$$\frac{\partial^2 Z(x,\ y)}{\partial x\ \partial y} \in \bar{F}(x,\ y) = F(x,\ y,\ Z(x,\ y)),\ \text{a.e. for}\ (x,\ y) \in D.$$

Therefore, Z satisfies the equation (2) as claimed.

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CONTROLLING THE FREE BOUNDARIES IN SOLIDIFICATION PROCESSES

BY

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1. **Introduction.** Let Ω be an open bounded region in R^N lying between two fixed closed smooth surfaces Γ_1 and Γ_2, and let $T>0$. Consider the following two-phase Stefan problem modelling the solidification process of some material in Ω:

$$\text{(1.1)}\qquad \begin{aligned} &\theta^1_t - k_1 \Delta \theta^1 = 0,\ \text{in}\ Q^+ \subset \Omega \times (0,\ T),\\ &\theta^2_t - k_2 \Delta \theta^2 = 0,\ \text{in}\ Q^- = (\Omega \times (0,\ T)) \setminus \overline{Q^+}, \end{aligned}$$

$$\text{(1.2)}\qquad \begin{aligned} &\theta^1(x,\ 0) = \theta_0(x),\ \text{if}\ (x,\ 0) \in \overline{Q^+},\\ &\theta^2(x,\ 0) = \theta_0(x),\ \text{if}\ (x,\ 0) \in \overline{Q^-}, \end{aligned}$$

$$\text{(1.3)}\qquad \begin{aligned} &\theta^1 \geqslant 0,\ \text{in}\ Q^+,\\ &\theta^2 \leqslant 0,\ \text{in}\ Q^-, \end{aligned}$$

$$\text{(1.4)}\qquad \theta^1 = \theta^2 = 0,\ \text{on}\ \bar{\Sigma} = \overline{Q^+} \cap \overline{Q^-},$$

$$\text{(1.5)}\qquad k_1 \frac{\partial \theta^1}{\partial \eta_x} - k_2 \frac{\partial \theta^2}{\partial \eta_x} = L\eta_t,\ \text{on}\ \bar{\Sigma},$$

$$\text{(1.6)}\qquad \frac{\partial \theta^1}{\partial n} = 0,\ \text{on}\ \Sigma_1 = \Gamma_1 \times (0,\ T),$$

$$\text{(1.7)}\qquad \alpha \frac{\partial \theta^2}{\partial n} + \theta^2 = v,\ \text{on}\ \Sigma_2 = \Gamma_2 \times (0,\ T).$$

Here Q^+ and Q^- are the liquid and the solid phase respectively; θ^1 is the temperature of the liquid and θ^2 is the temperature of the solid; k_1, k_2, L and α are positive constants; $\eta = (\eta_x,\ \eta_t)$ denotes the outward normal to ∂Q^- which is required to be sufficiently smooth on Σ, and n denotes the outward normal to $\Gamma = \partial\Omega$, θ_0 is the temperature at time $t=0$, $\theta_0 \in H^1(\Omega)$: the function v is the controlled temperature of the surrounding near Σ_2,

$$v \leqslant 0,\ \text{on}\ \Sigma_2.$$

Note that Q^+ is limited (on the lateral side) by Σ_1 and Σ while Q^- is limited by Σ_2 and Σ; ($\Sigma = \overline{Q^+} \cap \overline{Q^-}$ $(\Omega \times (0,\ T))$ is called the interface). By (1.3), we should have

$$\text{(1.8)}\qquad \begin{aligned} &\theta_0(x) \geqslant 0,\ \text{if}\ (x,\ 0) \in \overline{Q^+},\\ &\theta_0(x) \leqslant 0,\ \text{if}\ (x,\ 0) \in \overline{Q^-}. \end{aligned}$$

Suppose that there is a Hilbert space $\mathcal{U}$ and a closed, bounded, convex subset $\mathcal{U}_{ad}$ of $\mathcal{U}$ such that v is controlled by the rule:

$$\text{(1.9)}\qquad v = Bu + v_0,\ u \in \mathcal{U}_{ad},$$

where $B: \mathcal{U} \to L^2(\Sigma_2)$ is a linear continuous operator and $v_0 \in L^2(\Sigma_2)$. Naturally, we must require that

$$Bu + v_0 \leq 0, \quad \forall\, u \in \mathcal{U}_{ad}. \tag{1.10}$$

For any $u \in \mathcal{U}_{ad}$, denote by $\theta^1(u)$, $\theta^2(u)$, $\Sigma(u)$, $Q^+(u)$ and $Q^-(u)$ the solution to (1.1)—(1.7) (assume it exists) where v is given by (1.9). Given a smooth surface Σ_0 in $\Omega \times (0, T)$ which separates $\Omega \times (0, T)$ into two disjoint parts Q_0^+ and Q_0^-: Q_0^+ limited by Σ_0 and Σ_1 and Q_0^- limited by Σ_0 and Σ_2. Assume that the normal $\eta = (\eta_x, \eta_t)$ to Σ_0 is nowhere parallel to the t-axis, i.e. $\eta_x \neq 0$ on Σ_0, and that

$$\begin{aligned} &\theta_0 \geq 0, \text{ in } \{t=0\} \cap \overline{Q_0^+} \\ &\theta_0 \leq 0, \text{ in } \{t=0\} \cap \overline{Q_0^-} \end{aligned} \tag{1.11}$$

We will be concerned with the following inverse problem:

(P) *Find $u \in \mathcal{U}_{ad}$ such that*

$$Q^+(u) = Q_0^+, \quad Q^-(u) = Q_0^-.$$

This kind of problems have been studied in [1], [3], [4], [5], [6] and [9] by minimizing the „distance" between the interface $\Sigma(u)$ and Σ_0 or some penalizing functional of $(\theta^1(u), \theta^2(u))$. The methods used in these papers are very complicated due to the difficulties in solving the state equations, especially in the case $N > 1$. Here we will use a different approach. The idea is to transform the original problem into an equivalent optimal control problem where the state equations become linear. This equivalent problem will be approximated by minimizing some quadratic functional over all pairs of state and (admissible) control subject to some linear state equations.

Using the same method, one may generalize the results of this paper to similar optimal control problems in one-phase Stefan problems or in melting-solidification processes with convection (for formulation of these problems, see [9]).

The paper is organized as follows. In §2, we will reformulate the problem (P) and then give the approximating problem (referred as problem (Q)). We will see that problem (Q) is a „good" approximation to problem (P). The existence of the optimal controls will be shown. In §3, we will derive the optimality conditions for optimal controls of problem (Q). Based on the results of §3, we will give in §4 an algorithm for approximating the solutions of problem (Q). The convergence of the algorithm will be demonstrated.

The notations in this paper are the same with [7].

2. Transformation and approximation of the original problem. Let $z^1 \in L^2(Q_0^+)$ be the solution to

$$\begin{aligned} &z_t^1 - k_1 \Delta z_1 = 0, \text{ in } Q_0^+, \\ &z^1(x, 0) = \theta_0(x), \text{ on } \{t=0\} \cap \overline{Q_0^+}, \; z^1 = 0, \text{ on } \Sigma_0, \\ &\frac{\partial z^1}{\partial n} = 0 \text{ on } \Sigma_1. \end{aligned} \tag{2.1}$$

For any $u \in \mathcal{U}$, let $z^2(u) = z^2 \in L^2(Q^-)$ be the solution to

$$\begin{aligned} &z_t^2 - k_2 \Delta z^2 = 0, \text{ in } Q_0^-, \\ &z^2(x, 0) = \theta_0(x), \text{ on } \{t=0\} \cap \overline{Q_0^-}, \\ &z^2 = 0, \text{ on } \Sigma_0, \; \frac{\partial z^2}{\partial n} + z^2 = Bu + v_0, \text{ on } \Sigma_2. \end{aligned} \tag{2.2}$$

The existence and uniqueness of solutions to (2.1) and (2.2) will be shown later (see Lemma 2.2). By (1.11), (1.10) and the maximum principle, we obtain that:

$$z^1 \geq 0, \text{ in } Q_0^+, \; z^2 \leq 0, \text{ in } Q_0^-.$$

Then it is immediate that problem (P) is equivalent to the following one (P′) *Find $u \in \mathcal{U}_{ad}$ such that*

$$\frac{\partial z^2(u)}{\partial \eta_x} = g_d, \text{ on } \Sigma_0,$$

where

$$g_d = \left(\frac{k_1}{k_2} \frac{\partial z^1}{\partial \eta_x} - \frac{L}{k_2} \eta_t\right) \Big|\, \Sigma_0. \tag{2.3}$$

We will see (Lemma 2.3 and Remark 2.1) that $\left.\dfrac{\partial z^1}{\partial \eta_x}\right|_{\Sigma_0}$, $\left.\dfrac{\partial z^2(u)}{\partial \eta_x}\right|_{\Sigma_0}$ are well-defined and belong to $L^2(\Sigma_0)$.

Consider the following approximating problem:

(Q) *Minimize* $J(u) = \displaystyle\int_{\Sigma_0} \left|\frac{\partial z^2(u)}{\partial \eta_x} - g_d\right|^2 d\sigma$ *over all* $u \in \mathcal{U}_{ad}$.

Let $u \in \mathcal{U}_{ad}$, define $y(u)$ by

$$(y(u))(x, t) = \begin{cases} k_1 \theta^1(u)(x, t), \text{ if } (x, t) \in Q^+(u), \\ k_2 \theta^2(u)(x, t), \text{ if } (x, t) \in Q^-(u), \end{cases} \tag{2.4}$$

where as before, $\theta^1(u)$, $\theta^2(u)$, $Q^+(u)$ and $Q^-(u)$ denote the solution to (1.1)—(1.7) and (1.9). The following proposition shows that problem (Q) is a „good" approximation for problem (P).

Proposition 2.1. *If $\{u_n\}$ is a sequence in $\mathcal{U}_{ad}$ satisfying*

$$J(u_m) \to 0,$$

then any weak limit of $\{u_m\}$ is an optimal control for problem (P). *Furthermore, if in addition*

$$u_m \to u^* \textit{ weakly in } \mathcal{U}, \tag{2.5}$$

then

(2.6) $y(u_m) \to y(u^*)$ *weakly in* $H^1(\Omega \times (0, T))$ *weak-star in* $L^\infty(0, T; H^1(\Omega))$.

Proof. By [8] Ch. 2, §3.3 or [2], we may show that $y(u_m) = y_m$ is the solution of

$$\begin{aligned} &\frac{\partial \beta(y_m)}{\partial t} - \Delta y_m = 0 \text{ in } \Omega \times (0, T), \\ &y_m(x, 0) = y_0(x) = \begin{cases} k_1 \theta_0(x) \text{ if } (x, 0) \in \overline{Q^+}, \\ k_2 \theta_0(x) \text{ if } (x, 0) \in \overline{Q^-}, \end{cases} \\ &\frac{\partial y_m}{\partial n} = 0 \text{ on } \Sigma_1, \; \frac{\partial y_m}{\partial n} + y_m = k_2 (Bu_m + v_0) \text{ on } \Sigma_2, \end{aligned} \tag{2.7}$$

where β is the enthalpy function:

$$\beta(y)=\begin{cases} k_1^{-1}y+L, & y>0 \\ [0,\ L], & y=0 \\ k_E^{-1}\,y, & y<0. \end{cases}$$

As $\mathcal{U}_{ad}$ is bounded, we may show that $\{y_m\}$ is bounded in $H^1(\Omega\times(0,\ T))\cap L^\infty(0,\ T\ ;\ H^1(\Omega))$. Now, if (2.5) holds, and if y^* is a weak limit point of $\{y_m\}$, then we have $y^*=y(u^*)$. So (2.6) is satisfied. Moreover, since $J(u_n)\to 0$, we must have $J(u^*)=0$ (see Lemma 2.3 below). Therefore, u^* is a solution for problem (P′) which is equivalent to problem (P). The rest of the proposition follows immediately.

In the following, we will study problem (Q). For convenience, we will denote $z=z^2$, $k=k_2$, $Q=Q_0^-$,

$$\Gamma_0(t)=\{x\in\Omega\ ;\ (x,\ t)\in\in\overline{\Sigma_0}\}\ ,$$

and

$$D(t)=\mathrm{int}\ \{x\in\overline{\Omega}\ ;\ (x,\ t)\in\overline{Q}=Q_0^-\}\ .$$

The set $D(t)$ is a bounded domain in R^N lying between $\Gamma_0(t)$ and Γ_2. Problem (Q) may be written as :

(Q) *Minimize*

$$J(u)=\int_{\Sigma_0}\left|\frac{\partial z(u)}{\partial\tau_{ix}}-g_d\right|^2 d\Sigma_0, \tag{2.8}$$

over $u\in\mathcal{U}_{ad}$, *where* $z(u)=z$ *is the solution of*

$$\begin{gathered} z_t-k\Delta z=0,\ \ in\ \ Q, \\ z(x,\ 0)=\theta_0(x),\ \ in\ \ D(0), \\ z=0,\ \text{on}\ \Sigma_0,\ \varkappa\,\frac{\partial z}{\partial n}+z=Bu+v_0,\ \text{on}\ \Sigma_2. \end{gathered} \tag{2.9}$$

The first remark is that the state equations (2.9) do not admit a smooth solution. However we will use the following definition :

Definition. *Let* $\psi\in C^1(\overline{\Sigma}_2)$, $\psi\geqslant 0$. *Let* $w\in L^2(\Sigma_0)$, $v\in L^2(\Sigma_1)$. *A function* $z\in L^2(Q)$ *is called a solution to the following equations*

$$\begin{gathered} z_t-k\Delta z=0\ \text{ in }\ Q,\ z(x,0)=\theta_0(x)\ \text{ in }\ D(o), \\ z=w,\ \text{on}\ \Sigma_0,\ \frac{\partial z}{\partial n}+\psi_z=v,\ \text{on}\ \Sigma_2, \end{gathered} \tag{2.10}$$

if for any test function $\varphi\in H^{2,1}(Q)$ *satisfying*

$$\varphi(x,\ T)=0\ \text{ in }\ D(T)\ \ \varphi=0,\ \text{on}\ \Sigma_0,\ \ \frac{\partial\varphi}{\partial n}+\psi\varphi=0\ \text{ on }\ \Sigma_2, \tag{2.11}$$

the following relation is satisfied :

$$\int_Q z(\varphi_t+k\Delta\varphi)\,dxdt=-\int_{D(o)}\theta_0(x)\ \varphi(x,\ 0)\ dx+k\int_{\Sigma_0}w\frac{\partial\varphi}{\partial\tau_{ix}}d\Sigma-k\int_{\Sigma_2}v\varphi d\Sigma_2. \tag{2.12}$$

The justification of this definition is standard.

Lemma 2.2. *For any* $w\in L^2(\Sigma_0)$, $v\in L^2(\Sigma_2)$, *there exists a unique solution* $z=z(w,\ v)$ *to* (2.10). *Moreover, the mapping*

$$(w,\ v)\in L^2(\Sigma_0)\times L^2(\Sigma_2)\to z(w,\ v)\in L^2(Q)$$

is affine and continuous.

Proof. We will proceed by transposition as in [7], Ch. 3, § 9.3. So let Φ be the Hilbert space of all functions in $H^{2,1}(Q)$ verifying (2.11). Since the normal $\tau_i=(\tau_{ix},\ \tau_{it})$ to Σ is nowhere parallel to the t-axis, we may show easily that the mapping $\mathcal{L}$: $\Phi\to L^2(Q)$ defined by

$$\mathcal{L}\Phi=\varphi_t+k\Delta\varphi \tag{2.13}$$

is an isomorphism. Define the linear continuous functional $F:\Phi\to R$ by

$$F(\varphi)=-\int_{D(o)}\theta_0(x)\ \varphi(x,\ 0)\ dx+k\int_{\Sigma_0}w\frac{\partial\varphi}{\partial\tau_{ix}}d\Sigma-k\int_{\Sigma_2}v\varphi d\Sigma. \tag{2.14}$$

Then by definition, Eq. (2.10) is equivalent to

$$\int_Q z(\mathcal{L}\varphi)\ dxdt=F(\varphi),\ \ \forall\varphi\in\ \Phi, \tag{2.15}$$

which obviously admits a unique solution in $L^2(Q)$.

Clearly, the mapping $(w,\ v)\to z(w,\ v)$ is affine and continuous.

As a direct consequence of this lemma, Eqns. (2.1) and (2.2) admit unique solutions.

Because $\theta_0\in H^1(\Omega)$ satisfies (1.11), and since Σ_0 is smooth and τ_{ix} is nowhere zero, we may deduce that the solution $z(u)$ to (2.9) satisfies the following regularity property

$$z(u)\in H^{2,1}(\Sigma_0^\varepsilon),\ \text{ for some }\ \varepsilon>0, \tag{2.16}$$

where Σ_0^ε denotes a ε-neighbourhood of Σ_0 in Q, i.e.,

$$\Sigma_0^\varepsilon=\{(x,\ t)\in Q\ ;\ \exists(\tilde{x},\ \tilde{t})\in\Sigma_0\ \text{such that}\ |\ x-\tilde{x}\ |+|\ t-\tilde{t}\ |<\varepsilon\}\ .$$

Therefore Lemma 2.2 implies :

Lemma 2.3. *The mapping*

$$u\to\left.\frac{\partial z(u)}{\partial\tau_{ix}}\right|_{\Sigma_0}$$

is affine and compact from $\mathcal{U}$ *to* $L^2(\Sigma_0)$.

Remark 2.1. Using the above argument, we may also show that the solution z^1 to Eq. (2.1) is regular enough in a region in Q_0^+ near Σ_0, so $\left.\frac{\partial z^1}{\partial\tau_{ix}}\right|_{\Sigma_0}$ exists and belongs to $L^2(\Sigma_0)$.

Now, as $\mathcal{U}_{ad}$ is closed, bounded and convex, it is weakly compact in $\mathcal{U}$. Using Lemma 2.3, we may obtain

Theorem 2.4. *Problem* (Q) *admits at least one optimal control.*

3. The optimality conditions. For any $u\in\mathcal{U}$, let $p(u)=p\in L^2(Q)$ be the solution of

$$\begin{gathered} p_t+k\Delta p=0\ \text{ in }\ Q, \\ p(x,\ T)=0\ \text{ in }\ D(T), \\ \varkappa\frac{\partial p}{\partial n}+p=0\ \text{ on }\ \Sigma_2,\ \ p=2\left(\frac{\partial z(u)}{\partial\tau_{ix}}-g_d\right)\ \text{ on }\ \Sigma_0. \end{gathered} \tag{3.1}$$

By Lemma 2.2, there exists a unique solution p to (3.1). Furthermore, one has

(3.2) $$p(u)\in H^{2,1}(\Sigma_2^\varepsilon), \text{ for some } \varepsilon>0.$$

Lemma 3.1. *Let u be any element of $\mathcal{U}$. Then*

(3.3)
$$J(v)-J(u)=-<B^*\left(\frac{1}{\alpha}p(u)\Big|_{\Sigma_2}\right),\ v-u>+$$
$$+\iint_{\Sigma_0}\left(\frac{\partial z(u)}{\partial\gamma_{1x}}-\frac{\partial_z(u)}{\partial\gamma_{1x}}\right)^2 d\Sigma,\ \forall\ v\in\mathcal{U},$$

where $B^*: L^2(\Sigma_2)^*=L^2(\Sigma_2)\to\mathcal{U}^*=\mathcal{U}$, is the dual of B, and $<.,.>$ denotes the scalar product in $\mathcal{U}$.

Proof. Let $u, v\in\mathcal{U}$. We set $y=z(v)-z(u)$. Then we have

(3.4)
$$y_t-k\Delta y=0 \text{ in } Q,\ y(x,0)=0, \text{ in } D(0),$$
$$y=0, \text{ on } \Sigma_0,\ \alpha\frac{\partial y}{\partial n}+y=B(v-u) \text{ on } \Sigma_2$$

and

(3.5) $$\frac{\partial y}{\gamma_{1x}}=\frac{\partial z(u)}{\partial\gamma_{1x}}-\frac{\partial_z(u)}{\partial\gamma_{1x}}, \text{ on } \Sigma_0.$$

Multiplying the first equation in (3.1) by y, and then integrating over Q, we obtain in virtue of (2.16) and (3.2) that

$$\int_Q p(u)\,(-y_t+k\Delta y)\,dxdt+\int_{D(T)}p(u)\,y\,dx-$$
$$-\int_{D(0)}p(u)\,y\,dx+\int_{\Sigma_1}p(u)\,y\gamma_{1t}d\Sigma+\int_{\Sigma_1}k\frac{\partial p(u)}{\partial\gamma_{1x}}\,y\,d\Sigma+$$
$$+\int_{\Sigma_2}k\frac{\partial p(u)}{\partial n}\,yd\Sigma-\int_{\Sigma_0}k\frac{\partial y}{\partial\gamma_{1x}}\,p(u)d\Sigma-\int_{\Sigma_2}k\frac{\partial y}{\partial\gamma_1}\,p(u)\,d\Sigma=0.$$

So, in virtue of (3.1), (3.4) and (3.5), we get

$$\iint_{\Sigma_2}\left(-\frac{1}{\alpha}p(u)\right)y\,d\Sigma-\int_{\Sigma_0}2\left(\frac{\partial z(u)}{\partial\gamma_{1x}}-\frac{\partial z(u)}{\partial\gamma_{1x}}\right)\left(\frac{\partial z(u)}{\gamma_{1x}}-g_d\right)d\Sigma-$$
$$-\iint_{\Sigma_2}\left(\frac{1}{\alpha}B(v-u)-\frac{1}{\alpha}y\right)p(u)\,d\Sigma=0.$$

Hence

$$\int_{\Sigma_0}2\left(\frac{\partial z(v)}{\partial\gamma_{1x}}-\frac{\partial z(u)}{\partial\gamma_{1x}}\right)\left(\frac{\partial z(u)}{\partial\gamma_{1x}}-g_d\right)d\Sigma=$$
$$=-\int_{\Sigma_2}B(v-u)\frac{1}{\alpha}p(u)\,d\Sigma=-<B^*\left(\frac{1}{\alpha}p(u)\Big|_{\Sigma_2}\right),\ v-u>,$$

which implies (3.3). Then by Lemma 2.3, we get.

Lemma 3.2. *There exists a constant $M>0$ such that*

(3.6) $$\iint_{\Sigma_0}\left(\frac{\partial z(u)}{\partial\gamma_{1x}}-\frac{\partial z(v)}{\partial\gamma_{1x}}\right)^2 d\Sigma\leqslant M\ \|u-v\|^2,$$

for all $u, v\in\mathcal{U}$, where $\|\ .\ \|$ denotes the norm of $\mathcal{U}$.

Then as an immediate consequence of Lemmas 3.1 and 3.2 we get.

Theorem 3.3. *$u^*\in\mathcal{U}_{ad}$ is an optimal control for problem (Q) if and only if*

(3.7) $$<B^*\left(\frac{1}{\alpha}p(u^*)\Big|_{\Sigma_2}\right),\ v-u^*>\ \leqslant 0,\ \forall\ u\in\mathcal{U}_{ad}.$$

4. **An algorithm.** As g_d may be obtained by (2.1) and (2.3), we assume that it is known. Let $M>0$ be any constant satisfying (3.6). We propose the following algorithm for approximating the solutions of problem (Q).

Algorithm (A).

Step (i). Choose any $u\in\mathcal{U}_{ad}$.

Step (ii). Solve Eqs. (2.9). to obtain $\frac{\partial z(u)}{\partial\gamma_{1x}}\Big|_{\Sigma_0}$, then solve (3.1), to obtain $p(u)$. Compute

(4.1) $$\mu(u)=B^*\left(\frac{1}{\alpha}p(u)\Big|_{\Sigma_2}\right).$$

Step (iii). If $\mu(u)\in N(u)=\{\gamma_i\in\mathcal{U};\ <\gamma_i, v-u>\ \leqslant 0,\ \forall\, v\in\mathcal{U}_{ad}\}$, Stop; (by Theorem 3.3) u is an optimal control for problem (Q); otherwise, let $w(u)$ be any element of $(N)^{-1}(\mu(u))$ that is $w(u)\in\mathcal{U}_{ad}$ and $\mu(u)\in N(w(u))$. Calculate

(4.2) $$\lambda(u)=\mathrm{Min}\left\{\frac{<\mu(u),\ w(u)-u>}{2M\ \|w(u)-u\|^2},\ 1\right\},$$

and set

(4.3) $$\tilde{u}(u)=(1-\lambda(u))\,u+\lambda(u)\,w(u).$$

Step (iv). Replace u by $\tilde{u}(u)$ and go to *Step* (ii).

Lemma 4.1. *Suppose that $u\in\mathcal{U}_{ad}$ is not optimal (for problem (Q)). Let $\mu(u)$, $w(u)$, $\lambda(u)$ and $\tilde{u}(u)$ be constructed as in Algorithm (A), then $\tilde{u}(u)\in\mathcal{U}_{ad}$ and*

(4.4) $$J(\tilde{u}(u))\leqslant J(u)-\tfrac{1}{2}\lambda(u)<\mu(u),\ w(u)-u>.$$

Proof. Firstly, we note that $(N)^{-1}(\mu(u))=\partial I^*_{\mathcal{U}_{ad}}(\mu(u))=(\partial I_{\mathcal{U}_{ad}})^{-1}(\mu(u))$, where $I_{\mathcal{U}_{ad}}:\mathcal{U}\to[0,+\infty]$ is the indicator function of $\mathcal{U}_{ad}$ and $I^*_{\mathcal{U}_{ad}}:\mathcal{U}\to(-\infty\ +\infty)$ is its conjugate. So $w(u)\in(N)^{-1}(\mu(u))$ implies that $<\mu(u),\ w(u)-u>\ \geqslant 0$. It follows that $\lambda(u)\geqslant 0$, and hence $\tilde{u}(u)\in\mathcal{U}_{ad}$ by the convexity of $\mathcal{U}_{ad}$. Now, by Lemmas 3.1 and 3.2 we get

$$J(\tilde{u}(u))\leqslant J(u)-<\mu(u),\ \tilde{u}(u)-u>+M\ \|\tilde{u}(u)-u\|^2=$$
$$=J(u)-\lambda(u)<\mu(u),\ w(u)-u>+M\lambda^2\ \|w(u)-u\|^2,$$

which in virtue of (4.2) implies (4.4). Q.E.D.

Lemma 4.2. *Let u, $\mu(u)$ and $w(u)$ be as in Lemma 4.1, and let u^* be any optimal control of problem (Q). Then*

(4.5) $$0<J(u)-J(u^*)<\ <\mu(u),\ w(u)-u>.$$

Proof. The first inequality in (4.5) is trivial.
Using Lemma 3.1, we have

$$J(u^*)=J(u)-<\mu(u),\ u^*-u>+\iint_{\Sigma_0}\left(\frac{\partial z(u^*)}{\partial n_x}-\frac{\partial z(u)}{\partial n_x}\right)^2 d\Sigma$$
$$>J(u)-<\mu(u),\ u^*-u>$$
$$=J(u)-<\mu(u),\ u^*-w(u)>-<\mu(u),\ w(u)-u>$$
$$\geqslant J(u)-<\mu(u),\ w(u)-u>,$$

which is just the second inequality in (4.5).
Theorem 4.3 bellow shows that algorithm A is convergent.

Theorem 4.3. *Let $u_1\in\mathcal{U}_{ad}$, $u_2=\tilde{u}(u_1)$, ..., $u_{m+1}=\tilde{u}(u_m)$, ..., be the sequence constructed in algorithm* (A). *If $\{u_m\}$ is finite, then the last element is an optimal control for problem* (Q). *If $\{u_m\}$ is infinite, then any weak limit of $\{u_m\}$ is an optimal control for problem* (Q). *Furthermore, we have*

$$J(u_{m+1})-J(u^*)\leqslant(J(u_m)-J(u^*))\ \mathrm{Max}\left\{1-\frac{J(u_m)-J(u^*)}{4M\,\|w(u_n)-u_n\|^2},\ \frac{1}{2}\right\}\tag{4.6}$$

where u^* is any optimal control for problem (Q).

If M_1 is the radius of $\mathcal{U}_{ad}$, then (4.6) *implies that*

$$J(u_{m+1})-J(u^*)\leqslant(J(u_m)-J(u^*))\ \mathrm{Max}\left\{1-\frac{J(u_m)-J(u^*)}{4\,MM_1^2},\ \frac{1}{2}\right\}.\tag{4.7}$$

Proof. The theorem is evident if $\{u_m\}$ is finite. Now assume that $\{u_m\}$ is infinite. Then (4.6) follows by (4.4), (4.2) and (4.5). Relation (4.7) is a trivial consequence of (4.6). From (4.7), we deduce that $J(u_m)-J(u^*)$ is strictly decreasing and tends to zero. So by Lemma 2.3, we deduce that any weak limit of $\{u_m\}$ is an optimal control for problem (P).

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ON THE DISCRETE INEQUALITIES OF GRONWALL-BELLMAN TYPE

BY

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1. **Introduction.** Gronwall-Bellman inequalities play an important role in the study of various problems of integro-differential and summary-difference equations (see [8], [9]). Many types of the discrete analogies of integral and differential inequalities known in the continuous case have been studied recently. Among them, there have been considered discrete inequalities of one independent variable [4, 6], two variables [1, 10, 11] also involving higher order partial differences [5], many variables [3], in the distributive lattice [7]. Finite systems of difference inequalities have been studied in [2] using comparison theorems. In the present note we shall consider finite systems of discrete inequalities of one variable, however not using comparison methods strictly.

2. **Definitions and Notations.** We denote by N the set $\{n_0, n_0+1, n_0+2, \dots\}$ where n_0 is any fixed nonnegative integer, by S the set $\{1, 2, \dots, s\}$, and by R, R_0, R_+ the set of real numbers, nonnegative real numbers and positive real numbers, respectively.

For the function $y: N\to R$ the difference operator Δ is defined as usually $\Delta y(n)=y(n+1)-y(n)$, $\Delta^2 y(n)=\Delta(\Delta y(n))$. In the following section we shall use the following functions with their prescribed properties:

(i) $g_k,\ u_k: N\to R_0$, $f_k: N\to R$, $k\in S$,

(ii) $B: N\times R^s\to R$, B is a continuous function of s-variables for every fixed first variable n, $n\in N$

(ii)$_1$ $0\leqslant B(n, y_1, \dots y_s)\leqslant B(n, x_1, \dots, x_s)$ for $n\in N$, $0\leqslant y_k\leqslant x_k$, $k\in S$,

(ii)$_2$ $B(n, ay_1, \dots, ay_s)\leqslant A(a)\,B(n, y_1, \dots, y_s)$ for arbitrary $a\geqslant 1$,

where the function A is continuous, nondecreasing and positive on $[1, \infty)$. Instead to write $B(n, y_1, \dots, y_s)$ we shall use $B(n, [y_i]s)$.

(iii) $G: N\times R_0\to R_0$, G is continuous for every fixed $n\in N$,

$0\leqslant G(n, y)\leqslant G(n, x)$ for $n\in N$, $0\leqslant y\leqslant x$,

$G(n, ay)\leqslant E(a)\,G(n, y)$ for nonnegative numbers a,

where the function E is continuous, nonnegative and nondecreasing on R_0,

(iv) $h_k: N\to R_0$, $k\in S$.

Furthermore it is supposed

$$\prod_{j=n_0}^{n_0-1} y(j)=1 \quad\text{and}\quad \sum_{j=n_0}^{n_0-1} y(j)=0.$$