The rest of the proof is similar to that of Theorem 1 and hence is omitted.

The following example is illustrative.

Example 4. Consider the equation

$$\ddot{x}(t)+\frac{1}{t^{1+\alpha}}x[\sigma_1(t)]\dot{x}^2[\sigma_2(t)]=e(t),\ t>0,\ \alpha>0, \tag{18}$$

where $c(t)$ satisfies condition (4), $\sigma_1(t)=ct$ or t^c or $t\pm\sin t$ and $c>0$ and $\sigma_2(t)=c_1t$ or t^{c_1}, $0<c_1\leqslant 1$. Here we let $\sigma(t)=c_1t$ or t^{c_1} or $t-1$ and $\alpha=c_1$. All conditions of Theorem 2 are satisfied and hence every solution $x(t)$ of equation (18) is either oscillatory or $\lim\limits_{t\to\infty}\dot{x}(t)=0$.

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Received 21 I 1986

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Analele ştiinţifice ale Universităţii „Al. I. Cuza" din Iaşi
Tomul XXXIII, s. I a, Matematică, 1987, f. 1

A CHARACTERIZATION OF THE SOLUTIONS OF THE DARBOUX PROBLEM FOR THE EQUATION $\frac{\partial^2 z}{\partial x\partial y}\in F(x,y,z)$

BY

GEORGETA TEODORU

In this note we introduce the concept of solution of Darboux problem associated with the multivalued hyperbolic equation $\partial^2 z/\partial x\partial y\in F(x,y,z)$ and we prove a characterization of the solutions, using the Aumann integral defined for multivalued functions, [1], [2], [3].

Definition 1. *Let $D=[0,a]\times[0,b]\subset\mathbf{R}^2$. For each $(x,y)\in D$, let $H(x,y)$ be a nonempty subset of $\mathbf{R}^n$. Let $\mathscr{H}$ be the set of functions $h:D\to\mathbf{R}^n$ integrable on D and $h(x,y)\in H(x,y)$ for each $(x,y)\in D$. Then, by the integral of the multivalued function $H:D\to 2^{\mathbf{R}^n}$, we mean the set*

$$\iint_D H(x,y)dx\,dy=\Big\{\iint_D h(x,y)dx\,dy \,\Big|\, h\in\mathscr{H}\Big\}.$$

In the following we list some properties of the defined integral above.

Proposition 1. *If $H:D\to 2^{\mathbf{R}^n}$ is an upper semicontinuous multivalued function and there exists a positive real number Φ such that $|H(x,y)|=\sup\{\|\xi\| \mid \xi\in H(x,y)\}\leqslant\Phi$ for each $(x,y)\in D$, then*

$$\iint_D H(x,y)dx\,dy=\iint_D \operatorname{conv} H(x,y)dx\,dy.$$

Proposition 2. *If $H_k:D\to 2^{\mathbf{R}^n}$, $k\in\mathbf{N}$, are upper semi-continuous and there exists a positive real number Φ such that $|H_k(x,y)|\leqslant\Phi$ for each $(x,y)\in D$ and $k\in\mathbf{N}$, then*

$$\iint_D \underline{\lim}\, H_k(x,y)dx\,dy\subset\underline{\lim}\iint_D H_k(x,y)dx\,dy.$$

Taking into account the definition 2 in [2], we have $(x,y)\in\underline{\lim}\, H_k(x,y)$ iff each neighbourhood of (x,y) intersects all the sets $H_k(x,y)$ with k large enough.

Proposition 3. *If A is a compact subset of $\mathbf{R}^n$, independent on (x,y), then*

$$\int_{x_1}^{x_2}\int_{y_1}^{y_2} A\,dx\,dy=(x_2-x_1)(y_2-y_1)\operatorname{conv} A,$$

where $(x_1,y_1),\ (x_2,y_2)\in D$.

Moreover we need the following proposition :

Proposition 4. *If K is a convex set in a Banach space X, then the set $\mathcal{X}_\varepsilon=\bigcup_{x\in K} B[x,\varepsilon]$ is convex.*

We assume that the following hypotheses are satisfied:

(H_1) $F: D\times\Omega\to 2^{\mathbf{R}^n}$ is an upper semi-continuous multivalued function whose values are nonempty, compact and convex sets in $\mathbf{R}^n$, where $D=[0,a]\times[0,b]$, $\Omega\subset\mathbf{R}^n$ being an open bounded set,

(H_2) For each $z\in\Omega$, the mapping $(x,y)\to F(x,y,z)$ is measurable with respect to the Lebesgue measure on D,

(H_3) There exists a Lebesgue integrable function $g: D\to\mathbf{R}_+$ on D, such that $\|\xi\|\leqslant g(x,y)$, $\forall\xi\in F(x,y,z)$, $\forall(x,y)\in D$, $\forall z\in\Omega$,

(H_4) The functions $\sigma\in AC([0,a];\mathbf{R}^n)$, $\tau\in AC([0,b];\mathbf{R}^n)$, satisfy the condition $\sigma(0)=\tau(0)$ and the function $\alpha: D\to\mathbf{R}^n$ defined by the relation

(1) $$\alpha(x,y)=\sigma(x)+\tau(y)-\sigma(0),\ (x,y)\in D$$

is valued in M for $(x,y)\in\Delta=[0,x_0]\times[0,y_0]\subseteq D$.

Remark. According to the hypothesis (H_5) it follows that the function α is absolutely continuous in the Carathéodory sense [4, §565—§568] on D.

Definition 2. *The Darboux problem for the equation*

(2) $$\frac{\partial^2 z}{\partial x\partial y}\in F(x,y,z),\ (x,y)\in D,\ z\in\Omega,$$

consists in the determination of a solution of the equation (2) satisfying the conditions

(3) $$\begin{cases} z(x,0)=\sigma(x), & 0\leqslant x\leqslant a \\ z(0,y)=\tau(y), & 0\leqslant y\leqslant b,\end{cases}\qquad \sigma(0)=\tau(0).$$

Following A. Cellina [2], we have:

Definition 3. *A function $Z: D\to\mathbf{R}^n$ is called a solution of the Darboux problem (2)+(3), if it is absolutely continuous in the Carathéodory sense on D, it satisfies the equation (2) a.e. for $(x,y)\in D$, and conditions (3) hold for each $x\in[0,a]$ and $y\in[0,b]$.*

Theorem. *Let the hypotheses $(H_1)-(H_4)$ be satisfied. Then, the continuous function $Z: D\to\mathbf{R}^n$ is a solution of the Darboux problem (2)+(3) iff for each $(x_1,y_1)\in D$, $(x_2,y_2)\in D$ the relation*

(4) $$Z(x_2,y_2)-Z(x_1,y_2)-Z(x_2,y_1)+Z(x_1,y_1)\in\int_{x_1}^{x_2}\int_{y_1}^{y_2}F(\xi,\eta,Z(\xi,\eta))\,d\xi d\eta$$

holds, and Z satisfies the conditions (3).

Proof. The necessity of (4) is a consequence of the following arguments. Let Z be a solution of (2)+(3) on D. Then, there exists a selection f of the multivalued function F, $f: D\to\mathbf{R}^n$, such that

(5) $$f(x,y,Z(x,y))\in F(x,y,Z(x,y)),\ (x,y)\in D,$$

with the property that Z satisfies the equation

(6) $$\frac{\partial^2 Z(x,y)}{\partial x\partial y}=f(x,y,Z(x,y)),\ \text{a.e. for } (x,y)\in D$$

and conditions (3).

Choose two points (x_1,y_1), (x_2,y_2) in D; and integrating the equation (6) on $\delta=[x_1,x_2]\times[y_1,y_2]$, we get

(7) $$Z(x_2,y_2)-Z(x_1,y_2)-Z(x_2,y_1)+Z(x_1,y_1)=\int_{x_1}^{x_2}\int_{y_1}^{y_2}f(\xi,\eta,Z(\xi,\eta))d\xi d\eta.$$

Thereby, from (5) and Definition 1, we have

(8) $$\int_{x_1}^{x_2}\int_{y_1}^{y_2}f(\xi,\eta,Z(\xi,\eta))d\xi d\eta\in\int_{x_1}^{x_2}\int_{y_1}^{y_2}F(\xi,\eta,Z(\xi,\eta))d\xi d\eta.$$

According to (7), (8), we have (4), as claimed.

In order to prove the sufficiency, we prove first that the continuous function Z, satisfying (4) and (3), has the derivative $\partial^2 Z(x,y)/\partial x\partial y$ a.e. for $(x,y)\in D$. For this, we prove that Z is absolutely continuous in the Carathéodory sense on D. We associate to the continuous function Z, the interval function [4, § 453, § 565],

(9) $$\Phi(\delta)=Z(x_2,y_2)-Z(x_1,y_2)-Z(x_2,y_1)+Z(x_1,y_1).$$

We prove that $\Phi(\delta)$ is absolutely continuous, using the Theorem 1 in [4, § 453]. From (4), (9) we get

(10) $$\Phi(\delta)\in\int_{x_1}^{x_2}\int_{y_1}^{y_2}F(\xi,\eta,Z(\xi,\eta))d\xi d\eta.$$

In view of Definition 1, there exists a integrable function $f:\delta\times\Omega\to\mathbf{R}^n$, such that (5) holds for $(x,y)\in\delta$. Then (4), (10) yield

$$\Phi(\delta)=\int_{x_1}^{x_2}\int_{y_1}^{y_2}f(\xi,\eta,Z(\xi,\eta))d\xi\,d\eta.$$

According to the hypothesis (H_3), we obtain

(11) $$\|\Phi(\delta)\|\leqslant\int_{x_1}^{x_2}\int_{y_1}^{y_2}\|f(\xi,\eta,Z(\xi,\eta))\|\,d\xi d\eta\leqslant\int_{x_1}^{x_2}\int_{y_1}^{y_2}g(\xi,\eta)d\xi d\eta.$$

We set

(12) $$\eta(\lambda)=\sup_{\mu(\delta)\leqslant\lambda}\|\Phi(\delta)\|,\ \text{for any } \lambda\in\mathbf{R}_+.$$

In view of the absolute continuity of the integral, for each $\varepsilon>0$ there exists $\delta_1(\varepsilon)$ such that

(13) $$\iint_\delta g(\xi,\eta)d\xi d\eta=\int_{x_1}^{x_2}\int_{y_1}^{y_2}g(\xi,\eta)d\xi d\eta<\varepsilon,\ \text{whenever } \mu(\delta)<\delta_1(\varepsilon).$$

Let $\lambda<\delta_1(\varepsilon)$. According to (11), (12), (13) we obtain

$$\eta(\lambda) \leqslant \sup \int_{x_1}^{x_2}\int_{y_1}^{y_2} g(\xi, \eta)d\xi d\eta < \varepsilon, \text{ whenever } \mu(\delta) \leqslant \lambda < \delta_1(\varepsilon),$$

or

$$\lim_{\lambda\to 0} \eta(\lambda) = 0.$$

According to Theorem in [4, § 453] the interval function $\Phi(\delta)$ is absolutely continuous.

As the continuous function Z satisfies the relations (3), $Z(x, 0) = \sigma(x)$, $0 \leqslant x \leqslant a$, $Z(0, y) = \tau(y)$, $0 \leqslant y \leqslant b$, from (H_1), $Z(x, 0)$ and $Z(0, y)$ are absolutely continuous.

In view of [4, § 567] the function Z is absolutely continuous in the Carathéodory sense. From theorems 5, 6 [4, § 569 – § 570], the function Z has the derivative $\partial^2 Z(x, y)/\partial x\partial y$ a.e. for $(x, y) \in D$.

It remains to prove that the function Z satisfies the equation (2).

Taking into account the hypothesis (H_1) and the continuity of the function Z, it follows that the multivalued function $\tilde{F} : D \to \mathbf{R}^n$, given by

$$(14) \qquad \tilde{F}(x, y) = F(x, y, Z(x, y)), \ (x, y) \in D,$$

is upper semi-continuous on D. Then, by Theorem 9.3.1 [5, p. 69], we deduce

$$(15) \qquad \tilde{F}(B[(x, y), \delta_2]) \subset B[\tilde{F}(x, y), \varepsilon], \ (x, y) \in D,$$

where $B[(x, y), \delta_2]$ is the open ball with the centre $(x, y) \in D$ and radius $\delta_2 = \delta_2(\varepsilon)$ and

$$(\ast) \qquad B[\tilde{F}(x, y), \varepsilon] = \{\xi \in \mathbf{R}^n / d(\xi, \tilde{F}(x, y)) < \varepsilon\}.$$

Fix $(x, y) \in D$. If $(x', y') \in B[(x, y), \delta_2]$, then

$$(16) \qquad \tilde{F}(x', y') \subset B[\tilde{F}(x, y), \varepsilon],$$

because, by the definition 9.1.2 [5, p. 67], we have

$$\tilde{F}(B[(x, y), \delta_2]) = \{\cup \tilde{F}(x', y'), (x', y') \in B[(x, y), \delta_2]\}.$$

The condition (4) may be rewritten as

$$(4') \qquad Z(x', y') - Z(x', y) - Z(x, y') + Z(x, y) \in \int_x^{x'}\int_y^{y'} F(\xi, \eta, Z(\xi, \eta))d\xi d\eta$$

for the domain $[x, x'] \times [y, y']$. According to (14), from (4') we deduce

$$(17) \qquad Z(x', y') - Z(x', y) - Z(x, y') + Z(x, y) \in \int_x^{x'}\int_y^{y'} \tilde{F}(\xi, \eta)d\xi d\eta.$$

By $(x', y') \in B[(x, y), \delta_2]$, we obtain $|x - x'| < \delta_2$, $|y - y'| < \delta_2$. Moreover, $|\xi - x| < \delta_2$, $|\eta - y| < \delta_2$ for $x \leqslant \xi \leqslant x'$, $y \leqslant \eta \leqslant y'$. Similarly to (16) we have

$$(18) \qquad \tilde{F}(\xi, \eta) \subset B[\tilde{F}(x, y), \varepsilon].$$

Then, through (18), the relation (17) yields

$$(19) \qquad Z(x', y') - Z(x', y) - Z(x, y') + Z(x, y) \in \int_x^{x'}\int_y^{y'} B[\tilde{F}(x, y), \varepsilon]d\xi d\eta.$$

As the multivalued function $\tilde{F}$, given by (14) is upper semicontinuous on D, the set $B[\tilde{F}(x, y), \varepsilon]$ is closed in $\mathbf{R}^n$.

In view of $(\ast)$ it follows that $B[\tilde{F}(x, y), \varepsilon]$ is also bounded in $\mathbf{R}^n$, and therefore it is a compact set. Then we can use the Proposition 3 setting $A = B[\tilde{F}(x, y), \varepsilon]$ and $[x, x'] \times [y, y']$ instead of $\delta = [x_1, x_2] \times [y_1, y_2]$ we obtain

$$(20) \qquad \int_x^{x'}\int_y^{y'} B[\tilde{F}(x, y), \varepsilon]d\xi d\eta = (x' - x)(y' - y)\operatorname{conv} B[\tilde{F}(x, y), \varepsilon].$$

According to (20), the relation (19) yields

$$(21) \quad Z(x', y') - Z(x', y) - Z(x, y') + Z(x, y) \in (x' - x)(y' - y)\operatorname{conv} B[\tilde{F}(x, y), \varepsilon]$$

By Proposition 4, the set $B[\tilde{F}(x, y), \varepsilon]$ is convex and therefore

$$(22) \qquad \operatorname{conv} B[\tilde{F}(x, y), \varepsilon] = B[\tilde{F}(x, y), \varepsilon].$$

Using (22), the relation (21) yields

$$(23) \quad Z(x', y') - Z(x', y) - Z(x, y') + Z(x, y) \in (x' - x)(y' - y)B[\tilde{F}(x, y), \varepsilon].$$

From (23) we get

$$(24) \qquad \frac{Z(x', y') - Z(x', y) - Z(x, y') + Z(x, y)}{x' - x} \in (y' - y)B[\tilde{F}(x, y), \varepsilon].$$

Taking into account that $B[\tilde{F}(x, y), \varepsilon]$ is closed, the relation (24) yields

$$\lim_{x'\to x}\left[\frac{Z(x', y') - Z(x, y')}{x' - x} - \frac{Z(x', y) - Z(x, y)}{x' - x}\right] = \left[\frac{\partial Z}{\partial x}(x, y') - \frac{\partial Z}{\partial x}(x, y)\right] \in$$

$$\in (y' - y)B[\tilde{F}(x, y), \varepsilon].$$

Hence

$$\frac{\dfrac{\partial Z}{\partial x}(x, y') - \dfrac{\partial Z}{\partial x}(x, y)}{y' - y} \in B[\tilde{F}(x, y), \varepsilon],$$

and

$$\lim_{y'\to y}\frac{\dfrac{\partial Z}{\partial x}(x, y') - \dfrac{\partial Z}{\partial x}(x, y)}{y' - y} \in B[\tilde{F}(x, y), \varepsilon],$$

or

$$(25) \qquad \frac{\partial^2 Z(x, y)}{\partial x\partial y} \in B[\tilde{F}(x, y), \varepsilon].$$

Since $\tilde{F}(x, y)$ is closed and F is upper semi-continuous, the relation (25) yields, for $\varepsilon \to 0$

$$\frac{\partial^2 Z(x, y)}{\partial x \, \partial y} \in \tilde{F}(x, y) = F(x, y, Z(x, y)), \text{ a.e. for } (x, y) \in D.$$

Therefore, Z satisfies the equation (2) as claimed.

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Received 26 III 1986

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Analele științifice ale Universității „Al. I. Cuza" din Iași
Tomul XXXIII, s. I a, Matematică, 1987, f. 1

CONTROLLING THE FREE BOUNDARIES IN SOLIDIFICATION PROCESSES

BY

HE ZHENG-XU

1. **Introduction.** Let Ω be an open bounded region in R^N lying between two fixed closed smooth surfaces Γ_1 and Γ_2, and let $T>0$. Consider the following two-phase Stefan problem modelling the solidification process of some material in Ω:

$$\begin{aligned} &\theta^1_t - k_1 \Delta \theta^1 = 0, \text{ in } Q^+ \subset \Omega \times (0, T), \\ &\theta^2_t - k_2 \Delta \theta^2 = 0, \text{ in } Q^- = (\Omega \times (0, T)) \setminus \overline{Q^+}, \end{aligned} \tag{1.1}$$

$$\begin{aligned} &\theta^1(x, 0) = \theta_0(x), \text{ if } (x, 0) \in \overline{Q^+}, \\ &\theta^2(x, 0) = \theta_0(x), \text{ if } (x, 0) \in \overline{Q^-}, \end{aligned} \tag{1.2}$$

$$\begin{aligned} &\theta^1 \geq 0, \text{ in } Q^+, \\ &\theta^2 \leq 0, \text{ in } Q^-, \end{aligned} \tag{1.3}$$

$$\theta^1 = \theta^2 = 0, \text{ on } \Sigma = \overline{Q^+} \cap \overline{Q^-}, \tag{1.4}$$

$$k_1 \frac{\partial \theta^1}{\partial \nu_x} - k_2 \frac{\partial \theta^2}{\partial \nu_x} = L \nu_t, \text{ on } \overline{\Sigma}, \tag{1.5}$$

$$\frac{\partial \theta^1}{\partial n} = 0, \text{ on } \Sigma_1 = \Gamma_1 \times (0, T), \tag{1.6}$$

$$\alpha \frac{\partial \theta^2}{\partial n} + \theta^2 = v, \text{ on } \Sigma_2 = \Gamma_2 \times (0, T). \tag{1.7}$$

Here Q^+ and Q^- are the liquid and the solid phase respectively; θ^1 is the temperature of the liquid and θ^2 is the temperature of the solid; k_1, k_2, L and α are positive constants; $\nu = (\nu_x, \nu_t)$ denotes the outward normal to ∂Q^- which is required to be sufficiently smooth on Σ, and n denotes the outward normal to $\Gamma = \partial\Omega$, θ_0 is the temperature at time $t=0$, $\theta_0 \in H^1(\Omega)$; the function v is the controlled temperature of the surrounding near Σ_2,

$$v \leq 0, \text{ on } \Sigma_2.$$

Note that Q^+ is limited (on the lateral side) by Σ_1 and Σ while Q^- is limited by Σ_2 and Σ; ($\Sigma = \overline{Q^+} \cap \overline{Q^-}$ $(\Omega \times (0, T))$ is called the interface). By (1.3), we should have

$$\begin{aligned} &\theta_0(x) \geq 0, \text{ if } (x, 0) \in \overline{Q^+}, \\ &\theta_0(x) \leq 0, \text{ if } (x, 0) \in \overline{Q^-}. \end{aligned} \tag{1.8}$$

Suppose that there is a Hilbert space $\mathcal{U}$ and a closed, bounded, convex subset $\mathcal{U}_{ad}$ of $\mathcal{U}$ such that v is controlled by the rule:

$$v = Bu + v_0, \quad u \in \mathcal{U}_{ad}, \tag{1.9}$$