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A CRITERION FOR THE OSCILLATORY AND ASYMPTOTIC BEHAVIOUR OF DIFFERENTIAL EQUATIONS WITH FORCING TERM

BY

S. R. GRACE and B. S. LALLI

Consider the nonlinear second order differential equation

$$\ddot{x}(t)+q(t)f(x(t))g(\dot{x}(t))=e(t), \qquad \left(\cdot=\frac{d}{dt}\right) \tag{1}$$

where e, q : $[t_0, \infty)\to R=(-\infty,\infty)$, $f, g : R\to R$ are continuous, $q(t)$ nonnegative and not identically zero on any ray of the form $[t^*, \infty)$ for some $t^*\geq t_0$.

We will assume that :

$$g(y)>0 \text{ for } y\neq 0,\ g(y)\geq 0 \text{ for } y\in(0,\infty) \text{ and } g'(y)\leq 0 \text{ for } y\in(-\infty,0), \qquad \left({}'=\frac{d}{dy}\right); \tag{2}$$

$$xf(x)>0 \text{ and } f'(x)\geq 0 \text{ for } x\neq 0; \tag{3}$$

(4) $-f(-xy)\geq f(xy)\geq Kf(x)f(y)$ for $x, y>0$ and K is a positive constant. Furthermore, we assume that there exists an oscillatory function η : $[t_0, \infty]\to R$ such that

$$e(t)=\ddot{\eta}(t), \quad \frac{\eta(t)}{t}\to 0 \text{ and } \dot{\eta}(t)\to 0 \text{ as } t\to\infty. \tag{5}$$

We are concerned only with continuable solutions of equation (1) which exist on some ray $[t_0, \infty)$. A solution $x(t)$ of equation (1) is called nonoscillatory if there exists a $t_1\geq t_0$ such that $x(t)\neq 0$ for $t>t_1$, and is called oscillatory if it has arbitrary large zeros.

Some qualitative behaviour of equation (1) (with $e(t)\equiv 0$) has been studied by many authors. For some contribution in this area see the papers of Lalli [1] and Wong and Burton [3]. However, nothing much of significance seems to be known about equation (1), therefore the purpose of this paper is to establish a criterion for the oscillatory and asymptotic behaviour of all solutions of equation (1).

Our main result is the following theorem :

Theorem 1. *Let conditions* (2)—(5) *hold. If*

$$\limsup_{t\to\infty} \int_{t_0}^{t} q(s)f(s)\,ds = \infty, \tag{6}$$

then every solution $x(t)$ of equation (1) *is either oscillatory or* $\lim_{t\to\infty} \dot{x}(t)=0$.

Proof. Let $x(t)$ be a nonoscillatory solution of equation (1), say $x(t)>0$ for $t>t_0$. Let $y(t)=x(t)-\eta(t)$. Then

$$\ddot{y}(t)+q(t)f(y(t)+\eta(t))g(\dot{y}(t)+\dot{\eta}(t))=0. \tag{7}$$

If $y(t)<0$ for all large t, then $y(t)+\eta(t)>0$ implies $\eta(t)>-y(t)>0$, a contradiction to the oscillatory character of $\eta(t)$. Hence $y(t)>0$ for all large t. There exists a sufficiently large $t_1\geq t_0$ so that $y(t)>0$ for $t\geq t_1$ and this implies that $\dot{y}(t)>0$ and $\ddot{y}(t)\leq 0$ for $t\geq t_1$. Furthermore, consider an arbitrary number θ with $0<\theta<1$. Then, by applying Lemma 2 in [2], we conclude that there exists a $t_2\geq t_1$ such that

$$y(t)\geq \theta t\dot{y}(t) \quad \text{for all } t\geq t_2. \tag{8}$$

Using (3) and (8) we have

$$\ddot{y}(t)+q(t)f(\theta t\dot{y}(t)+\eta(t))g(\dot{y}(t)+\dot{\eta}(t))\leq 0.$$

By condition (4), we obtain

$$\ddot{y}(t)+K^2f(0)q(t)f(t)f\left(\dot{y}(t)+\frac{\eta(t)}{\theta t}\right)g(\dot{y}(t)+\dot{\eta}(t))\leq 0.$$

Now, let $u=\dot{y}$. Then

$$\dot{u}(t)+K^2f(0)q(t)f(t)f\left(u(t)+\frac{\eta(t)}{\theta t}\right)g(u(t)+\dot{\eta}(t))\leq 0. \tag{9}$$

It follows that $\dot{u}(t)<0$ so that $\lim_{t\to\infty} u(t)=c$, where $c\geq 0$ is a constant. If $c>0$, then we have

$$\dot{x}(t)=\dot{y}(t)+\dot{\eta}(t)\geq \frac{c}{2} \text{ and } \dot{y}(t)+\frac{\eta(t)}{\theta t}\geq \frac{c}{2} \quad \text{for } t\geq t_2.$$

From (9) we get

$$\dot{u}(t)+K^2f(0)q(t)f(t)f\left(\frac{c}{2}\right)g\left(\frac{c}{2}\right)\leq 0, \text{ for } t\geq t_2. \tag{10}$$

Integrating (10) from t_2 to t, we obtain

$$u(t)-u(t_2)+K^2f(0)f\left(\frac{c}{2}\right)g\left(\frac{c}{2}\right)\int_{t_2}^{t} q(s)f(s)\,ds\leq 0. \tag{11}$$

By taking the limit superior in (11) as $t\to\infty$, we get a contradiction to (6). Hence we conclude that $c=0$. From this we get that $\lim_{t\to\infty}\dot{x}(t)=0$. This completes the proof.

For illustration we consider the following examples:

Example 1. The equation

$$\ddot{x}(t)+\frac{1}{\ln t}x(t)\dot{x}^2(t)=0, \quad t\geq 3, \tag{12}$$

has the nonoscillatory solution $x(t)=\ln t$, $\dot{x}(t)=\frac{1}{t}\to 0$ as $t\to\infty$.

All conditions of Theorem 1 are satisfied.

Example 2. The equation

$$\ddot{x}(t)+\frac{1}{2}\,\frac{1}{(2-\sin t)(1+\cos^2 t)}\,x(t)(1+\dot{x}^2(t))=\frac{1}{2}+\sin t, \quad t\geq 0, \tag{13}$$

has the nonoscillatory $x(t)=2-\sin t$. Clearly $\dot{x}(t)=-\cos t\not\to 0$ as $t\to 0$. Here only condition (5) is violated.

Example 3. Consider the equation

$$\ddot{x}(t)+e^{2t}\,x(t)\dot{x}^2(t)=e^{-t}\left[\sin t-\cos t+\frac{1}{2}\cos 3t\right], \quad t\geq 0. \tag{14}$$

It is easy to check that the hypotheses of Theorem 1 are satisfied and hence every solution $x(t)$ of equation (14) is either oscillatory or $\lim_{t\to\infty}\dot{x}(t)=0$. One such solution is $x(t)=e^{-t}\sin t$.

It is easy to extend our result to differential equations with deviating arguments of the form:

$$\ddot{x}(t)+q(t)f(x[\sigma_1(t)])g(\dot{x}[\sigma_2(t)])=e(t), \tag{15}$$

where e, q, f and g satisfy conditions (2)—(4), $\sigma_i: [t_0,\infty)\to R$, $i=1,2$ are continuous, $\sigma_2(t)\leq t$ and $\sigma_i(t)\to\infty$ as $t\to\infty$ $i=1,2$. We assume that there is a positive continuous function $\sigma(t)$ on $[t_0,\infty)$ such that

$$\sigma(t)\leq \inf_{s\geq t}\{\sigma_1(s),\ \sigma_2(s),\ s\}, \text{ and } \sigma(t)\to\infty \text{ as } t\to\infty. \tag{16}$$

Theorem 2. *Let conditions* (2)—(5) *and* (16) *hold. If*

$$\limsup_{t\to\infty}\int_{t_0}^{t} q(s)f(\sigma(s))\,ds=\infty, \tag{17}$$

then every solution $x(t)$ of equation (15) *is either oscillatory or* $\lim_{t\to\infty}\dot{x}(t)=0$.

Proof. Let $x(t)$ be a nonoscillatory solution of equation (1), say $x(t)>0$ for $t\geq t_0$. There exists a $t_1\geq t_0$ so that $x[\sigma(t)]>0$ for $t\geq t_1$. Let $x(t)=y(t)+\eta(t)$. As in the proof of Theorem 1, we obtain

$$x[\sigma(t)]\geq \theta\sigma(t)\dot{x}[t] \quad \text{for} \quad t\geq t_2$$

and

$$\ddot{y}(t)+K^2f(\theta)q(t)f(\sigma(t))f\left(\dot{y}(t)+\frac{\eta[\sigma(t)]}{\theta\sigma(t)}\right)g(\dot{y}(t)+\dot{\eta}[\sigma_2(t)])\leq 0.$$

The rest of the proof is similar to that of Theorem 1 and hence is omitted.

The following example is illustrative.

Example 4. Consider the equation

$$x(t) + \frac{1}{t^{1+\alpha}} x[\sigma_1(t)] x^{2}[\sigma_2(t)] = e(t), \quad t > 0, \quad \alpha > 0, \tag{18}$$

where $e(t)$ satisfies condition (4), $\sigma_1(t) = ct$ or t^c or $t \pm \sin t$ and $c > 0$ and $\sigma_2(t) = c_1 t$ or t^{c_1}, $0 < c_1 \leqslant 1$. Here we let $\sigma(t) = c_1 t$ or t^{c_1} or $t - 1$ and $\alpha = c_1$. All conditions of Theorem 2 are satisfied and hence every solution $x(t)$ of equation (18) is either oscillatory or $\lim\limits_{t \to \infty} \dot{x}(t) = 0$.

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A CHARACTERIZATION OF THE SOLUTIONS OF THE DARBOUX PROBLEM FOR THE EQUATION $\frac{\partial^2 z}{\partial x \partial y} \in F(x, y, z)$

BY

GEORGETA TEODORU

In this note we introduce the concept of solution of Darboux problem associated with the multivalued hyperbolic equation $\partial^2 z / \partial x \partial y \in F(x, y, z)$ and we prove a characterization of the solutions, using the Aumann integral defined for multivalued functions, [1], [2], [3].

Definition 1. *Let* $D = [0, a] \times [0, b] \subset \mathbf{R}^2$. *For each* $(x, y) \in D$, *let* $H(x, y)$ *be a nonempty subset of* $\mathbf{R}^n$. *Let* $\mathcal{H}$ *be the set of functions* $h : D \to \mathbf{R}^n$ *integrable on* D *and* $h(x, y) \in H(x, y)$ *for each* $(x, y) \in D$. *Then, by the integral of the multivalued function* $H : D \to 2^{\mathbf{R}^n}$, *we mean the set*

$$\iint_D H(x, y)\,dx\,dy = \Big\{ \iint_D h(x, y)\,dx\,dy \;\Big|\; h \in \mathcal{H} \Big\}.$$

In the following we list some properties of the defined integral above.

Proposition 1. *If* $H : D \to 2^{\mathbf{R}^n}$ *is an upper semicontinuous multivalued function and there exists a positive real number* Φ *such that* $|H(x, y)| = \sup\{\|\xi\| \mid \xi \in H(x, y)\} \leqslant \Phi$ *for each* $(x, y) \in D$, *then*

$$\iint_D H(x, y)\,dx\,dy = \iint_D \operatorname{conv} H(x, y)\,dx\,dy.$$

Proposition 2. *If* $H_k : D \to 2^{\mathbf{R}^n}$, $k \in \mathbf{N}$, *are upper semi-continuous and there exists a positive real number* Φ *such that* $|H_k(x, y)| \leqslant \Phi$ *for each* $(x, y) \in D$ *and* $k \in \mathbf{N}$, *then*

$$\iint_D \varlimsup H_k(x, y)\,dx\,dy \subset \varlimsup \iint_D H_k(x, y)\,dx\,dy.$$

Taking into account the definition 2 in [2], we have $(x, y) \in \varlimsup H_k(x, y)$ iff each neighbourhood of (x, y) intersects all the sets $H_k(x, y)$ with k large enough.

Proposition 3. *If* A *is a compact subset of* $\mathbf{R}^n$, *independent on* (x, y), *then*

$$\int_{x_1}^{x_2}\int_{y_1}^{y_2} A\,dx\,dy = (x_2 - x_1)(y_2 - y_1) \operatorname{conv} A,$$

where $(x_1, y_1), (x_2, y_2) \in D$.

Moreover we need the following proposition :