $x \mapsto T'(x)$ and possessing a contractivity property of the form (xii)) in which case, they may be deemed as an alternative to Rheinboldt's convergence theory (see the above reference as well as O r t e g a [16]) which is, until now, the basic tool in studying such processes from a convergence viewpoint especially. Secondly, the transformation $P$ having the form

$$t' = (t/\tau)^{\alpha+1}/(\alpha+1),\ \ \tau' = \tau - (t/\tau)^\alpha,\ \ r' = r^{\alpha+1}/(\alpha+1)\tau$$

a candidate for (xiii)' +(xiv) is the subset $B$ of all triplets $(t, \tau, r)$ in $(0, 1] \times (0, 1] \times (0, 1]$ satisfying $t^\alpha \leqslant H\tau^{\alpha+1}$, $r \leqslant K(\alpha+1)\tau$, where $(H, K)$ is any couple in $(0, 1) \times (0, 1)$ for which (10) and

$$(11) \qquad 1/(\alpha+1) \leqslant K \leqslant (1-H)^{1/\alpha}$$

hold. For example, an immediate solution of (10) +(11) is $H = 1 - (1/(\alpha+1))^\alpha$, $K = 1/(\alpha+1)$ in which case, the correspondig version of the uniqueness property we already stated may be compared with Theorem 3 of *Janko* [11, ch. II] as well as Theorem 1 (the non-perturbed case) of *Rokne* [22]. Of course, this hölderian choice of the function $c$ is by no means the only possible one which is to be treated by these methods ; some further aspects of the problem will be discussed elsewhere.

4. **Altman transfinite processes.** As already precised in the Introduction, the result of Kantorovich subsumed to Theorem 1 has a non-constructive counterpart (stated as Theorem 2) obtained by A l t m a n through a transfinite induction argument [1] or, equivalently, by ordering techniques [4]. Concerning the relationships between these two statements it was observed in [3] that under some restrictive assumptions about the function $f$ (including the standard quadratic case

$$f(t)=\lambda(s^*-t)^2,\ s_0\leqslant t\leqslant s^*,$$

with $\lambda>0$ suitably chosen) Theorem 1 may be identified with a certain constructive portion of Theorem 2 under some appropriate hypotheses concerning the Fréchet derivative $T'$. Since, however, a general reduction result of this type seems to be not possible, it is natural to ask, in a first approximation, until what extent (modulo the results of Section 1) can Theorem 2 be generalized and then — under an appropriate positive answer — to investigate the relationships between their corresponding extensions

Let $(X, d, \leqslant)$ be a quasi-ordered metric space with $\leqslant$ being self-closed and $(X, d)$ complete. Given the subset $D$ of $X$ and the Banach space $(Z, \|.\|)$, remember that a mapping $T$ from $D$ into $Z$ is said to be $\leqslant$*-closed* when $(x_n\,;\,n\in N)$ increasing in $D$, $x_n\to x$ and $Tx_n\to z$ imply $x\in D$ and $Tx=z$. With $T$ having such a property and the element $y_0\in D$ satisfying $0<\|Ty_0\|\leqslant\leqslant s_0$ for some $s_0>0$, we are interested to determine sufficient conditions of a local type under which the equation $Tx=0$ should have a solution in a $D$-sphere around $y_0$. In this case, as an immediate application of the developments of Section 1, we have

**Theorem 6.** *Suppose a couple of functions* $a$: $(0,\infty)\to(0,\infty)$, $\varphi:(0,\infty)\to\to[0,\infty)$ *with* $a(t)<t$ *and* $a^n(t)\to 0$, *for all* $t>0$ *(where* $a^n$ *denotes the n-th iterate of* $a$*) satisfying*

$$\text{(xv)}\quad (t+a(t))/(t-a(t))\leqslant(\varphi(t)-\varphi(s))/(t-s),\ t,s>0,\ a(t)\leqslant s<t$$

*as well as a couple of functions* $b:(0,\infty)\to[0,\infty)$, $\psi:(0,\infty)\to[0,\infty)$ *with* $b$ *not identically zero in any interval* $(0,t)$, $t>0$, *and*

$$\text{(xvi)}\quad b(t)/(t-a(t))\leqslant(\psi(t)-\psi(s))/(t-s),\ t,s>0,\ a(t)\leqslant s<t$$

*may be found with the property : for each* $x$ *in*

$$D_0^*=\{x\in D\,;\,y_0\leqslant x,\ d(y_0,x)<\psi(s_0),\ \|Ty_0-Tx\|<\psi(s_0)\}$$

*and each* $t$ *in* $(0,s_0]$ *with* $0<\|Tx\|\leqslant t$ *there exist* $x'\geqslant x$ *in* $D$ *and* $\varepsilon$ *in* $(0,1]$ *with*

$$(2)'\qquad \|Tx'-(1-\varepsilon)Tx\|\leqslant\varepsilon a(t),\ d(x,x')\leqslant\varepsilon b(t).$$

*Then,* $Tx=0$ *has at least a solution in* $D_0=\{x\in D\,;\,y_0\leqslant x,\ d(y_0,x)\leqslant\psi(s_0),\ \|Ty_0-Tx\|\leqslant\varphi(s_0)\}$.

*Proof.* Denote $M=[0,s_0]$ endowed with the usual metric and the usual dual ordering and put $Fx=\|Tx\|$, $Gx=Tx$, $x\in D$. From the above relations (2)',

$$\|Tx'\|\leqslant(1-\varepsilon)\|Tx\|+\varepsilon a(t)\leqslant(1-\varepsilon)t+\varepsilon a(t)=t'<t$$
$$\|Tx-Tx'\|\leqslant\varepsilon(\|Tx\|+a(t))\leqslant\varepsilon(t+a(t)).$$



Adding the fact that $\varepsilon=(t-t')/(t-a(t))$ one gets, by (xv) + (xvi)

$$Fx'\leqslant t',\ d(x,x')\leqslant\psi(t)-\psi(t'),\ \|Gx-Gx'\|\leqslant\varphi(t)-\varphi(t')$$

so, if we let $f=i$ (the identity) and $g=\psi$, $h=\varphi$ (observe at this moment that, without any loss ($\varphi$ and $\psi$ being increasing on their existence domain and $\varphi(t)>0$, $\psi(t)>0$ at each $t>0$) one may assume $\varphi(0+)=\psi(0+)=0$, which enables us to complete the definition of these functions at the origin by putting $\varphi(0)=\psi(0)=0$) conditions of Corollary 2 are fulfilled and the proof is complete. q.e.d.

It as now clear that Altman's result subsumed to Theorem 2 follows from the above one if we let $a(t)=qt$, $t>0$ (hence $\varphi(t)=((1+q)/(1-q))t$, $t>0$) for some $q\in(0,1)$ and assume $b$ is increasing and strictly positive on its existence domain and $b^*(t)=\int_0^t(b(s)/s)ds<\infty$, $t>0$ (in which case $\psi(t)==b^*(t/q)$, $t>0$) ; see also C r a m e r and R a y [8]. However, the result of T u r i n i c i [28] seems to be not reductible to Theorem 6 because the function $b$ appearing in that statement need not be increasing. Concerning the possibility of attaining a solution of the equation $Tx=0$ through a constructive process of the form $x\vdash x'$, the following practical criterion may be formulated. Given $x_0\in D_0^*$ with

$$0<\|Tx_0\|\leqslant t_0,\ d(y_0,x_0)\leqslant\psi(s_0)-\psi(t_0),\ \|Ty_0-Tx_0\|\leqslant\varphi(s_0)-\varphi(t_0)$$

for some $t_0$ in $(0,s_0]$, let $x_1\geqslant x_0$ in $D$ and $\varepsilon_0$ in $(0,1]$ be such that (2)' holds with $(x,x')=(x_0,x_1)$, $\varepsilon=\varepsilon_0$ : denoting $t_1=(1-\varepsilon_0)t_0+\varepsilon_0 a(t_0)$ assume $Tx_1\neq 0$ (hence $x_1\in D_0^*$ and $0<\|Tx_1\|\leqslant t_1$) then, again by (2)' one may select a couple $x_2\geqslant x_1$ in $D$ and $\varepsilon_1$ in $(0,1]$ satisfying that relation with $(x,x')=(x_1,x_2)$, $\varepsilon=\varepsilon_1$, and so on. By an ordinary induction procedure an increasing (modulo $\leqslant$) sequence $(x_n\,;\,n\in N)$ in $D_0^*$, a decreasing sequence $(t_n\,;\,n\in N)$ in $(0,s_0]$ and a sequence $(\varepsilon_n\,;\,n\in N)$ in $(0,1]$ can be constructed with (see the reasonings of Theorem 6)

$$0<\|Tx_n\|\leqslant t_n,\ t_{n+1}=(1-\varepsilon_n)t_n+\varepsilon_n a(t_n),\ n\in N$$

$$d(x_n,x_m)\leqslant\psi(t_n)-\psi(t_m),\ \|Tx_n-Tx_m\|\leqslant\varphi(t_n)-\varphi(t_m),\ n\leqslant m.$$

As $(\varphi(t_n)\,;\,n\in N)$ and $(\psi(t_n)\,;\,n\in N)$ are decreasing — hence Cauchy —sequences in $(0,\infty)$ it is clear that $x_n\to x^*$, $x_n\leqslant x^*$, $n\in N$, and $Tx_n\to Tx^*$ for some $x^*\in D_0$ and therefore, to draw the conclusion $Tx^*=0$ it suffices that the sequence $(t_n\,;\,n\in N)$ be zero-convergent. Suppose on the contrary $t_n\to t^*>0$, and let us admit

$$\text{(xvii)}\quad \limsup_{s\to t+}(a(s))<t,\ \textit{for each}\ \ t>0$$

(remark at this moment that, by a result of the author [23], $a(t)<t$, $t>0$, plus (xvii), have as immediate consequence $a^n(t)\to 0$, $t>0$). From this fact, $a(s)<\lambda s$ (hence $s-a(s)>(1-\lambda)s>(1-\lambda)t^*$) for $t^*<s<t^*+\eta$, where $\eta>0$ is sufficiently small and $\eta$ suitably chosen in $(0,1)$ and therefore, in view of

$$\varepsilon_n(t_n-a(t_n))\leqslant t_n-t_{n+1},\ n\in N$$

it is clear that, for a sufficiently large $n(t^*)\in N$

$$(1-\lambda)t^* \varepsilon_n \leq t_n - t_{n+1}, \quad n \geq n(t^*).$$

Now, the series $\sum_{n \in N} (t_n - t_{n+1})$ being convergent one obtains a contradiction provided we stipulate that

(xviii) *the series* $\sum_{n \in N} \varepsilon_n$ *is divergent.*

Summing up these developments we found that, given the sequence $(x_n ; n \in N)$ constructed under the requirements (2)', the associated sequence $(\varepsilon_n ; n \in N)$ of which satisfying (xviii), its limit $x^* \in D_0$ is necessarily a solution of the underlying equation provided the supplementary hypothesis (xvii) is accepted. Particulary, when $a(t) = qt$, $t > 0$, for some $q \in (0, 1)$ the corresponding variant of this criterion reduces to that of Altman [3] obtained by a different method.

Concerning the relationships between Theorems 4 and 6 remember that for each $x \in D^0$ with

$$0 < \|Tx\| \leq f(t), \quad \|y_0 - x\| \leq t - s_0 \text{ for some } t_0 \in [s_0, s^*)$$

there exist $x'$ in $D$ and $t' = t - f(t)/f'(t)$ in $[s_0, s^*)$ with

$$\|Tx'\| \leq f(t') = f(t - f(t)/f'(t)), \quad \|x - x'\| \leq t' - t = -f(t)/f'(t)$$

so, conditions (2)' will be surely fulfilled with $\varepsilon = 1$ provided

$$(12) \qquad f(t - f(t)/f'(t)) \leq a(f(t)), \quad -f(t)/f'(t) \leq b(f(t)), \quad s_0 \leq t < s^*.$$

These relations turn out to be satisfied in many particular cases. For example, when $f$ is described as

$$f(t) = \lambda(s^* - t)^{\alpha+1}, \quad s_0 \leq t \leq s^*, \text{ for some } \alpha > 0, \ \lambda > 0$$

the second of them, having the form

$$(s^* - t)/(\alpha + 1) \leq b(\lambda(s^* - t)^{\alpha+1}), \quad s_0 \leq t < s^*$$

will be satisfied (with equality) if we take

$b(t) = \mu t^{1/(\alpha+1)}$, $t > 0$, for a suitably chosen $\mu > 0$

while the first one becomes

$$((s^* - t)\alpha/(\alpha + 1))^{\alpha+1} \leq a(\lambda(s^* - t)^{\alpha+1}), \quad s_0 \leq t < s^*$$

a solution of it (again with equality) being of the form $a(t) = qt$, $t > 0$, for an appropriate $q \in (0, 1)$. At the same time, concerning the relationships between Theorems 5 and 6, we recall that, given $x \in D^0$ and $(t, \tau)$ in $A$ with

$$0 < \|Tx\| \leq t, \quad \gamma(T'(x)) \geq \tau, \quad \|y_0 - x\| \leq c^{-1}(1) - c^{-1}(\tau)$$

there exists $x'$ in $D$ satisfying

$$\|Tx'\| \leq C(t/\tau), \quad \gamma(T'(x')) \geq \tau - c(t/\tau), \quad \|x - x'\| \leq t/\tau$$

and therefore (2)' will take place (with $\varepsilon = 1$) in case

$$(12)' \qquad C(t/\tau) \leq a(t), \quad t/\tau \leq b(t).$$

To discuss this eventuality under the standard choice $c(t) = t^\alpha$, $t \geq 0$, for some $\alpha$ in $(0, 1)$ observe that, in such a case these relations become



$$(t/\tau)^{\alpha+1}/(\alpha+1) \leq a(t), \quad t/\tau \leq b(t)$$

and since, as a candidate for $A$, we may take the subset of all $(t, \tau)$ in $(0, 1] \times (0, 1]$ satisfying

$$t^\alpha \leq H\tau^{\alpha+1} \text{ (or, equivalently, } (t/\tau)^{\alpha+1} \leq Ht),$$

where $H$ is any solution in $(0, 1)$ of (10), the above inequalities will be surely fulfilled if we put

$$a(t) = (H/(\alpha+1))t, \quad b(t) = (Ht)^{1/(\alpha+1)}, \quad t > 0.$$

Of course, the possibility of satisfying (12)+(12)' in general, is still an open problem but, anyway, the standard cases are contained in this reduction scheme. For a number of interesting applications to nonlinear analysis of these problems we refer to Altman [2].

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# A CRITERION FOR THE OSCILLATORY AND ASYMPTOTIC BEHAVIOUR OF DIFFERENTIAL EQUATIONS WITH FORCING TERM

BY

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Consider the nonlinear second order differential equation

(1) $$\ddot{x}(t)+q(t)f(x(t))g(\dot{x}(t))=e(t), \qquad \left(\cdot=\frac{d}{dt}\right)$$

where $e$, $q:[t_0,\infty)\to R=(-\infty,\infty)$, $f,g:R\to R$ are continuous, $q(t)$ nonnegative and not identically zero on any ray of the form $[t^*,\infty)$ for some $t^*\geq t_0$.

We will assume that:

(2) $$g(y)>0 \text{ for } y\neq0,\ g'(y)\geq0 \text{ for } y\in(0,\infty) \text{ and } g'(y)\leq0 \text{ for } y\in(-\infty,0), \qquad \left('=\frac{d}{dy}\right);$$

(3) $$xf(x)>0 \text{ and } f'(x)\geq0 \text{ for } x\neq0;$$

(4) $-f(-xy)\geq f(xy)\geq Kf(x)f(y)$ for $x,y>0$ and $K$ is a positive constant. Furthermore, we assume that there exists an oscillatory function $\eta:[t_0,\infty]\to R$ such that

(5) $$e(t)=\ddot{\eta}(t), \quad \frac{\eta(t)}{t}\to0 \text{ and } \dot{\eta}(t)\to0 \text{ as } t\to\infty.$$

We are concerned only with continuable solutions of equation (1) which exist on some ray $[t_0,\infty)$. A solution $x(t)$ of equation (1) is called nonoscillatory if there exists a $t_1\geq t_0$ such that $x(t)\neq0$ for $t>t_1$, and is called oscillatory if it has arbitrary large zeros.

Some qualitative behaviour of equation (1) (with $e(t)\equiv0$) has been studied by many authors. For some contribution in this area see the papers of Lalli [1] and Wong and Burton [3]. However, nothing much of significance seems to be known about equation (1), therefore the purpose of this paper is to establish a criterion for the oscillatory and asymptotic behaviour of all solutions of equation (1).

Our main result is the following theorem:

**Theorem 1.** *Let conditions* (2)—(5) *hold. If*